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SOME SUBMANIFOLDS OF COMPLEX PROJECTIVE SPACES

茨城大学教育学部 木村真琴 (MAKOTO KIMURA)

We discuss a generalization of ruled surfaces in $\mathbb{R}^3$ to submanifolds in complex projective spaces. A ruled surface is generated by $1-$parameter family of lines in $\mathbb{R}^3$. Examples are: hyperboloid of one sheet, hyperbolic paraboloid, circular cylinder, circular conic and right helicoid. It is classically known that a ruled surface M in $\mathbb{R}^3$ is minimal if and only if M is a part of a plane $\mathbb{R}^2$ or a right helicoid.

We denote by $\mathbb{P}^n(\mathbb{C})$ an n-dimensional complex projective space with Fubini-Study metric of holomorphic sectional curvature 1 unless otherwise stated. It is known that a totally geodesic submanifold of complex projective space $\mathbb{P}^n(\mathbb{C})$ is one of the following:

(a) Kähler submanifold $\mathbb{P}^k(\mathbb{C})$ $(k < n)$,
(b) Totally real submanifold $\mathbb{P}^k(\mathbb{R})$ $(k \leq n)$.

Now we study the following submanifolds in $\mathbb{P}^n(\mathbb{C})$:

(1) Kähler submanifold M^{k+r} on which there is a holomorphic foliation of complex codimension r and each leaf is a totally geodesic $\mathbb{P}^k(\mathbb{C})$ in $\mathbb{P}^n(\mathbb{C})$ (for simplicity, we said that M is holomorphically $k-$ruled).
(2) Totally real (Lagrangian) submanifold M^n on which there is a (real) foliation of real codimension $n-k$ and each leaf is a totally geodesic $\mathbb{P}^k(\mathbb{R})$ in $\mathbb{P}^n(\mathbb{C})$.

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1. The set of k dimensional totally geodesic complex projective subspaces in $\mathbb{P}^n(\mathbb{C})$ is identified with the complex Grassmann manifold $\mathbb{G}_{k+1,n-k}(\mathbb{C})$ of $k+1$ dimensional complex linear subspaces in $\mathbb{C}^{n+1}$:

$$\{\mathbb{P}^k(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})\} \cong \{\mathbb{C}^{k+1} \subset \mathbb{C}^{n+1}\} = \mathbb{G}_{k+1,n-k}(\mathbb{C}).$$

Hence there is a natural correspondence between Kähler submanifolds M^{k+r} holomorphically foliated by totally geodesic $\mathbb{P}^k(\mathbb{C})$ in $\mathbb{P}^n(\mathbb{C})$ and Kähler submanifolds Σ^r in the complex Grassmann manifold $\mathbb{G}_{k+1,n-k}(\mathbb{C})$.

Here we note that a holomorphically $k-$ ruled Kähler submanifold M^{k+r} in $\mathbb{P}^n(\mathbb{C})$ is totally geodesic if and only if the corresponding Kähler submanifold $\Sigma^r \hookrightarrow \mathbb{G}_{k+1,n-k}(\mathbb{C})$ is contained in some totally geodesic $\mathbb{G}_{k+1,r}(\mathbb{C})$ in $\mathbb{G}_{k+1,n-k}(\mathbb{C})$ (cf. [CN]).

Examples of holomorphically k−ruled submanifolds in $\mathbb{P}^n(\mathbb{C})$.

(1) totally geodesic $\mathbb{P}^{k+r}(\mathbb{C})$,

(2) Segre imbedding $\mathbb{P}^k(\mathbb{C}) \times \mathbb{P}^r(\mathbb{C}) \hookrightarrow \mathbb{P}^{kr+k+r}(\mathbb{C})$.

In the case $k = r = 1$, we can see that

Theorem 1. *Let M^2 be a Kähler surface in $\mathbb{P}^n(\mathbb{C})$ holomorphically foliated by totally geodesic projective lines $\mathbb{P}^1(\mathbb{C})$ in $\mathbb{P}^n(\mathbb{C})$. If the scalar curvature of M^2 is constant and M^2 is not totally geodesic, then the Gauss curvature of the corresponding holomorphic curve Σ^1 in $\mathbb{G}_{2,n-1}(\mathbb{C})$ is constant.*

Remark. The converse of the above Theorem does not hold. Consider the following holomorphic imbedding:

$$\mathbb{P}^1(\mathbb{C}) \longrightarrow \mathbb{P}^k(\mathbb{C}) \times \mathbb{P}^\ell(\mathbb{C}) \longrightarrow \mathbb{G}_{2,k+\ell}(\mathbb{C}),$$

where the first imbedding is the product of Veronese imbeddings of degree k and ℓ, and the second one is the totally geodesic holomorphic imbedding of $\mathbb{P}^k(\mathbb{C}) \times \mathbb{P}^\ell(\mathbb{C})$ into $\mathbb{G}_{2,k+\ell}(\mathbb{C})$ (cf. [CN]). By direct calculations, we can see that the Gauss curvature of the induced metric on $\mathbb{P}^1(\mathbb{C})$ is $\frac{1}{k+\ell}$. But the scalar curvature of the corresponding holomorphically 1−ruled Kähler surface M^2 in $\mathbb{P}^{k+\ell+1}(\mathbb{C})$ is constant only when $k = \ell$. In this case, the Kähler surface M^2 is holomorphically congruent to

$$\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1_{1/k}(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^k(\mathbb{C}) \to \mathbb{P}^{2k+1}(\mathbb{C}),$$

where the first one is the product of the identity map and the Veronese imbedding of degree k, and the second one is the Segre imbedding. These examples show that contrary to Kähler submanifolds in $\mathbb{P}^n(\mathbb{C})$ (cf. [E]), the rigidity for Kähler submanifolds in complex Grassmann manifolds (of rank ≥ 2) does not hold (cf. [CZ]).

2. The set $W_{k,n}$ of k dimensional totally geodesic real projective subspaces $\mathbb{P}^k(\mathbb{R})$ in $\mathbb{P}^n(\mathbb{C})$ is:

$$W_{k,n} = \begin{cases} \frac{SU(n+1)}{SO(n+1)}, & \text{if } n = k, \\ \frac{SU(n+1)}{K_{k+1,n-k}}, & \text{if } n > k, \end{cases}$$

where

$$K_{k+1,n-k} = \left\{ \begin{pmatrix} e^{\frac{i\theta}{k+1}} P & 0 \\ 0 & e^{-\frac{i\theta}{n-k}} Q \end{pmatrix} \middle| \; \theta \in \mathbb{R},\ P \in SO(k+1),\ Q \in SU(n-k) \right\}.$$

Let $G = SU(n+1)$, $\mathfrak{g} = \mathfrak{su}(n+1)$, and

$$\mathfrak{k} = \left\{ i \begin{pmatrix} \frac{\theta}{k+1} E_{k+1} & 0 \\ 0 & -\frac{\theta}{n-k} E_{n-k} \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \middle| \; \theta \in \mathbb{R},\ U \in \mathfrak{so}(k+1),\ V \in \mathfrak{su}(n-k) \right\}, \tag{2}$$

where E denotes identity matrix. Then $\mathfrak{k}$ is the Lie algebra of the Lie group $K = K_{k+1,n-k}$. Put

$$\mathfrak{m} = \left\{ \begin{pmatrix} iW & Z \\ -Z^* & 0 \end{pmatrix} \,\middle|\, W \in S_0(k+1,\mathbb{R}),\ Z \in M(k+1, n-k, \mathbb{C}) \right\},$$

where $S_0(k+1,\mathbb{R})$ denotes the set of $(k+1)\times(k+1)$ (real) symmetric matrices with trace $= 0$. Then we have $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ (direct sum) and $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$. Since K is connected, the homogeneous space G/K is reductive. Note that G/K is naturally reductive with respect to some metric.

As §1, we can see that there is a one-to-one correspondence between $k+r$ $(0 < r < 2n-k)$ dimensional submanifolds foliated by totally geodesic, totally real $\mathbb{P}^k(\mathbb{R})$ in $\mathbb{P}^n(\mathbb{C})$ and r dimensional submanifolds in G/K.

There is a natural fibration on G/K as follows: Let $\pi : G/K \to G/H$ be the projection defined by $gK \mapsto gH$, where $K = K_{k+1,n-k}$ and $H = S(U(k+1)\times U(n-k))$. Note that $K \subset H$ and $G/H = \mathbb{G}_{k+1,n-k}(\mathbb{C})$. Then it can be seen that π is a Riemannian submersion with which each fibre $H/K \cong SU(k+1)/SO(k+1)$ is totally geodesic in G/K.

Since $G/H = \mathbb{G}_{k+1,n-k}(\mathbb{C})$ is a Kähler manifold, there is the almost complex structure on the horizontal distribution of G/K compatible to π and the canonical complex structure of G/H.

We claim

Proposition 2. *Let M^n be a submanifold in $\mathbb{P}^n(\mathbb{C})$ foliated by totally geodesic $\mathbb{P}^k(\mathbb{R})$. Then M^n is totally real (Lagrangian) if and only if the corresponding submanifold Σ^{n-k} in G/K is "horizontal" and "totally real" with respect to the almost complex structure on the horizontal distribution of G/K as above.*

Hence if M^n is a totally real submanifold in $\mathbb{P}^n(\mathbb{C})$ foliated by totally geodesic $\mathbb{P}^k(\mathbb{R})$, then there is a corresponding totally real submanifold Σ^{n-k} in $G/H = \mathbb{G}_{k+1,n-k}(\mathbb{C})$. So we consider the following question: If Σ^{n-k} is a (totally real) submanifold in $G/H = \mathbb{G}_{k+1,n-k}(\mathbb{C})$, then does there exist a "horizontal lift" of Σ^{n-k} in G/K with respect to π? Note that a totally real submanifold M^n foliated by totally geodesic $\mathbb{P}^k(\mathbb{R})$ in $\mathbb{P}^n(\mathbb{C})$ is totally geodesic if and only if the corresponding submanifold Σ^{n-k} in G/K is contained in some $\pi^{-1}(\mathbb{G}_{k+1,n-k}(\mathbb{R}))$, where $\mathbb{G}_{k+1,n-k}(\mathbb{R})$ is a maximal totally geodesic, totally real submanifold in $G/H = \mathbb{G}_{k+1,n-k}(\mathbb{C})$.

Hereafter we assume that: Let G be a linear Lie group, let $H \supset K$ be closed subgroups of G such that G/K is a reductive homogeneous space, $G/H$is a Riemannian symmetric space and the projection $\pi : G/K \to G/H$ defined by $gK \mapsto gH$ is a Riemannian submersion with which each fibre H/K is totally geodesic in G/K. We denote the projections as $\pi_H : G \to G/H$ and $\pi_K : G \to G/K$. Let $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{k}$ be the Lie algebras of the Lie groups G, H, K, respectively, and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m} = \mathfrak{h} + \mathfrak{p}$ be the canonical decompositions of $\mathfrak{g}$.

Let $f : \Sigma \to G/H$ be an isometric immersion. We would like to find the "condition" for existence of a "horizontal lift" of f. By using a local cross section $G/H \to G$, we always have a "framing" $\Phi : \Sigma \to G$ satisfying $f = \pi_H \circ \Phi$ **locally**. To construct a "horizontal lift" of f, we may find the map $\Psi : \Sigma \to H$ such that $F := \pi_K \circ (\Phi \cdot \Psi) : \Sigma \to G \to G/K$ is the desirable horizontal lift. Here $\Phi \cdot \Psi$ is defined by the product of $\Phi(x) \in G$ and $\Psi(x) \in H \subset G$ for $x \in \Sigma$ as elements of the Lie group G.

Let $\alpha := \Phi^{-1} d\Phi$ (resp. $\beta := \Psi^{-1} d\Psi$) be the $\mathfrak{g}$−valued (resp. $\mathfrak{h}$−valued) 1-form on Σ which is the pull-back of the Maurer-Cartan form of G (resp. H). The following fact is well known (cf. [G]): Let ω be the Maurer-Cartan form on G. Suppose that α is a $\mathfrak{g}$−valued 1-form on a connected and simply connected manifold Σ. Then there exists a C^∞ map $\Phi : \Sigma \to G$ with $\Phi^* \omega = \alpha$ if and only if $d\alpha + \frac{1}{2}[\alpha, \alpha] = 0$. Moreover, the resulting map is unique up to left translation.

We see that

$$\begin{aligned} & F = \pi_K \circ (\Phi \cdot \Psi) : \Sigma \to G/K \text{ is horizontal,} \\ \iff \quad & (\mathfrak{h} \cap \mathfrak{m}) - \text{part of } (\Phi \cdot \Psi)^{-1} d(\pi_k \circ (\Phi \cdot \Psi)) = 0, \\ \iff \quad & \beta_{\mathfrak{m}} + (\mathrm{ad}(\Psi^{-1})\alpha)_{\mathfrak{h} \cap \mathfrak{m}} = 0. \end{aligned}$$

Taking an exterior derivation and using the integrability conditions of α and β, we get that the condition for existence of a horizontal lift of the isometric immersion $f : \Sigma \to G/H$ is

$$[\alpha_{\mathfrak{p}} \wedge \alpha_{\mathfrak{p}}]_{\mathfrak{h} \cap \mathfrak{m}} = 0.$$

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