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REMARKS ON ISOMORPHISMS OF REGRESSIVE TRANSFORMATION SEMIGROUPS

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For a (finite or infinite) set X, let $T(X)$ be *the full transformation semigroup on* X, i.e. the set of all maps from X to X, the semigroup operation being composition of maps.

When X is a partially ordered set, we let

$$T_{RE}(X) = \{ f \in T(X) \mid f(x) \leq x \text{ for all } x \in X \},$$

$$T_{OP}(X) = \{ f \in T(X) \mid f(x) \leq f(y) \text{ if } x \leq y \text{ for } x, y \in X\}.$$

Then, both of them are subsemigroups of $T(X)$ with the identity $id_{T(X)}$. We call $T_{RE}(X)$ *the full regressive transformation semigroup on* X, and $T_{OP}(X)$ *the full order-preserving transformation semigroup on* X.

Recently, some interesting results on $T_{RE}(X)$ have been obtained (cf. [1], [4], [5]).

It is known that, for partially ordered sets X, Y, if $T_{OP}(X)$ and $T_{OP}(Y)$ are isomorphic as semigroups, then X and Y are isomorphic or anti-isomorphic as ordered sets (see [3], Theorem V.8.9).

It is natural to ask whether the above result holds or not for regressive transformation semigroups. In general, it does not hold. However, we obtain a necessary and sufficient condition on partially ordered sets X and Y for $T_{RE}(X)$ and $T_{RE}(Y)$ to be isomorphic.

Umar showed in [6] that, when X and Y are totally ordered sets, any idempotent in $T_{RE}(X)$ whose image is an order-ideal is mapped to an idempotent in $T_{RE}(Y)$ with the same property by isomorphisms from $T_{RE}(X)$ to $T_{RE}(Y)$, and he considered the above problem through this result. If the result holds even if "an order-ideal" in it is replaced by "a principal order-ideal", then it can be shown that if $T_{RE}(X) \cong T_{RE}(Y)$ as semigroups then $X \cong Y$ as ordered sets. At the present time, this is unsolved.

In here, we achieve our purpose by showing that any idempotent of defect 1 in $T_{RE}(X)$ is mapped to an idempotent of defect 1 in $T_{RE}(Y)$ by isomorphisms from $T_{RE}(X)$ to $T_{RE}(Y)$, where the defect of α in $T_{RE}(X)$ means the cardinality of the set of idempotents in X which do not belong to the image of α.

For partially ordered set X, an element in X is said to be *isolated* if it is incomparable with every element in X except itself. Let $Is(X)$ be the set of all isolated elements in X. Then, it is easy to see that $T_{RE}(X)$ and $T_{RE}(X \setminus Is(X))$ are isomorphic. Therefore, we may assume that every partially ordered set, treated in this paper, does not contain any isolated elements.

Let X be a partially ordered set under the order relation $\leq$.

For $a \in X$, the set of (resp. *strict*) *upper bounds* of a is denoted by $U(a)$ (resp. $SU(a)$), i. e.

$$U(a) = \{x \in X \mid x \geq a\} \quad \text{and} \quad SU(a) = \{x \in X \mid x > a\},$$

and the set of all minimal elements in X is denoted by $Min(X)$,

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This is an abstract and the details will be published in Semigroup Forum.

b, we have that $j'(j(a, b), k(a, b)) = a$ and $k'(j(a, b), k(a, b)) = b$.

Lemma 2. (1) *$k(a, b) = k(c, d)$ if and only if $b = d$.*
(2) *If $a < c < b$, then $k(a, c) = j(c, b)$ and $j(a, b) = j(a, c)$.*
(3) *$j(a, b) = j(c, d)$ if and only if $a = c$.*
Proof. (1) It is easy to see that
$b = d \Leftrightarrow \lambda^d{}_c \circ \lambda^b{}_a = \lambda^b{}_a \Leftrightarrow \lambda^{k(c, d)}{}_{j(c, d)} \circ \lambda^{k(a, b)}{}_{j(a, b)} = \lambda^{k(a, b)}{}_{j(a, b)} \Leftrightarrow k(a, b) = k(c, d)$.
This assertion means that $k(a, b)$ depends only on b.
(2) The proof is omitted.
(3) To show the assertion, we need that X and Y are adjusted. Let $a = c$. If a is not minimal in X, then $e < a$ for some $e \in X$. From (2), we have that $j(a, b) = k(e, a) = j(a, d) = j(c, d)$.
If a is minimal in X, then b and d are connected in $SU(a)$, since X is adjusted, so that there exist $e_1, e_2, \dots, e_n \in SU(a)$ such that $b = e_1 \leq^s e_2 \leq^s \dots \leq^s e_n = d$. Since e_i and e_{i+1} are comparable, by (2) we have that $j(a, e_i) = j(a, e_{i+1})$ $(i = 1, 2, \dots, e_{n-1})$. Thus, we have that $j(a, b) = j(a, d) = j(c, d)$.
Let $j(a, b) = j(c, d)$. If we apply the above fact to j', then we have that $a = j'(j(a, b), k(a, b)) = j'(j(c, d), k(c, d)) = c$.
This assertion means that $j(a, b)$ depends only on a.

We write $j(a, b) = j(a)$ and $k(a, b) = k(b)$ for $a, b \in X$ with $a < b$. In this case, if a is maximal in X, then $j(a)$ is undefined, and if b is minimal in X, then $k(b)$ is undefined. Since $j(a) < k(b)$ if $a < b$, we have that if a is not maximal in X, then neither is $j(a)$ in Y. By (2) of Lemma 2, if c is neither maximal nor minimal in X, then $j(c) = k(c)$.
Similarly, we write $j'(a', b') = j'(a')$ and $k'(a', b') = k'(b')$ for $a', b' \in Y$ with $a' < b'$. Then, we have that $j'(j(a)) = a$, $k'(k(b)) = b$, $j(j'(a')) = a'$ and $k(k'(b')) = b'$.
Let a be maximal in X. Then, we can show that $k(a)$ is maximal in Y.

Define a map $h : X \to Y$ by $h(a) = j(a)$ if a is not maximal in X, and $h(a) = k(a)$ if a is maximal in X. Then, we can show that the h is an order-isomorphism of X onto Y.

Since any totally ordered set is clearly adjusted, we obtain :

Corollary 3.
Let X and Y be totally ordered sets. Then, $T_{RE}(X)$ and $T_{RE}(Y)$ are isomorphic as semigroups if and only if X and Y are isomorphic as ordered sets.

Let X, Y be partially ordered sets. From Theorem 1 and Theorem 2, we have that
$$T_{RE}(X) \cong T_{RE}(Y) \Leftrightarrow T_{RE}(A(X)) \cong T_{RE}((A(Y)) \Leftrightarrow A(X) \cong A(Y).$$
Thus, we obtain the following main theorem :

Theorem 4.
Let X and Y be partially ordered sets. Then, $T_{RE}(X)$ and $T_{RE}(Y)$ are isomorphic as semigroups if and only if their adjusted sets $A(X)$ and $A(Y)$ are isomorphic as ordered sets.

and the set of all minimal elements in X is denoted by $Min(X)$.

Let $\leq^s$ be *the symmetric relation generated by* $\leq$, i.e. $a \leq^s b$ if and only if $a \leq b$ or $b \leq a$, and let $\leq^e$ be *the equivalent relation generated by* $\leq$, i.e. $a \leq^e b$ if and only if there exist $c_1, c_2, \ldots, c_n \in X$ such that $a = c_1 \leq^s c_2 \leq^s \ldots \leq^s c_n = b$ (see [2], I). In this case, we say that a and b are *connected* in X. A subset Y of X is *connected* if every $a, b \in Y$ are connected in Y, i. e. there exist $c_1, c_2, \ldots, c_n \in Y$ such that $a = c_1 \leq^s c_2 \leq^s \ldots \leq^s c_n = b$

A partially ordered set X is said to be *adjusted* if it does not contain any minimal elements , or for every $m \in Min(X)$, $SU(m)$ is connected.

Theorem 1.

Let X be a partially ordered set. Then, there exists an adjusted partially ordered set A such that $T_{RE}(A)$ is isomorphic to $T_{RE}(X)$ as semigroups.

We can construct an adjusted partially ordered set $A(X)$ from X such that $T_{RE}(A(X))$ is isomorphic to $T_{RE}(X)$. In this case, the $A(X)$ is called *the adjusted partially ordered set of* X.

Theorem 2.

Let X, Y be adjusted partially ordered sets. Then, $T_{RE}(X)$ and $T_{RE}(Y)$ are isomorphic as semigroups if and only if X and Y are isomorphic as ordered sets.

Suppose that X and Y are isomorphic. Let h be an isomorphism from X onto Y. Then, it is easy to show that the map $i : T_{RE}(X) \to T_{RE}(Y), f \to i(f)$ defined by $i(f)(h(x)) = h(y)$ if $f(x) = y$, is an isomorphism.

To show the only if–part, we need two lemmas (Lemmas 1 and 2).

For each pair $a, b \in Z$ with $a < b$, where Z is a partially ordered set, we define $\lambda^b{}_a$ in $T_{RE}(Z)$ by

$$\lambda^b{}_a(b) = a, \quad \lambda^b{}_a(x) = x \text{ if } x \neq b.$$

From now until the end of the proof of Theorem 2, X *and* Y will denote adjusted partially ordered sets, and i will denote an isomorphism from $T_{RE}(X)$ onto $T_{RE}(Y)$.

Lemma 1. *For each pair $a, b \in X$ with $a < b$, there exist $a', b' \in Y$ such that $i(\lambda^b{}_a) = \lambda^{b'}{}_{a'}$.*

The assertion can be easily shown by using the following facts :

For $g \in T_{RE}(X)$,

$$\lambda^b{}_a \circ g = \lambda^b{}_a \text{ if and only if } g = id_{T(X)} \text{ or } g = \lambda^b{}_a,$$

$$g \circ \lambda^b{}_a = g \text{ if and only if } g(a) = g(b) \text{ and } a < b.$$

For each pair $a, b \in X$ with $a < b$, the pair a', b' in Lemma 1 is clearly unique. So we write

$$a' = j(a, b) \text{ and } b' = k(a, b), \text{ namely } i(\lambda^b{}_a) = \lambda^{k(a,b)}{}_{j(a,b)}.$$

We similarly have that for each pair a', b' in Y with $a' < b'$, there exist unique elements $j'(a', b')$, $k'(a', b')$ in X such that $i^{-1}(\lambda^{b'}{}_{a'}) = \lambda^{k'(a',b')}{}_{j'(a',b')}$. Then, for each a, b in X with $a <$

Corollary 5.

Let X and A be as in Theorem 1. Then A is uniquely determined by X up to isomorphisms.

We next aim to refine Theorem 2 to the following :

Theorem 6.

Let X and Y be as in Theorem 2, and let i be a semigroup isomorphism from $T_{RE}(X)$ *onto* $T_{RE}(Y)$. *Then, there exists an order isomorphism h from X onto Y such that* $h(f(a)) = i(f)(h(a))$ *for all* $f \in T_{RE}(X)$ *and all* $a \in X$.

Let h be an isomorphism from X onto Y determined by i in Theorem 6 as in the proof of Theorem 2. Thus, $i(\lambda^b{}_a) = \lambda^{h(b)}{}_{h(a)}$ for each $a, b \in X$ with $a < b$. We show that this h serves as a desired h in Theorem 6. To show the theorem, again we need two lemmas (Lemmas 3 and 4).

For each $f \in T_{RE}(X)$ and each $a \in X$, we define f^a and f_a, as follows :

$f^a(x) = x$ if $x \geq a$, $f^a(x) = f(x)$ otherwise, and $f_a(x) = f(x)$ if $x > a$, $f_a(x) = x$ otherwise.

Then, it is easy to check that $f = f_a \circ \lambda^a{}_{f(a)} \circ f^a$ for all $a \in X$, where $\lambda^a{}_{f(a)} = id_{T(X)}$ if $a = f(a)$.

Lemma 3. *For any* $f \in T_{RE}(X)$ *and any* $b, c \in X$ *with* $c \leq b$,

(1) $f(b) = f(c)$ *if and only if* $i(f)(h(b)) = i(f)(h(c))$. *In particular,*

(2) *if* $a \leq b$, *then* $i(f^a)(h(b)) = i(f^a)(h(c))$ *implies that* $h(b) = h(c)$, *and*

(3) *if* $b \not> a$, *then* $i(f_a)(h(b)) = i(f_a)(h(c))$ *implies that* $h(b) = h(c)$, *where* $b \not> a$ *means that* $b \leq a$ *or a and b are incomparable.*

From Lemma 3, we have :

Lemma 4. *For every* $a, b \in X$,

(1) *if* $h(b) \geq h(a)$, *then* $i(f^a)(h(b)) = h(b)$,

(2) *if* $h(b) \not> h(a)$, *then* $i(f_a)(h(b)) = h(b)$.

Since $f = f_a \circ \lambda^a{}_{f(a)} \circ f^a$ for all $a \in X$, and since $h(f(a)) \leq h(a)$, we have that

$$\begin{aligned} i(f)(h(a)) &= i(f_a) \circ i(\lambda^a{}_{f(a)}) \circ i(f^a)(h(a)) = i(f_a) \circ \lambda^{h(a)}{}_{h(f(a))}(h(a)) \\ &= i(f_a) \circ \lambda^{h(a)}{}_{h(f(a))}(h(a)) = i(f_a)(h(f(a))) = h(f(a)) . \end{aligned}$$

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