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ASYMPTOTICS FOR SOLUTIONS OF THE TWO DIMENSIONAL NONLINEAR ELLIPTIC EQUATIONS WITH CRITICAL GROWTH

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§1 Introduction and Results.

We study asymptotics for solutions of semilinear elliptic equations which have a critical growth in two dimensions. Let Ω be a simply connected bounded domain in $\mathbf{R}^2$ with smooth boundary $\partial\Omega$. We consider the following particular equation on Ω.

$$(E)\qquad \begin{cases} -\Delta u = \lambda u e^{u^2}, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where λ is a positive parameter.

The equation (E) arise from a variational problem related to the two-dimensional Sobolev type inequality. Suppose that $u \in H_0^1(\Omega)$ with $\|\nabla u\|_2 \leq 1$, then Trudinger [**25**] proved that there are two positive constants α and C_0 such that

$$(TM)\qquad \int_\Omega \{\exp(\alpha u^2) - 1\} dx \leq C_0 |\Omega|.$$

Moser [**15**] refined this inequality as (TM) is true if $\alpha \leq 4\pi$ and not true if $\alpha > 4\pi$ (cf. [**17**]). The extremal function in $H_0^1(\Omega)$ which maximize the left hand side of (TM) under the restriction $\|\nabla u\|_2 \leq 1$, if it exists, satisfies the equation (E) with Lagrange Multiplier λ. In fact, this problem was solved by Carleson-Chang [**6**] for a radially symmetric case and by Flucher [**9**] for a general domain case. There are more existence results to (E). Atkinson-Peletier studied the radially symmetric case and gave a fine analysis of the behavior of a solution of (E) ([**2**], [**3**]). Shaw [**23**] and Adimurthi [**1**] considered some variational problems involving (E) and obtained a solution in some situations. Among these existence results, Adimurthi constructed a positive and in fact smooth solution to (E) for $0 < \lambda < \lambda_0$, where λ_0 is the first eigen value of $-\Delta$ with zero Dirichlet boundary condition in Ω.

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Our main concern is an asymptotic behavior of the solutions (E) as $\lambda \to 0$. If a smooth solution of (E) exists, the L^∞-norm of the solution must blow-up as $\lambda \to 0$. In fact, by multiplying (E) by u and integrating by parts, we have

$$\|\nabla u\|_2^2 = \lambda \int_\Omega u^2 e^{u^2} dx. \tag{1}$$

Therefore by Poincáre's inequality,

$$\frac{1}{\lambda} = \frac{1}{\|\nabla u\|_2^2} \int_\Omega u^2 e^{u^2} dx \le e^{\|u\|_\infty^2} \frac{\|u\|_2^2}{\|\nabla u\|_2^2} \le \frac{1}{\lambda_0} e^{\|u\|_\infty^2} \to \infty \tag{2}$$

as $\lambda \to 0$.

We firstly consider the unit disk case. Since by the famous result of Gidas-Ni-Nirenberg [**10**], a positive smooth solution is necessarily radially symmetric. Then it has already shown ([**18**], [**19**]) that:

Proposition 0. *Let u be any positive smooth solution of (E) on $\Omega = D \equiv \{x \in \mathbf{R}^2, |x| < 1\}$. Then we have*

$$u(x) \to 0 \text{ as } \lambda \to 0 \text{ locally uniformly on } D \setminus \{0\}, \tag{3}$$

$$\lim_{\lambda \to 0} \lambda \int_D u e^{u^2} dx = 0, \tag{4}$$

$$\lim_{\lambda \to 0} \lambda \int_D (e^{u^2} - 1) dx = 0, \tag{5}$$

$$\varliminf_{\lambda \to 0} \int_D |\nabla u|^2 dx \ge 4\pi. \tag{6}$$

In this case, a microscopic structure for the above asymptotics can be stated under an assumption of the finite energy.

Theorem 1. *Let $\{(u, \lambda)\}$ be a solutions of (E) satisfying the finite energy condition:*

$$\varlimsup_{\lambda \to 0} \int_D |\nabla u|^2 dx \equiv E_0 < \infty. \tag{7}$$

Then there is a subsequence $\{(u_m, \lambda_m)\}$ and a scaling sequence $\gamma_m \to 0$ as $\lambda_m \to 0$, satisfying

$$u_m^2(\gamma_m x) - u_m^2(\gamma_m) \to 2\log\left(\frac{2}{1+|x|^2}\right) \quad \text{as } \lambda_m \to 0 \tag{8}$$

locally uniformly on $\mathbf{R}^2$.

The left hand side of (8) is a scaling sequence of the solution on the scaling parameter $\{\gamma_m\}$, while the limit function in (8) is a unique explicit solution of

$$\begin{cases} -\Delta v = 2e^v, & x \in \mathbf{R}^2, \\ v = 0, & x \in \partial D \end{cases} \tag{9}$$

(see Chen-Li [7]).

This kind of structure was implicitly suggested by [6]. In [24], Struwe explicitly pointed out a similar behavior for non-compact maximizing sequence to the left hand side of (TM). By using a result by Brezis-Merle [4], we have shown ([20]) that there is a subsequence of solutions of the same behavior as the above under smallness condition of the asymptotic energy; $E_0 < 6\pi$. Theorem 1 removes this restriction.

Next we discuss about general cases. Let Ω be a bounded domain and simply connected. Set the blow up set $\mathcal{S}$ as

$$\mathcal{S} = \{x \in \bar{\Omega} |^{\exists} x_n \to x \text{ such that } u(x_n) \to \infty \text{ as } \lambda \to 0\}.$$

We extend Proposition 0 as in the followings.

Theorem 2. *Let Ω be a simply connected bounded domain. Suppose that u be a smooth solution of (E), then we have*

$$\|u\|_\infty \to \infty \text{ as } \lambda \to 0, \tag{10}$$

$$\lim_{\lambda \to 0} \lambda \int_\Omega u e^{u^2} dx = 0, \tag{11}$$

$$\lim_{\lambda \to 0} \lambda \int_\Omega (e^{u^2} - 1) dx = 0. \tag{12}$$

Moreover if we assume

$$E_0 = \overline{\lim_{\lambda \to 0}} \int_\Omega |\nabla u|^2 dx < \infty,$$

then for every $x \in \mathcal{S}$ and for any $\delta > 0$ such that $B_\delta(x) \subset \Omega$, we have

$$\underline{\lim_{\lambda \to 0}} \int_{B_\delta(x)} |\nabla u|^2 dx \geq 4\pi \tag{13}$$

and

$$u(x) \to 0 \text{ locally uniformly on } \Omega \setminus \mathcal{S} \tag{14}$$

as $\lambda \to 0$.

Remark that under the condition $E_0 < \infty$, $\mathcal{S}$ is a finite set and

$$\#\mathcal{S} \leq \frac{1}{4\pi} E_0. \tag{15}$$

We also note that the lower bound like (13) has been proved for the higher dimensional cases (see e.g., [12]).

We apply the above result to the solution obtained by the variational method. Due to Nehari's critical point theory, Adimurthi [1] constructed a solution by finding a minimizer of

$$J_\lambda(v) \equiv \frac{1}{2} \|\nabla v\|_2^2 - \frac{\lambda}{2} \int_\Omega (e^{v^2} - 1) dx \tag{16}$$

in $H_0^1(\Omega)$ under the Nehari constraint (1). That is

Proposition 3 (Adimurthi [1]). *For $0 < \lambda < \lambda_0$, there is a minimizer $u \in H_0^1(\Omega)$ of $J_\lambda(v)$ which attains*

$$\tilde{J}_\lambda \equiv \inf\{J_\lambda(v)| \quad v \in H_0^1(\Omega) \setminus \{0\}, \|\nabla v\|_2^2 = \lambda \int_\Omega v^2 e^{v^2} dx\}. \tag{17}$$

The minimizer satisfies solution (E). Moreover we have

$$0 < \tilde{J}_\lambda < 2\pi \text{ for all } 0 < \lambda < \lambda_0. \tag{18}$$

Remark 1. The regularity of the above solution is directly obtained by the similar argument in [**4**].

Remark 2. There is no positive solution of (E) for $\lambda \leq 0$ or $\lambda \geq \lambda_1$.

According to Theorem 2, we have the following for the solution in Proposition 3:

Corollary 4. *For the solution u of (E) obtained by a variational formulation in (14), we have*

$$\lim_{\lambda \to 0} \|\nabla u\|_2^2 = 4\pi \tag{19}$$

and the blow-up set $\mathcal{S}$ consists of one interior point of Ω, i.e., $\mathcal{S} = \{^\exists x_0 \in \Omega\}$.

In the similar problem of the higher dimensions, it has shown that the blow-up point appearing the singular limit coincides the critical point of the regular part of the Green function of $-\Delta$ (see [**5**], [**11**], [**22**] and [**26**]). Therefore it is expected that the singular point x_0 in the above theorem would be a maximum point of the regular part of the Green function of $-\Delta$ or in the other word, it might coincide the conformal center of the domain (cf. [**9**]).

In what follows, we shall show the sketch of proofs of Theorems and discuss about the relation between the solution obtained other variational methods.

§2 Outline of Proofs.

We show sketch of the proofs of theorems. For the radially symmetric case, we note the following fact, which plays a crucial role of proof of Theorem 1.

Lemma 5 ([18]). *Let u be any solution of (E) on $\Omega = D$. Then we have*

$$ru_r(r) \to 0 \quad \text{uniformly on } D \text{ as } \lambda \to 0, \tag{20}$$

where $r = |x|$.

This lemma is obtained by a simple use of the Pohozaev identity ([**21**]) to (E) and implies (3)-(5) in Proposition 0.

We now formulate to show theorem 1.

Proof of Theorem 1. For some scaling constant $\gamma > 0$, which will be determined later, we transform the equation (E) by putting

$$v(r) = u^2(\gamma r) - u^2(\gamma), \tag{21}$$

into

$$\begin{cases} -\Delta v = 2k(r)e^v - 2\gamma^2|\nabla_\rho u(\rho)|^2, & 0 \le r < \gamma^{-1}, \rho = \gamma r, \\ v(1) = 0, \end{cases} \tag{22}$$

where $k(r) = \lambda\gamma^2 e^{u(\gamma)^2} u^2(\gamma r)$.

For each u, the scaling parameter γ are chosen as

$$u^2(0) - u^2(\gamma) = 2\log 2. \tag{23}$$

Then

$$\|v\|_{L^\infty} \le 2\log 2 \tag{24}$$

and $\gamma \to 0$ by (3) and (23).

Lemma 6. *By passing a subsequence, we observe that for some constant $\mu > 0$, we have*

$$k(r) \to \mu, \tag{25}$$

$$\gamma^2|\nabla_\rho u(\rho)|^2 \to 0 \tag{26}$$

locally uniformly in $\mathbf{R}^2 \setminus \{0\}$.

In fact, (26) is an immediate consequence of Lemma 5. For the convergence of (25), we need to show that

$$k(r)|_{r=1} \le C(E_0), \tag{27}$$

where the constant $C(E_0)$ is independent of λ. This bound is obtained by making use of the Pohozaev identity for the equation (22). Using Lemma 5 and (27), we have

$$\mu \min(r^{-2\eta}, 1) + o(\lambda) \le k(r) \le \mu \max(r^{-2\eta}, 1) + o(\lambda),$$

which implies (25). By the apriori bound (24) and the Ascori-Arzela theorem yields that there exists a limit function $v_0 \in C(B) \cap C^2(\mathbf{R}^2 \setminus \{0\})$ satisfying, for some subsequence of v,

$$v(r) \to v_0(r) \text{ locally uniformly on } (0, \infty).$$

Moreover v_0 satisfies

(28)
$$\begin{cases} -\Delta v_0 = 2\mu e^{v_0}, & x \in \mathbf{R}^2 \setminus \{0\}, \\ v_0 = 0, & x \in \partial B. \end{cases}$$

Since

(29)
$$\|v_0\|_{L^\infty(B)} \leq 2\log 2,$$

we conclude
$$v_0 = 2\log\frac{2}{1+r^2}$$
and $\mu = 1$. The uniform convergence
$$v(r) \to v_0(r)$$
on any compact set $K \subset\subset \mathbf{R}^2$ follows immediately. This shows the sketch of proof Theorem 1.

Next we consider the general case.

Before proving Theorem 2, we need the following inequalities.

Lemma 7. *For any positive smooth solution u of (E),*

(30)
$$4\pi\lambda\int_\Omega (e^{u^2}-1)dx \leq (\lambda\int_\Omega ue^{u^2}dx)^2,$$

(31)
$$(\lambda\int_\Omega ue^{u^2}dx)^2 \leq \sigma_\Omega\lambda\int_\Omega (e^{u^2}-1)dx,$$

where σ_Ω is a constant determined by the conformal map from D to Ω.

The relation (30) is a consequence of the isoperimetric inequality and simple argument of the distribution functions to u (c.f. Chen-Li [7]). The second inequality (31) is obtained by the Pohozaev identity.

Proof of Theorem 2. Since (10) has already shown, we show (11) and (12). From the inequality (31), it follows for any $t > 0$,
$$\begin{aligned}(\lambda\int_\Omega ue^{u^2}dx)^2 &\leq \sigma_\Omega\lambda\left\{\int_{u>t}(e^{u^2}-1)dx + \int_{u\leq t}(e^{u^2}-1)dx\right\} \\ &\leq \frac{C_\Omega}{t}\lambda\int_{u>t} ue^{u^2}dx + \lambda\sigma_\Omega|\Omega|(e^{t^2}-1)\end{aligned}$$

Then by letting $\lambda \to 0$,
$$\overline{\lim_{\lambda\to 0}}\,\lambda\int_\Omega ue^{u^2}dx \leq \frac{C_\Omega}{t},$$
which goes to 0 as $t \to \infty$. This proves (11) and (12) by (30).

To show (13) and (14), we need the following:

Lemma 8. *For any $K \subset\subset \Omega$ and $1 \leq p < \infty$, we have*

$$\int_K u^p dx \to 0 \text{ as } \lambda \to 0. \tag{32}$$

In fact, using the first eigen function ϕ_1 of $-\Delta|_0$,

$$\lambda_1 \int_\Omega \phi_1 u dx = \lambda \int_\Omega \phi_1 u e^{u^2} dx \leq C\lambda \int_\Omega u e^{e^2} dx \to 0$$

by Theorem 3, which shows $\int_K u dx \to 0$. The conclusion follows from $E_0 < \infty$ and the Gagliardo-Nirenberg inequality.

Note that $\mathcal{S} \subset\subset \Omega$ by the boundary condition and Hopf's lemma (see [**8**] and [**11**]). For $x \in \mathcal{S}$, let $2\delta = d(x, \partial\Omega)$. We then assume for the contrary that

$$\int_{B_{2\delta}} |\nabla u|^2 dx < 4\pi. \tag{33}$$

Introducing a cut off function $\phi_\delta(\cdot) = \phi(\frac{\cdot - x}{\delta})$, where $\phi \in C_0^\infty(B_2)$, $\phi = 1$ on B_1, we see by the Schwartz inequality, Lemma 8 and (~~31~~ 33) that for small $\varepsilon > 0$,

$$\begin{aligned} \|\nabla(\phi_\delta u)\|_2^2 &\leq (1+\varepsilon) \int \phi_\delta^2 |\nabla u|^2 dx + (1 + \frac{1}{\varepsilon}) \int u^2 |\nabla \phi_\delta|^2 dx \\ &\leq (1+\varepsilon) \int_{B_{2\delta}} |\nabla u|^2 + C(1+\frac{1}{\varepsilon})\delta^{-2} \int_{B_{2\delta}\setminus B_\delta} u^2 dx \\ &< 4\pi \end{aligned} \tag{34}$$

as $\lambda \to 0$. Then consider the localized equation;

$$\begin{cases} -\Delta(\phi_{\delta/2} u) = \lambda \phi_{\delta/2} u e^{u^2} - F(u, \phi_{\delta/2}), & x \in B_\delta, \\ \phi_{\delta/2} u = 0, & x \in \partial B_\delta, \end{cases}$$

where $F(u, \psi) = 2\nabla u \cdot \nabla \psi_{\delta/2} + u\Delta\phi_{\delta/2} \in L^2(B_\delta)$. Since by (TM) and (34) $e^{\phi_\delta^2 u^2} \in L^{1+\eta'}(B_{2\delta})$, we see for $\eta > 0$,

$$\lambda \phi_{\delta/2} u e^{u^2} \in L^{1+\eta}(B_\delta).$$

The standard elliptic estimate implies

$$\|u\|_{L^\infty(B_{\delta/2})} \leq C,$$

which impossible since $x \in \mathcal{S}$. Hence we have

$$\int_{B_{2\delta}} |\nabla u|^2 dx \geq 4\pi. \tag{35}$$

For any compact set $K \subset\subset \Omega \setminus \mathcal{S}$, we have $\|u\|_{L^\infty(K)} \leq C$ and hence by the equation (E),

$$\|\Delta u\|_{L^\infty(K)} \leq C.$$

By Lemma 8, this implies $\|u\|_{L^\infty(K)} \to 0$ as $\lambda \to 0$. This proves (14).

Proof of Corollary 4. Since (12), (13) and (18) we see

$$4\pi \leq \varliminf_{\lambda \to 0} \|\nabla u\|_2^2 \leq \varlimsup_{\lambda \to 0} \|\nabla u\|_2^2 = \varlimsup_{\lambda \to 0} 2J_\lambda(u) \leq 4\pi.$$

This shows (19) and by (15), $\mathcal{S} = \{x_0\}$.

§3 Relations of Variational Solutions.

Finally we remark the relation between the solution in Proposition 3 and the solution obtained by the variational problem by Shaw. In **[23]**, Shaw considered a different kind of variational solution to (E).

Proposition 9 (Shaw). *For $u \in H_0^1(\Omega)$ with $\|\nabla u\|_2^2 < 4\pi$ with*

$$I(u) = \int_\Omega (e^{u^2} - 1)dx = \mu,$$

there exists a minimizer of $\|\nabla u\|_2^2$.

The "dual" of Shaw's formulation is the natural variational problem associate to (TM).

Proposition 10 ([6], [9]). *There exists a maximizer of*

$$I(u) = \int_\Omega (e^{u^2} - 1)dx$$

in $H_0^1(\Omega)$ with $\|\nabla u\|_2^2 \leq 4\pi$. The maximizer solves (E) with a certain Lagrange multiplier λ.

The solution obtained in Proposition 10 is in fact a minimizer of $J_\lambda(u)$.

Proposition 11. *Let u be a solution obtained in Proposition 10 with some λ. Then it is a solution obtained in Proposition 3, i.e.,*

$$J(u)_\lambda = \tilde{J}_\lambda.$$

Proof of Proposition 11. Suppose u be a solution which maximize $I(u)$ with $\|\nabla u\|_2^2 \leq 4\pi$. For any $v \in H_0^1(\Omega)$, we can choose some $t_0 > 0$ such that

$$\|\nabla v\|_2^2 = \lambda \int_\Omega v^2 e^{t_0^2 v^2} dx.$$

Note that for all $t > 0$, $J_\lambda(tv) \le J_\lambda(t_0 v)$ since $\frac{\partial}{\partial t} J_\lambda(tv)|_{t=t_0} = 0$. By setting $s^2 = \|\nabla u\|_2^2/\|\nabla v\|_2^2$, we have

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2}\|\nabla u\|_2^2 - \frac{\lambda}{2}\int_\Omega (e^{u^2} - 1)dx \\ &\le \frac{1}{2}\|\nabla (sv)\|_2^2 - \frac{\lambda}{2}\int_\Omega (e^{(su)^2} - 1)dx = J_\lambda(sv) \le J_\lambda(t_0 v). \end{aligned}$$

Therefore $J_\lambda(u) = \inf_{v\in V} J_\lambda(v)$, where $V = \{v \in H_0^1(\Omega)\backslash\{0\}, \|\nabla v\|_2^2 = \lambda \int_\Omega v^2 e^{v^2} dx\}$.

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