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Monoids with finite (regular) complete presentation

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If a monoid is defined by a finite complete rewriting system, then it has some good properties such as solvability of the word problem (see [1]). So, it is an important problem to find good conditions for monoids to have finite complete presentation.

A rewriting system over a (finite) alphabet Σ is a subset R of $\Sigma^* \times \Sigma^*$. An element (u, v) of R is called a rule and written $u \to v$. A word $x \in \Sigma^*$ is rewritten to $y \in \Sigma^*$ under R, if $x = x_1ux_2$, $y = x_1vx_2$ for some $x_1, x_2 \in \Sigma^*$ and some $u \to v \in R$. In this case we write $x \to_R y$. The reflexive transitive closure and the reflexive symmetric transitive closure of the relation $\to_R$ is denoted by $\to_R^*$ and $\leftrightarrow_R^*$, respectively. The quotient monoid $M = \Sigma^*/\leftrightarrow_R^*$ is a monoid presented by (Σ, R).

A subset S of Σ^* is called <u>s-closed</u>, if any subword of an element of S is also in S, equivalently, if the complement of S forms an ideal of Σ^*. S is s-closed if and only if S is expressed as

$$S = \Sigma^* - \Sigma^* T \Sigma^* \tag{1}$$

with a subset T of Σ^*. A set S is <u>finitely</u> <u>s-closed</u> if S is expressed as (1) with a finite set T, or equivalently, S is the complement of a finitely generated ideal.

Proposition 1. For an s-closed subset S of Σ^* the following are equivalent.

(1) S is finitely s-closed.

(2) There is a positive n such that any subword of length $\leq n$ of $x \in \Sigma^*$ is in S, then x is in S.

Corollary. A finite s-closed set is finitely s-closed.

Let $R \subset \Sigma^* \times \Sigma^*$ and let $M = M(\Sigma, R)$ be the monoid presented by (Σ, R). A subset S of Σ^* is a <u>transversal</u> for M (or for (Σ, R)), if S forms a complete set of representatives for M, that is, for any $x \in \Sigma^*$ there is a unique $\hat{x} \in S$ with $x \leftrightarrow^* \hat{x}$.

M has a <u>complete presentation</u>, if M is presented by (Σ, R) such that R is a complete (i.e. noetherian and confluent) rewriting system.

Proposition 2. If a monoid has a finite complete presentation, then it has a finitely s-closed transversal.

We are interested in the converse problem of this results.

Problem 1. If a monoid has a finitely s-closed transversal, does it admit a finite complete presentation ?

Proposition 3. Let $S = \Sigma^* - \Sigma^* T \Sigma^*$ be an s-closed transversal for M, and let $R(T) = \{u \to \hat{u} | u \in T\}$. If $R(T)$ is noetherian, then R is a complete system defining M.

Unfortunately, the system $R(T)$ given above is not always noetherian even when S is finitely s-closed and T is chosen minimally.

Example 1. Let $\Sigma = \{a, b\}$ and $M = M(\Sigma, E)$, where

$$E = \{(baab, aba)\} \cup \{(x, y) \in \Sigma^* \times \Sigma^* \,|\, |x| = |y| = 4\}.$$

Then M is a finite monoid with finite s-closed transversal

$$S = \{x \in \Sigma^* \,|\, |x| \leqq 3,\ x \neq aba\} \cup \{baab\}.$$

A set T with which S is expressed as (1) must contain the rule aba. Hence, $R = R(T)$ contains the rule $aba \rightarrow baab$. Thus

$$abaa \rightarrow_R baaba \rightarrow_R babaab \rightarrow_R bbaabab \rightarrow_R bbabaabb \rightarrow_R \dots,$$

and R is not noetherian.

If M has a transversal $S = \Sigma^* - \Sigma^* T^*$ with T a singleton, then we expect that the system R(T) would be noetherian, and M would admit a complete one-rule system.

Problem 2. Suppose M has a transversal S of the form
$S = \Sigma^* - \Sigma^* v \Sigma^*$, $v \in \Sigma^*$.
Is the one-rule system $\{v \rightarrow \hat{v}\}$ noetherian ?

A system R is regular, if the left hand sides of the rules from R forms a regular set (see [2]).

Theorem 1. Suppose M is presented by a regular complete system (Σ, R). Then,

(1) M has a regular sets of representative.

(2) Irr(R) grows either exponentially or polynomially.

(3) If Irr(R) grows exponentially, then the monoid $M = M(\Sigma, R)$ also grows exponentially and contains a free submonoid of rank 2.

Problem 3. Does M have a regular complete presentation, provided M has a regular s-closed transversal ?

A system (Σ, R) is polynomially mild, if there is a polynomial f such that $|y| \leq f(|x|)$ for any $x \in \Sigma^*$ and y such that $x \rightarrow^* y$.

Theorem 2. If (Σ, R) is a polynomially mild regular complete system such that Irr(R) grows polynomially, then the monoid $M = M(\Sigma, R)$ grows polynomially.

In theorem 2, the mildness of R cannot be removed. In fact, M can grows exponentially in general, even if Irr(R) grows polynomially.

Example 2. Let $\Sigma = \{a, b\}$ and $R_1 = \{abb \to ba\}$, $R_2 = \{ba \to abb\}$. Then, the both R_1 and R_2 are complete and define the same monoid. However, $Irr(R_1)$ grows exponentially, while $Irr(R_2)$ grows polynomially. So, the system R_2 is not mild.

There is a monoid with a very simple presentation which never admits a finite complete presentation.

Example 3. Let $\Sigma = \{a, b\}$ and $R = \{b^2a \to ab^2, ab^3 \to b^3, b^3a \to b^3\}$, then $M = M(\Sigma, R)$ has no finitely s-closed transversal. So, M has no finite complete presentation.

References

[1] K. Madlener and F. Otto, About the descriptive power of certain classes of finite string-rewriting systems, Theoret. Comp. Sci. **67**, (1989), 143-172.

[2] Y. Kobayashi, A finitely presented monoid which has solvable word problem but has no regular complete presentation, to appear in Theoret. Comp. Sci.