数理解析研究所講究録
908 巻 1995 年 49-60

Steady pulse solutions to an RLW equation with instability and dissipation

大信田 丈志 (*OOSHIDA Takeshi*), Faculty of Science
川原 琢治 (*Takuji Kawahara*), Faculty of Engineering
Kyoto University

Abstract

Properties of steady pulse solutions are investigated numerically and analytically for the regularized-long-wave(RLW) equation including the Kuramoto-Sivashinsky terms. Both positive and negative pulses with oscillatory or monotone tails are found to be available depending on the effects of instability, dissipation, and "base line". The solutions are classified in terms of the two parameters defined by the ratios of three effects.

1 Introduction

Typical wave phenomena involved in nonlinear dispersive media with both instability and dissipation, such as long waves on liquid films, have been investigated in terms of the Benney equation [1]

$$[\,\partial_t + \partial_x + \partial_x^3\,]\,u + u\partial_x u + [\,\alpha\partial_x^2 + \beta\partial_x^4\,]\,u = 0 \quad (\alpha > 0,\ \beta > 0). \tag{1}$$

Its linear dispersion relation is given by

$$\omega = k - k^3 + i(\alpha k^2 - \beta k^4) \equiv \mathrm{Re}\,\omega + i\mathrm{Im}\,\omega, \tag{2}$$

and $\mathrm{Re}\,\omega = k - k^3 \equiv \omega_{\mathrm{KdV}}$ is the dispersion of the Korteweg-de Vries(KdV) equation [2]. In some media, however, the dispersion relation is not necessarily approximated by the KdV term. It would be then interesting to investigate such non-KdV-like dispersion in comparison with the Benney equation.

In this paper, we take up the following equation

$$[\,\partial_t + \partial_x - \partial_t\partial_x^2\,]\,u + u\partial_x u + [\,\alpha\partial_x^2 + \beta\partial_x^4\,]\,u = 0 \quad (\alpha > 0,\ \beta > 0). \tag{3}$$

The first two parts of eq.(3) is the Regularized-Long-Wave(RLW) equation [3][4]

$$[\,\partial_t + \partial_x - \partial_t\partial_x^2\,]\,u + u\partial_x u = 0, \tag{4}$$

which approximately governs the drift waves in plasmas [5][6] or the void (volume fraction) waves in a general one-dimensional model of two-phase systems [7]. The linear dispersion relation of the RLW equation is given by

$$\omega_{\mathrm{RLW}} = k/(1 + k^2), \tag{5}$$

which is reduced to the KdV dispersion $\omega_{\rm KdV}$ when expanded for small k. The third part of eq.(3) is the Kuramoto-Sivashinsky(KS) terms representing instability (proportional to α) and dissipation (proportional to β) [8][9].

The original Benney equation (1) is known to possess a steady pulse solution thanks to a balance between instability and dissipation [10]. The attempt to understand the dynamics(i.e., the time evolutions) of solutions to eq.(1) in terms of such soliton-like pulses weakly interacting with one another was quite successful [11][12].

Now the question is whether the same treatment is applicable to the case of eq. (3), i.e., the RLW equation with the KS terms, and to seek what differences arise due to the change in the linear dispersion term. As the first step to this aim, in this paper, we explore steady pulse solutions of eq.(3).

In § 2, we summarize some existing results for the RLW equation and the Benney equation. Perturbation solutions and numerical results of steady pulse solutions are given in § 3. Effect of the boundary conditions at infinity is discussed in § 4. Conclusions are given in § 5.

2 Preliminary Remarks

For convenience of later discussion we summarize several results for the RLW equation and the Benney equation.

2.1 RLW equation

The RLW equation (4) was derived first by Peregrine [3] by means of shallow water approximation of an "undular bore" in a canal. As is well known, an analysis of essentially the same problem has given birth to the KdV equation [2]; in fact, *as far as extremely long waves are concerned, the RLW equation is equivalent to the KdV equation.* This is clear if we expand (5) for $k \ll 1$, as $\omega_{\rm RLW} = k(1 - k^2 + k^4 - \cdots)$, to find the leading two terms are nothing but ω_{KdV}. Another way to show the equivalence between the RLW equation and the KdV equation is to note that for unidirectional propagation of long waves $\partial_t u \simeq -\partial_x u$ [4].

The RLW equation, however, is different from the KdV equation in the following points:

1. The propagation speed is finite for short wave components, because it follows from (5) that

$$|\omega_{\rm RLW}/k| = 1/(1+k^2) < +\infty. \tag{6}$$

In contrast, $\omega_{\rm KdV}/k = 1-k^2$ goes to $-\infty$ as $k \to +\infty$. It means that the RLW equation is a "regularization" of the KdV equation in the sense that the former is free from the non-locality of the latter; this is why Benjamin, Bona and Mahony named eq.(4) "*Regularized-Long-Wave equation*" [4].

2. The RLW equation admits two distinct families of the solitary wave solutions characterized by positive or negative amplitude. For the boundary conditions $u(z \to \pm\infty) = 0$, they are provided by

$$u_{\rm RLW} = A\,{\rm sech}^2\,[(x-ct)/\lambda] \quad (\lambda > 0), \tag{7}$$

$$A = A(\lambda) = 12/(\lambda^2 - 4), \qquad c = c(\lambda) = \lambda^2/(\lambda^2-4). \tag{8}$$

This solution consists of two distinct branches, namely (i) KdV branch ($\lambda > 2$, $A > 0$, $c > 1$) and (ii) plasma branch ($0 < \lambda < 2$, $A < -3$, $c < 0$). When λ passes across

2, the pulse solution (7) suffers discontinuous changes. The branch of negative amplitude solitary waves is called "plasma branch", because the RLW equation is applicable to drift waves in plasma. In that case, solitary waves with $c < 0$, as well as those with positive velocity, must be observable [13][14].

3. The RLW equation is not invariant under the so called Galilei transform. This fact leads us to consider "base line"(i.e., u_b) which defines the boundary conditions by $u(x \to \pm\infty) = u_b$. Concerning the KdV equation, the boundary value problem with arbitrary u_b is reduced to the zero-boundary value case by means of the so called *Galilei transform*:

$$u(x,t) = u_b + v(x',t'), \quad x = x' + u_b t', \quad t = t', \tag{9}$$

so that a solution v solved under the boundary conditions $v(x' \to \pm\infty) = 0$ can provide a solution u under non-zero boundary conditions. However, such Galilei invariance does not hold for the RLW equation. Instead, the following rescaling

$$u = u_b + (1+u_b)\tilde{u}, \quad x = \pm\tilde{x} \quad (u_b \gtrless -1), \quad t = \tilde{t}/|1+u_b| \tag{10}$$

can be used to reduce the problem to the case of zero boundary conditions ($u(x \to \pm\infty) = 0$) conserving the form of (4) when $u_b \neq -1$.

4. The RLW equation is not completely integrable and seems to be not solvable by the inverse scattering method, since it admits only several numbers of conserved quantities. This fact indicates that the solitary waves given by (7) are not 'soliton' in an exact sense but they suffer changes in collisions [14].

For $u_b \neq -1$, solitary wave solutions are given by

$$u_{\mathrm{RLW}} = u_b + A\,\mathrm{sech}^2\,[(x-ct)/\lambda] \quad (\lambda > 0), \tag{11}$$

$$A = A(\lambda) = 12(u_b+1)/(\lambda^2-4), \qquad c = c(\lambda) = \lambda^2(u_b+1)/(\lambda^2-4). \tag{12}$$

For $u_b = -1$ we put $u(x,t) = -1 + v(x,t)$, then v satisfies the *Equal-Width equation*

$$[1-\partial_x^2]\partial_t u + u\partial_x u = 0. \tag{13}$$

A steady pulse solution to this equation is given by

$$u = 3c\,\mathrm{sech}^2\,[(x-ct)/2] \quad (\ u(z \to \pm\infty) = 0\ ), \tag{14}$$

and its "width" (characteristic length) remains constant for various values of c. This solution can readily be obtained from (11) by setting $u_b \to -1$ and $\lambda \to 2$, i.e., A and c can become finite for $u_b = -1$ only if $\lambda = 2$.

2.2 Steady solutions of the Benney equation

While the solitary wave solution to the KdV equation contains one arbitrary parameter, the Benney equation does not allow such a freedom [10]. When the KS terms are added as small perturbations to the KdV equation, then a pulse which propagates in constant form and speed under the KdV dynamics must generally grow or shrink unless the effect of perturbation balances each other and determines a special value for the parameter λ signifying the pulse width. Therefore a steady pulse solution of the Benney equation bears a definite amplitude, form and velocity corresponding to each fixed set of the control parameters.

A parallel argument suggests that one-parameter set of the steady pulses of the RLW equation is reduced to only one pulse with constant form under the effect of the added KS

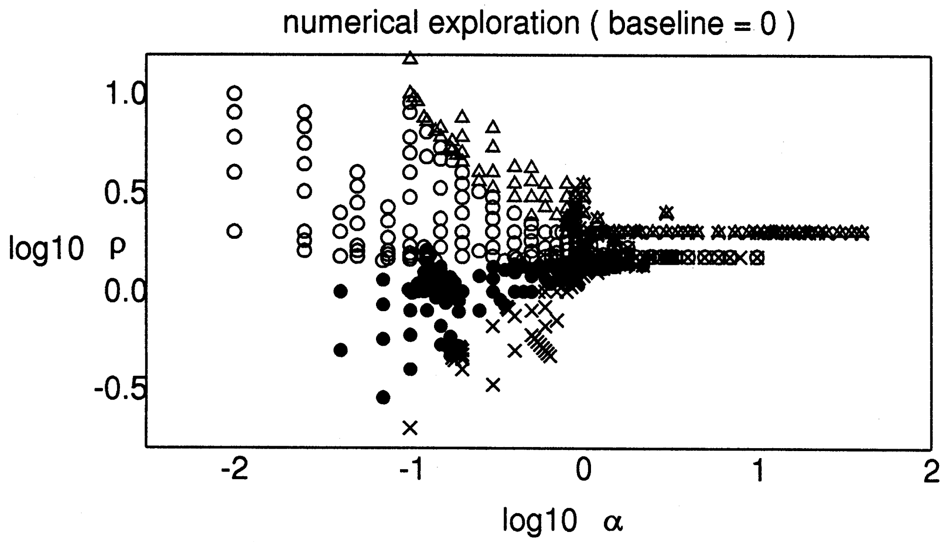

Figure 1: The range of (α, β) for numerical exploration of steady pulses: $\rho = \beta/\alpha$, $u_b = 0$. Symbols stand for: $\circ$ and $\bullet$, positive and negative pulses with monotone tails; and $\triangle$ and $\times$, positive and negative ones with oscillatory tails, respectively.

terms. Perturbation and numerical analyses done in the sequel support this suggestion, as long as the relative effect of the KS terms is not very large. When that effect amplified, pulses with oscillatory tails resembling those of the KS equation appear. As will be shown, *two* pulses (either positive or negative amplitude) are possible for the same set of the parameter values.

To explore such steady solutions, we put $u = u(z)$, $z = x - ct$ in eq.(3) leading to the fourth-order ordinary differential equation,

$$[(1-c)\partial_z + c\partial_z^3]\,u + u\partial_z u + [\alpha\partial_z^2 + \beta\partial_z^4]\,u = 0 \quad (\alpha > 0,\ \beta > 0), \tag{15}$$

of the same form as for the Benney equation (1). Then the form and the amplitude of the equilibrium pulse and the velocity c must be determined at the same time, if the two parameters (α, β) and the "base line" $u_b = u(z \to \pm\infty)$ are assigned. Then two questions, i.e., (i). How the solutions depend on (α, β)? and (ii). How the solutions depend on u_b? are posed. Now we investigate these two points by numerical and mathematical methods.

3 Dependence of pulse solutions on (α, β)

In this section we focus our attention to find how a pulse solution of eq.(15)(steady pulse solution of eq.(3)) depends on the values of α (instability) and β (dissipation).

3.1 Approximation from the RLW equation

First, we consider the case when the effect of the KS terms is small and deal with eq.(3) as the RLW equation with perturbation by rewriting as

$$[\partial_t + \partial_x - \partial_t\partial_x^2]\,u + u\partial_x u = \epsilon\,\hat{\Pi}u, \quad \hat{\Pi} \equiv -[\alpha'\partial_x^2 + \beta'\partial_x^4], \tag{16}$$

where $(\alpha, \beta) \equiv \epsilon\,(\alpha', \beta')$, so that $\alpha' \sim \beta' \sim O(1)$.

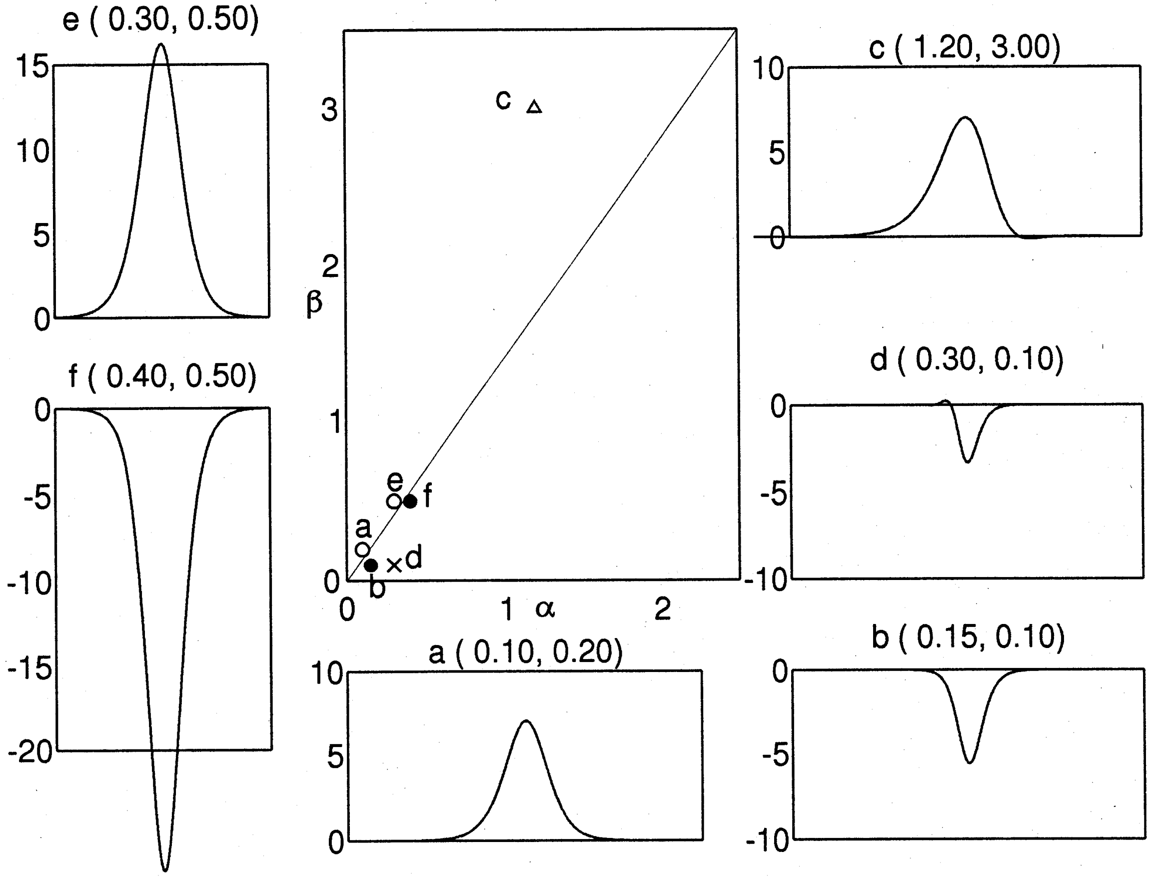

Figure 2: Pulse solutions of the equation (15), numerically obtained for the parameter values indicated in the graph at the center (and for $u_b = 0$). (a),(c),(e) are positive pulses, while (b),(d),(f) are negative. (c) and (d) have an oscillatory tail. The line ($\beta/\alpha = 7/5$) on the parameter plane is predicted by (22) to divide the territories of the positive and negative pulses. Symbols are the same as in Fig. 1.

The effect of perturbation is tractable by means of the *method of modified conservation laws* [10] [16]. We utilize the fact that the RLW equation conserves the following three quantities:

$$M = \int u_0 \, dx, \quad P = \int [\, u_0^2/2 + (\partial_x u_0)^2/2\,] \, dx, \quad H = \int (\, u_0^3/6 + u_0^2/2\,) \, dx, \tag{17}$$

where u_0 denotes a solution of the RLW equation (4). In the perturbed equation (16), P and H are no more conserved quantities and their changes are expressed by

$$dP/dT = \int u\hat{\Pi} u \, dx, \quad dH/dT = \int (u + u^2/2 - u_{xt})\hat{\Pi} u \, dx, \tag{18}$$

where $T \equiv \epsilon t$ denotes a slowly varying time. Note that $(u + \frac{1}{2}u^2 - u_{xt})$ can be equated to $-\partial_x^{-1}\partial_t u$ plus some correction of order ϵ.

We suppose that the zeroth-order solution to eq.(16) has the solution of the form (11). Assuming that the parameter λ changes slowly with respect to time (i.e., $\lambda = \lambda(T)$) and substituting (11) into (18), we obtain

$$dP/dT = [12/(\lambda^2 - 4)]^2 (u_b + 1)^2 (16\alpha'/15\lambda - 64\beta'/21\lambda^3. \tag{19}$$

For a steady pulse, the instability(α) and the dissipation(β) in (19) must be balanced and from $dP/dT = 0$ we obtain

$$\lambda^2 = {\lambda_e}^2 \equiv 20\beta/7\alpha = 20\rho/7 \qquad (\rho \equiv \beta/\alpha), \tag{20}$$

where the subscript e stands for the equilibrium. Then the equilibrium solution is given by

$$u_e = u_b + A_e \operatorname{sech}^2[(x - c_e t)/\lambda_e], \tag{21}$$

$$A = A_e \equiv 21(u_b + 1)/(5\rho - 7), \qquad c = c_e \equiv 5\rho(u_b + 1)/(5\rho - 7). \tag{22}$$

(Adoption of H instead of P simply duplicates the result.) Thus λ_e, A_e and c_e depend on ρ and u_b. Note that ρ is the ratio of (α, β). Perturbation expansion of eq.(15) shown in Appendix A yields the same result in the lowest order of approximation.

The results of the approximation for small α and β are summed up as follows when we consider the case $u_b = 0$. The equilibrium pulse is characterized in the lowest order *solely by the ratio* $\rho = \beta/\alpha$. Hence the amplitude and the velocity are changed discontinuously $(+\infty \to -\infty)$ when ρ is changed continuously passing across 7/5 $(7/5 + 0 \to 7/5 - 0)$. Intuitively this is a consequence of the equilibrium

$$\alpha \partial_x^2 u \sim \beta \partial_x^4 u, \tag{23}$$

which defines a representative wave length $\lambda_0 \sim (\beta/\alpha)^{1/2}$ (now its measure is the width of the pulse), and of those relations between λ, A and c inherited from the RLW pulses which incorporate discontinuity of the amplitude and the velocity at $\lambda = 2$. Thus a jump of $u \simeq u_{RLW}|_{\lambda=\lambda_0}$ takes place when ρ passes a certain value ρ_c, such that $\lambda_0(\rho_c) = 2$ and the above-mentioned result (20) indicates $\rho_c = 7/5$. It follows from the higher order of approximation given by eq.(33) in Appendix A that the odd part of u suffers a correction proportional to $\epsilon \sim (\alpha, \beta)$, but the velocity and the even part of u (and therefore the amplitude) are independent of ϵ to the first order of approximation.

3.2 Numerical calculation of steady solutions

Equation (15) was numerically solved as a nonlinear eigenvalue problem with eigenfunction u and eigenvalue c under the boundary conditions $u(z \to \pm\infty) = 0$ for various values of (α, β) shown in Fig. 1. The numerical method is explained in Appendix B. As is illustrated in Figure 2, we distinguish *positive* and *negative* pulses according to their sign of the amplitude, and distinguish the pulses by their tails, calling them *monotone* or *oscillatory* when the eigenequation (36) possesses three real roots or one real and two complex conjugate ones.

For small values of α and β, Fig. 3 supports the result of the approximation that the line $\beta - (7/5)\alpha = 0$ on the parameter plane divides regions for the positive and the negative pulses (the positive ones are situated on the β-axis side). It turns out that we can extend this dividing line beyond the limitation $\alpha \sim \beta \ll 1$, so that it still divides the positive and the negative pulses even for the values of α and β as large as 5, as far as the pulses with *monotone* tails are concerned. It is natural that the result of the approximation is not valid for the pulses with *oscillatory* tails. The invasion of negative, oscillatory-tailed pulses across the dividing line means that for relatively large values of α and β two distinct pulses are possible, namely a positive and a negative one, when $\rho > 7/5$ (it is not certain whether such ρ has an upper limit). Invasion of positive pulses with oscillatory tails into the region of negative pulses was not observed.

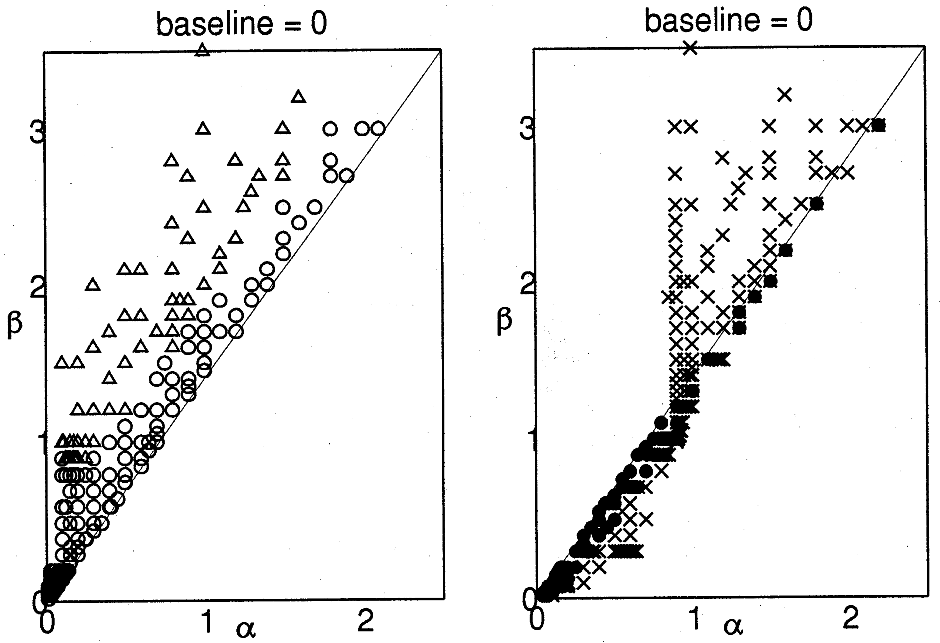

Figure 3: The range of the parameters (α, β) for which positive (left figure) or negative (right figure) steady pulses were found ($u_b = 0$). Symbols are the same as in Fig. 1.

The prediction of the perturbation approximation (i.e., the relations (22)) are proved by Fig. 4 for small and not very large values of α and β. Only negative, oscillatory-tailed pulses are found not to obey the relations.

Figure 5 shows deviation of the amplitude from (22) for increased value of α and β, with $\rho = 2$ fixed. While α and β are small, the deviation is of $O(\alpha^2)$ (i.e.,(22) is almost exact); when $(\alpha, \beta) \to \infty$ with $\rho = \text{const.}(> 7/5)$, then, in contrast, the amplitude grows asymptotically as $O(\alpha)$. (The velocity behaves similarly.) Intuitive interpretation is that when the KS terms are very large, then the nonlinear term $u^2/2$ must participate in the detailed balance among the terms, which postulates that the amplitude should be proportional to α. In the following section we will show that this situation can be identified with the case of $u_b \to -1$ and that in this limit $A \propto \alpha$ in fact.

4 Dependence of pulse solutions on base line

We mean by the word "base line" the asymptote of the pulse defined by $u_b = u(z \to \pm\infty)$. In contrast to the case of the original Benney equation, the dependence on u_b in not a trivial problem for eq.(3), because it lacks invariance under the Galilei transform.

Now we introduce the rescaling

$$u = u_b + (1 + u_b)\tilde{u}, \quad x = \pm\tilde{x} \quad (u_b \gtrless -1),$$
$$t = \tilde{t}/|1 + u_b|, \quad (\alpha, \beta) = |1 + u_b|(\tilde{\alpha}, \tilde{\beta}). \tag{24}$$

into a problem for u with arbitrary u_b

$$[\,\partial_t + \partial_x - \partial_t\partial_x^2\,]\, u + u\partial_x u + [\,\alpha\partial_x^2 + \beta\partial_x^4\,]\, u = 0 \quad (\, u(x \to \pm\infty) = u_b \,). \tag{25}$$

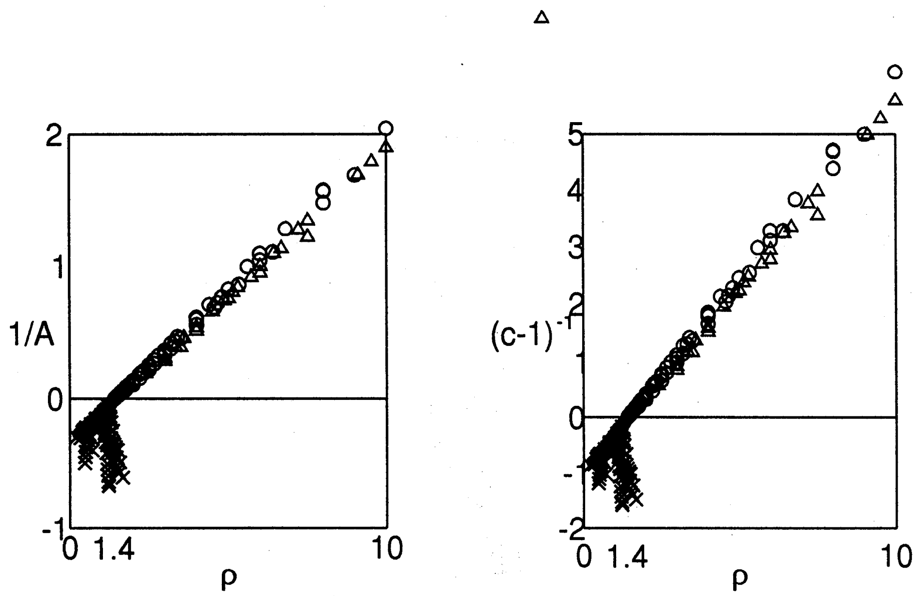

Figure 4: Dependence of A and c on $\rho = \beta/\alpha$, in the case when α and β are not very large.

Then eq.(25) is reduced to a problem for $\tilde{u}$ with zero-leveled base line

$$[\,\partial_{\tilde{t}} + \partial_{\tilde{x}} - \partial_{\tilde{t}}\partial_{\tilde{x}}^2\,]\,\tilde{u} + \tilde{u}\partial_{\tilde{x}}\tilde{u} + [\,\tilde{\alpha}\partial_{\tilde{x}}^2 + \tilde{\beta}\partial_{\tilde{x}}^4\,]\,\tilde{u} = 0 \quad (\,\tilde{u}(\tilde{x} \to \pm\infty) = 0\,). \tag{26}$$

The transform (24) tells that the multiplication of $(1 + u_b)$ by $1/k$ is equivalent to that of (α, β) by k, taking notice of the tilded variables in (24). In particular, $u_b \to -1$ and $(\alpha, \beta) \to \infty$ have the same effect $(\tilde{\alpha}, \tilde{\beta}) \to \infty$. This similarity law suggests that we should take $b = (1 + u_b)/\alpha$ and $\rho = \beta/\alpha$ as control parameters, rather than α, β and u_b. According to this idea we normalize the boundary value problem (25) as follows

$$[\,\partial_T - \partial_T\partial_X^2 + \partial_X^2 + \rho\partial_X^4\,]\,U + U\partial_X U = 0 \quad (\,U(x \to \pm\infty) = b\,), \tag{27}$$

where $U = (1 + u)/\alpha$, $T = \alpha t$ and $X = x$.

Now we ask whether a steady solution exists for $u_b \to -1$? The transform (24) together with (22) lead to an inference that when u_b goes to -1, then a pulse must disappear. This conjecture turns out to be incorrect. As is mentioned above, $\tilde{\alpha}$ and $\tilde{\beta}$ goes to ∞, which makes (22) cease to be valid. Numerical calculations prove that for $u_b = -1$, when $\rho > 7/5$, steady pulse solutions indeed exist. This is consistent with the divergence of the amplitude $\sim O(\alpha)$ shown in Fig. 5. In fact, a transform

$$u = \sqrt{\alpha^3/\beta}\,v - 1, \quad t = (\beta/\alpha^2)\,\tau, \quad x = \sqrt{\beta/\alpha}\,\xi, \tag{28}$$

rewrites eq.(3) into the KS equation with the RLW-like dispersion term as a perturbation,

$$\left[\partial_\tau + \partial_\xi^2 + \partial_\xi^4\right] v + v\partial_\xi v - \frac{1}{\rho}\partial_\tau\partial_\xi^2 v = 0, \tag{29}$$

where a steady pulse solution is possible for relatively large values of ρ. As is noted at the end of § 3, the transform (28) shows that $A = \max|u + 1|$ is proportional to α.

From the point of view of the normalized equation (27), the existence of pulses for $u_b = -1$ indicates that the curve of the relation between the amplitude A (relative to the

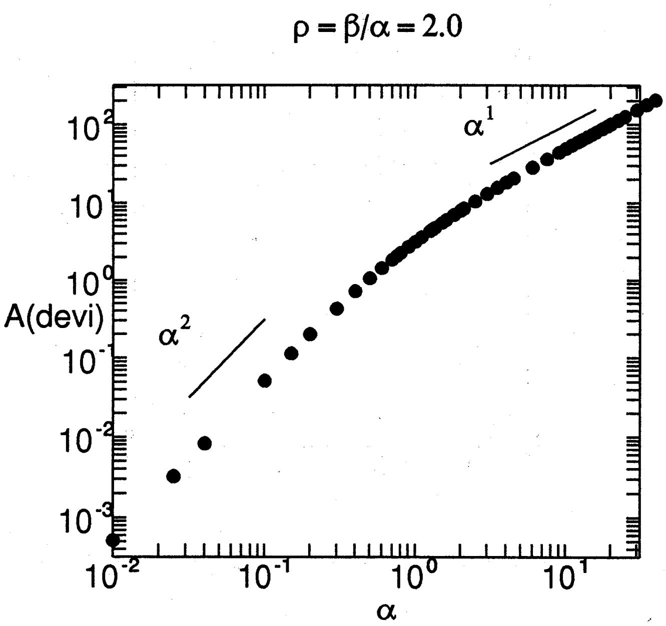

Figure 5: Deviation of A from (22), shown against α in log-log plot; $\rho = \beta/\alpha = 2.0$ is fixed.

base line) and the level of the base line b deviates from the line going through the original point (Fig. 6.1). The symmetry of Fig. 6.1 with respect to the origin is a manifestation of the invariance of (27) under the reflection

$$U \to -U, \quad X \to -X, \quad T \to T, \tag{30}$$

with b replaced by $-b$. Particularly, when $b = 0$ (i.e., $u_b = -1$), the whole problem (27) is symmetric with respect to the reflection, i.e., pulses with either sign can exist quite equally. The observed fact that two different pulses are possible for a certain range of (α, β) when $u_b = 0$ corresponds to the double-valuedness of $A = A(b)$ in Fig. 6.1. This figure suggests that either of the two solutions can be continuously deformed to the other, changing the control parameters (inclusive of u_b) and just once reflecting the solution with respect to some point on the line $u_b = -1$.

Figure 6.2 suggests, on the other hand, that if $\rho = 1.0 < 7/5$ a pulse solution cannot exist for $b = 0$. A transform $u = \sqrt{\frac{\alpha^2}{\beta}} w - 1$ yields an equation

$$\partial_\tau \left[1 - \partial_x^2\right] w + w\partial_x w + \rho\partial_x^2 w + \rho^2 \partial_x^4 w = 0 \quad (w(x \to \pm\infty) = 0). \tag{31}$$

For small ρ it is the Equal-Width equation (13) perturbed by the KS terms. An analysis similar to that for eq.(3) reveals that no equilibrium pulse is possible (unless $\rho = 7/5$), because the "wave length" of the pulses (14) cannot be changed.

5 Conclusion

In terms of the perturbation approximation and the direct numerical calculation, we explored how the steady travelling pulse solutions of eq.(3) depend on (α, β) and u_b. Main results are:

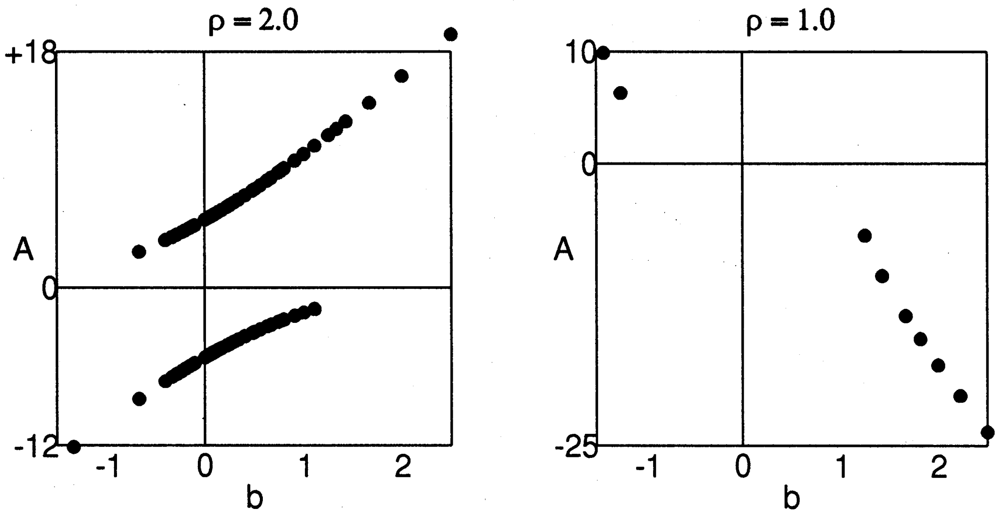

Figure 6: Dependence of the amplitude A on b in the normalized equation (27), for $\rho = 2.0$ (left figure) and for $\rho = 1.0$ (right figure).

1. The balance between instability and dissipation decides the representative length $\lambda \sim \sqrt{\beta/\alpha}$, just like in the original Benney equation. When the pulse is sech-like with monotone tails, it can bear only a fixed amplitude and a velocity owing to the dependence of the amplitude and the velocity on λ. In particular, $\rho > 7/5$ stands for positive equilibrium amplitude and $\rho < 7/5$ negative one.

2. The level of the base line, u_b, has a non-trivial meaning. It is not the simple magnitude of α ($\sim \beta$) but the magnitude of $b = (1 + u_b)/\alpha$ that decides the relative importance of the KS terms, i.e. whether the pulse is more RLW-like than KdV-like or not. When $u_b \sim -1$, the solutions are not RLW-like even for small (α, β) but rather KS-like as characterized by oscillatory tails and by the possibility of both signs for the same control parameters.

In Table 1, the features of the pulses in various cases are summarized according to the values of $\rho = \beta/\alpha$ and $b = (1 + u_b)/\alpha$.

We hope that the dynamics of eq.(3) can be described to some extent in terms of the soliton-like pulses weakly interacting one another. The results of this study suggest that the "base line" (or probably its local level), or an extremely long wave structure, must take part in such description of dynamics as well as the pulses with $\lambda \sim (\beta/\alpha)^{1/2}$, because the displacement of the base line, probably even locally, changes the character of the equilibrium pulse. In other words, when (positive) pulses grow from small buds collecting and incorporating the "mass" u (conserved under the evolution equation with a form $u_t + J_x = 0$), then they lower the "environmental" level of u, i.e., changes the uniform background on which the pulses exist. Our result says that this change influences the pulses in return. A lack of the Galilei invariance in eq.(3) expresses such back influence from the environment to the pulses, which may reflect one aspect of the reality in some physical systems.

	$\rho > 7/5$	$\rho \sim 7/5$	$\rho < 7/5$
$b > 0$	positive pulses with $\lambda > 2$ (when $b \to +0$, there are negative pulses with oscillatory tails as well)	amplitude $= \pm\infty$ ($\rho = \frac{7}{5} \pm 0$)	negative pulses with $\lambda < 2$; unstable?
$b \sim 0$	positive and negative pulses, being reflection of each other (with oscillatory tails)	positive and negative pulses, being reflection of each other (with monotone tails)	no steady pulse?
$b < 0$	negative pulses with $\lambda > 2$ (when $b \to -0$, there are positive pulses with oscillatory tails as well)	amplitude $= \mp\infty$ ($\rho = \frac{7}{5} \pm 0$)	positive pulses with $\lambda < 2$; unstable?

Table 1: Classification of steady travelling pulses in terms of $\rho = \beta/\alpha$ and $b = (1 + u_b)/\alpha$.

A Calculation of an equilibrium pulse by perturbation

We solve eq.(16) by means of a standard perturbation expansion [12]. Under the boundary conditions $u(z \to \pm\infty) = 0$, (16) is once integrated for the steady case, into which we substitute the following expansion:

$$u = u^{(0)} + \epsilon u^{(1)} + \epsilon^2 u^{(2)} + \cdots, \qquad c = c^{(0)} + \epsilon c^{(1)} + \epsilon^2 c^{(2)} + \cdots. \tag{32}$$

When we adopt the sech-pulse (7) for $O(\epsilon^0)$-equation, the *solvability condition* for $u^{(1)}$ gives $\lambda^2 = (20/7)\rho$, which then decides $u^{(0)}$ and $c^{(0)}$. This result coincides with that obtained by the modified conservation laws. From the solvability of $O(\epsilon^2)$-equation on $u^{(2)}$ we find $c^{(1)} = 0$. Now $O(\epsilon^1)$-equation is an inhomogeneous linear equation on a single unknown function $u^{(1)}$, whose solution under the restriction $(d/dz)^2 u^{(1)} = 0$ is given by

$$u^{(1)} = u^{(1)}(\rho; z) = \frac{36\alpha'}{5\lambda} \operatorname{sech}^2\frac{z}{\lambda} \tanh\frac{z}{\lambda} \ln\left[\operatorname{sech}^2\frac{z}{\lambda}\right] \quad \left(\lambda^2 = \frac{20}{7}\rho\right), \tag{33}$$

which turns out to be an odd (anti-symmetric) function of z.

B Numerical method to calculate steady pulses

Supposing $u_b = u(z \to \pm\infty)$ is given, we once integrate eq.(15) and rewrite the result in a matrix form:

$$\frac{d}{dz}\begin{bmatrix} u' \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ A_1 & A_2 & A_3 \end{bmatrix}\begin{bmatrix} u' \\ v \\ w \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ (u')^2/2\beta \end{bmatrix}, \tag{34}$$

where $u' = u - u_b$ and

$$[A_1 \quad A_2 \quad A_3] = -[(1 + u_b - c) \quad \alpha \quad c]/\beta. \tag{35}$$

The set of equations (34), regarded as a dynamical system with z as the time, has fixed points ${}^t(0,0,0)$ and ${}^t(0,0,-2(u_b+1-c))$. A pulse solution of eq.(15) corresponds to a homoclinic trajectory starting from ${}^t(0,0,0)$. The direction of the stable manifold and the unstable manifold is given as ${}^t(1,\mu,\mu^2)$ by the roots of the eigenequation of the matrix in (34), namely

$$\beta\mu^3 + c\mu^2 + \alpha\mu + 1 + u_b - c = 0. \tag{36}$$

Here u behaves like $u_b + \exp(\mu z)$ when $z \to \pm\infty$. The essence of the calculation is to integrate numerically the set of eqs.(34) along the one-dimensional unstable manifold (which depends on c), and to seek such value of c that the trajectory will return (along the stable manifold) to the original point as precisely as possible. This is known as the *shooting method* [15][16].

References

[1] D. J. Benney: J. Math. Phys. **45**, 150 (1957)

[2] D. J. Korteweg & G. de Vries: Phil. Mag. (v), **39**, 422 (1895)

[3] D. H. Peregrine: J. Fluid Mech. **25**, 321 (1966)

[4] T. B. Benjamin, J. L. Bona & J. J. Mahony: Phil. Trans. R. Soc. London, **A272**, 47 (1972)

[5] V. I. Petviashvili: Sov. J. Plasma Phys. **3**, 150 (1977) (English translation)

[6] J. D. Meiss & W. Horton: Phys. Fluids **25**, 1838 (1982)

[7] T. Kawahara: IUTAM Symposium on "Waves in liquid/gas and liquid/vapor two-phase system"(1994)

[8] Y. Kuramoto & T. Tsuzuki: Prog. Theor. Phys. **55**, 356 (1976)

[9] G. I. Sivashinsky: Acta Astronautica **4**, 1177 (1977)

[10] T. Kawahara: Phys. Rev. Lett. **51**, 381 (1983)

[11] T. Kawahara & S. Toh: Phys. Fluids **31**, 2103 (1988)

[12] S. Toh & T. Kawahara: J. Phys. Soc. Jpn. **54**, 1257 (1985)

[13] J. D. Meiss: in "Statistical Physics and Chaos in Fusion Plasmas" ed. by C. W.Horton, Jr & L. E. Reichl, John Wiley & Sons, 273 (1984)

[14] P. J. Morrison, J. D. Meiss & J. R. Cary: Physica **11**D, 324 (1984)

[15] S. Toh: J. Phys. Soc. Jpn. **56**, 949 (1987)

[16] T. Kawahara: "Soriton kara Kaosu e", Asakura Shoten, Tokyo (1993) (in Japanese)