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Convex entropy function and symmetrization of the relativistic Euler equation

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1 Introduction

The motion of a relativistic perfect fluid in the Minkowski space-time is governed by

$$(1.1)\begin{cases} \dfrac{\partial}{\partial t}\left(\dfrac{\rho c^2+p}{c^2-v^2}-\dfrac{p}{c^2}\right)+\displaystyle\sum_{k=1}^{3}\dfrac{\partial}{\partial x_k}\left(\dfrac{\rho c^2+p}{c^2-v^2}v_k\right)=0, \\ \dfrac{\partial}{\partial t}\left(\dfrac{\rho c^2+p}{c^2-v^2}v_i\right)+\displaystyle\sum_{k=1}^{3}\dfrac{\partial}{\partial x_k}\left(\dfrac{\rho c^2+p}{c^2-v^2}v_iv_k+p\delta_{ik}\right)=0, \quad i=1,2,3. \end{cases}$$

Here c denotes the speed of light, p the pressure, (v_1, v_2, v_3) the velocity of the fluid particle, ρ the mass-energy density of the fluid (as measured in units of mass in a reference frame moving with the fluid particle) and $v^2 = v_1^2+v_2^2+v_3^2$. The fluid is assumed to be bartropic, which means that the equation (1.1) is to be supplemented with the equation of state

$$(1.2) \qquad p = p(\rho),$$

where $p(\rho)$ is a given function of ρ only.

For the case of one space dimension, Smoller and Temple [7] constructed global weak solutions to (1.1) for the isentropic case $p(\rho) = a^2\rho$ with $0 < a < c$, and Chen [1] for the case $p(\rho) = a^2\rho^\gamma$ with $a > 0$ and $\gamma > 1$.

In our previous paper [6], the existence of local smooth solutions was proved for three space dimensions, with $p(\rho) = a^2\rho$, $0 < a < c$. Our objective here is to extend this results to the general equation of state (1.2), under the sole assumption that

$$(1.3) \qquad \begin{aligned} &p(\rho) \in C^\infty(\rho_*, \rho^*), \\ &p(\rho) > 0, \quad 0 < p'(\rho) < c^2 \quad \textit{for} \quad \rho \in (\rho_*, \rho^*), \end{aligned}$$

where ρ_* and ρ^* are some constants such that $0 \le \rho_* < \rho^* \le \infty$. Note that if $p(\rho) = a^2\rho^\gamma$, then $\rho_* = 0$ while $\rho^* = \infty$ if $\gamma = 1$ and $\rho^* = \{c^2/(\gamma a^2)\}^{1/(\gamma-1)}$ if $\gamma > 1$.

We consider the initial value problem to (1.1) with the initial condition

$$\left\{ \begin{array}{l} \rho|_{t=0} = \rho_0(x), \\ v_i|_{t=0} = v_{0i}(x), \quad i = 1,2,3. \end{array} \right. \tag{1.4}$$

The main result of this paper is,

Theorem 1.1. *Assume (1.3) for $p(\rho)$. Suppose that the initial data ρ_0 and (v_{01}, v_{02}, v_{03}) belong to the locally uniform Sobolev space $H^s_{ul} = H^s_{ul}(\mathbf{R}^3), s \ge 3$, ([3]) and that there exist a positive constant δ sufficiently small so that*

$$\rho_* + \delta \le \rho(x) \le \rho^* - \delta,$$

$$v_0^2(x) = v_{01}^2(x) + v_{02}^2(x) + v_{03}^2(x) \le (1-\delta)c^2,$$

hold for all $x \in \mathbf{R}^3$. Then, the Cauchy problem (1.1), (1.2) and (1.4) has a unique solution satisfying

$$(\rho, v_1, v_2, v_3) \in L^\infty(0,T; H^s_{ul}) \cap C([0,T]; H^s_{loc}) \cap C^1([0,T]; H^{s-1}_{loc}), \tag{1.5}$$

with $\rho_* < \rho(x,t) < \rho^*$ and $v^2(x,t) < c^2$, and moreover,

$$(\rho, v_1, v_2, v_3) \in C([0,T]; H^{s-\epsilon}_{ul}) \cap C^1([0,T]; H^{s-1-\epsilon}_{ul}), \tag{1.6}$$

for any $\epsilon > 0$. Here $T > 0$ depends only on δ and the H^s_{ul}-norm of the initial data.

As in [6], we shall prove the theorem by symmetrizing (1.1) and applying the Friedlichs-Lax-Kato theory [3], [5] of symmetric hyperbolic systems. According to Godunov [2], a suitable symmetrizer can be constructed if a strictly convex entropy function exists. In §3, it is shown that such an entropy function exists for (1.1), and in §2, the symmetrizer it induces is discussed. Finally in §4, the non-relativistic limit of the solutions to (1.1) as $c \to \infty$ is shown to be a solution of the non-relativistic Euler equation with the same equation of state (1.2).

2 Symmetrization

Theorem 1.1 can be proved if there is a change of variables

$$z = (\rho, v_1, v_2, v_3)^T \longrightarrow u = (u_0, u_1, u_2, u_3)^T, \tag{2.1}$$

which reduces the system (1.1) to a system of the form

$$A^0(u)\frac{\partial u}{\partial t} + \sum_{\ell=1}^{3} A^\ell(u)\frac{\partial u}{\partial x_\ell} = 0, \tag{2.2}$$

whose coefficent matrices $A^\alpha(u)$, $\alpha = 0, 1, 2, 3$, satisfy the condition

(2.3)
(*i*) *they are all real symmetric and smooth in* u, *and*
(*ii*) $A^0(u)$ *is positive definite.*

The system (2.2) satisfying (2.3) is called a symmetric hyperbolic system, see [3], [5]. We claim that for (1.1), one of such changes of variables is given by

$$\begin{cases} u_0 &= -\dfrac{c^3 K e^{\phi(\rho)}}{(\rho c^2 + p)(c^2 - v^2)^{1/2}} + c^2, \\ u_j &= \dfrac{cK e^{\phi(\rho)}}{(\rho c^2 + p)(c^2 - v^2)^{1/2}} v_j, \quad j = 1, 2, 3, \end{cases} \tag{2.4}$$

where

$$\phi(\rho) = \int_{\bar\rho}^{\rho} \frac{c^2}{\rho c^2 + p(\rho)} d\rho, \quad K = c^2\bar\rho + p(\bar\rho), \tag{2.5}$$

$\bar\rho$ being an arbitrarily fixed number in (ρ_*, ρ^*). The derivation of (2.4), based on the idea of Godunov [2], will be presented in §3. Here we shall check the condition (2.3). To this end, we shall find the matrices $A^\alpha, \alpha = 0, 1, 2, 3$. explicitly. First, note from (2.4) that

$$v^2 = \frac{c^4}{(c^2 - u_0)^2} u^2, \quad u^2 = u_1^2 + u_2^2 + u_3^2.$$

Substituting this into the first equation of (2.4) and putting

$$\Phi(\rho) = \frac{K e^{\phi(\rho)}}{\rho c^2 + p(\rho)}, \tag{2.6}$$

we get

$$(2.7)\qquad \Phi(\rho) = \frac{1}{c^2}((c^2 - u_0)^2 - c^2 u^2)^{1/2}.$$

Since $\Phi'(\rho) = -Kp'(\rho)e^{\phi(\rho)}/(\rho c^2 + p)^2 < 0$ from (2.5) and (2.6), (2.7) can be solved uniquely for $\rho \in (\rho_*, \rho^*)$ provided

$$(2.8)\qquad \Phi(\rho^* - 0)^2 < (1 - \frac{u_0}{c^2})^2 - \frac{u^2}{c^2} < \Phi(\rho_* + 0)^2.$$

Thus, the map (2.1) defined with (2.4) is a diffeomorphism from

$$(2.9)\qquad \Omega_z = \{\rho_* < \rho < \rho^*, v^2 < c^2\}$$

onto

$$(2.10)\qquad \Omega_u = \{u_0 < c^2, \quad (2.8)\ \textit{holds.}\}.$$

After a straight but tedious computation, we find the coefficients $A^\alpha(u) = (A^\alpha_{\beta\gamma})$, $\alpha, \beta, \gamma = 0, 1, 2, 3$, as follows :

$$(2.11)\qquad \begin{aligned} A^0_{00} &= A_1\Psi(\rho), \quad A^0_{0i} = A^0_{i0} = A_2\Psi(\rho)v_i, \\ A^0_{ij} &= A_3\Psi(\rho)v_iv_j + A_4\Psi(\rho)\delta_{ij}, \\ A^\ell_{00} &= A_2\Psi(\rho), \quad A^\ell_{0i} = A^\ell_{i0} = A_3\Psi(\rho)v_iv_\ell + A_5\Psi(\rho)\delta_{i\ell}, \\ A^\ell_{ij} &= A_3\Psi(\rho)v_iv_jv_\ell + A_4\Psi(\rho)(v_i\delta_{j\ell} + v_j\delta_{i\ell} + v_\ell\delta_{ij}), \end{aligned}$$

for $i, j, \ell = 1, 2, 3$, where

$$\Psi(\rho) = \frac{1}{K}(\rho c^2 + p)^2 e^{-\phi(\rho)},$$

and

$$(2.12)\qquad \begin{aligned} A_1 &= \frac{c^4 + 3p'v^2}{c^3p'(c^2 - v^2)^{3/2}}, \quad A_2 = \frac{c^4 + 2p'c^2 + p'v^2}{c^3p'(c^2 - v^2)^{3/2}}, \\ A_3 &= \frac{c^2 + 3p'}{cp'(c^2 - v^2)^{3/2}}, \quad A_4 = \frac{1}{c(c^2 - v^2)^{1/2}}, \\ A_5 &= \frac{1}{c(\rho c^2 + p)(c^2 - v^2)^{1/2}}. \end{aligned}$$

These coefficents can be calculated by the chain rule and the formula

$$\begin{aligned} \frac{\partial\rho}{\partial u_0} &= \frac{A_4}{p'}\Psi(\rho), \quad \frac{\partial\rho}{\partial u_j} = \frac{A_4}{p'}\Psi(\rho)v_j, \\ \frac{\partial v_i}{\partial u_0} &= A_6\Psi(\rho)v_i, \quad \frac{\partial v_i}{\partial u_j} = c^2A_6\Psi(\rho)\delta_{ij}, \quad i, j = 1, 2, 3, \end{aligned}$$

with

$$A_6 = \frac{(c^2 - v^2)^{1/2}}{c^3(\rho c^2 + p)}.$$

It is clear from (2.11) that the matrices $A^\alpha(u)$ are all real symmetric and smooth in Ω_z, and hence in Ω_u. To see that $A^0(u)$ is positive definite, let $\Xi = (\xi_0, \xi)^T \in \mathbf{R}^4$ be a 4-vector with $\xi \in \mathbf{R}^3$. We should calculate the inner product

$$(A^0(u)\Xi \mid \Xi) = \Psi(\rho)\{A_1\xi_0^2 + 2A_2\xi_0(v|\xi) + A_3(v|\xi)^2 + A_4\xi^2\},$$

A_j being those in (2.12). In the same way as in [6], we can get an estimate

$$(A^0\Xi|\Xi) \geq \frac{1}{2}(\kappa_0\xi_0^2 + \kappa\xi^2), \tag{2.13}$$

with

$$\kappa_0 = \frac{(c^2 - v^2)^{1/2}(c^4 - p'v^2)\Psi(\rho)}{c^3(c^4v^2 + 2c^2v^2p' + c^4p')},$$
$$\kappa = \frac{(c^2 - v^2)^{1/2}(c^4 - p'v^2)\Psi(\rho)}{c^3(c^4 + 3v^2p')},$$

which implies that (2.3)(ii) is also satisfied in Ω_u since (1.3) is fulfilled. Thus, (2.2) with (2.11) for the elements of the matrices $A^\alpha(u)$ is a symmetric hyperbolic system, which entails the existence of smooth local solutions to (2.2), thanks to the Friedrichs–Lax–Kato theory [3],[5]. Since (2.4) is a diffeomorphism, we can go back from(2.2) to the original system (1.1) to conclude Theorem 1.1.

3 Strictly convex entropy function

In this section, we shall follow Godunov [2] and explain how to find out the change of variables (2.4). First of all, we rewrite (1.1) in the form of the conservation laws,

$$w_t + \sum_{k=1}^{3}(f^k(w))_{x_k} = 0, \tag{3.1}$$

where $w = (w_0, w_1, w_2, w_3)^T$ and $f^k(w) = (w_k, f_1^k, f_2^k, f_3^k)^T$ are defined by

$$
(3.2) \qquad \begin{aligned} w_0 &= \frac{\rho c^2 + p}{c^2 - v^2} - \frac{p}{c^2}, \quad w_j = \frac{\rho c^2 + p}{c^2 - v^2} v_j, \\ f_i^k &= \frac{\rho c^2 + p}{c^2 - v^2} v_i v_k + p\delta_{ik}. \end{aligned}
$$

A scalar fuction $\eta = \eta(w)$ is called an entropy function to (3.1) if there exist scalar functions, $q^k = q^k(w)$, $k = 1, 2, 3$, satisfying

$$
(3.3) \qquad D_w \eta(w) D_w f^k(w) = D_w q^k.
$$

Then, the symmetrizing variable u can be given by

$$
(3.4) \qquad u = (D_w \eta)^T.
$$

For the detail, see Godunov [2] or Kawashima-Shizuta [4].

Now, we shall solve (3.3). To this end, it is convenient to employ $z = (\rho, v_1, v_2, v_3)$, instead of w of (3.2), as the independent variables in (3.3). This is possible since $D_z w$ is regular;

$$
\det D_z w = \frac{(\rho c^2 + p)(c^4 - v^2 p')}{c^2(c^2 - v^2)} > 0,
$$

which comes by noting

$$
(3.5) \qquad \begin{aligned} \frac{\partial w_0}{\partial \rho} &= B_1, \quad \frac{\partial w_0}{\partial v_j} = B_2 v_j, \\ \frac{\partial w_i}{\partial \rho} &= B_3 v_i, \quad \frac{\partial w_i}{\partial v_j} = B_2 v_i v_j + B_4 \delta_{ij}, \end{aligned}
$$

where

$$
\begin{aligned} B_1 &= \frac{c^2 + p'}{c^2 - v^2} - \frac{p}{c^2}, \quad B_2 = \frac{2(\rho c^2 + p)}{(c^2 - v^2)^2}, \\ B_3 &= \frac{c^2 + p'}{c^2 - v^2}, \qquad B_4 = \frac{\rho c^2 + p}{c^2 - v^2}. \end{aligned}
$$

Thus the mapping $z \to w$ is a diffeomorphism in a neighbourhood of each point of Ω_z. Moreover, using (3.5), we get

$$
(3.6) \qquad (D_z w)^{-1} = (e_{\alpha\beta})_{\alpha,\beta=0,1,2,3}
$$

as

$$e_{00} = c^2(c^2 + v^2)E_1, \qquad e_{0j} = -2c^2 E_1 v_j,$$
$$e_{i0} = -c^2(c^2 + p')E_1 E_2 v_i, \quad e_{ij} = 2p' E_1 E_2 v_i v_j + E_2 \delta_{ij},$$

with

$$E_1 = \frac{1}{c^4 - v^2 p'}, \quad E_2 = \frac{c^2 - v^2}{\rho c^2 + p}.$$

In view of (3.2) and (3.6), (3.3) can now be rewritten as

$$D_z \eta C^k = D_z q^k, \quad k = 1, 2, 3, \tag{3.7}$$

where

$$C^k = (D_z w)^{-1} D_z f^k = (c^k_{\alpha\beta})_{\alpha,\beta=0,1,2,3},$$

are given by

$$c^k_{00} = c^2 C_1 v_k, \quad c^k_{i0} = -C_1 C_2 v_i v_k + C_2 \delta_{kj},$$
$$c^k_{0j} = C_3 \delta_{ki}, \quad c^k_{ij} = C_4 v_i \delta_{kj} + v_k \delta_{ij},$$

with

$$\begin{aligned} C_1 &= \frac{c^2 - p'}{c^4 - p' v^2}, & C_2 &= \frac{p'(c^2 - v^2)}{\rho c^2 + p}, \\ C_3 &= \frac{c^2(\rho c^2 + p)}{c^4 - v^2 p'}, & C_4 &= \frac{p'(c^2 - v^2)}{c^4 - v^2 p'}. \end{aligned} \tag{3.8}$$

Let us solve (3.7) for (η, q^1, q^2, q^3). A quick count shows that (3.7) costitutes 12 equations for 4 unknowns, that is, it forms an over-determined system. We shall look for the solution of the form

$$\eta = H(\rho, y), \quad q^k = Q(\rho, y) v_k, \tag{3.9}$$

where

$$y = v^2 = v_1^2 + v_2^2 + v_3^2.$$

This ansatz reduces (3.7) to the following system of first order linear partial differential equations;

$$H_y = Q_y, \tag{3.10}$$

$$c^2 C_1 H_\rho + 2C_2(-C_1 y + 1)H_y = Q_\rho, \tag{3.11}$$

$$C_3 H_\rho - 2C_4 y H_y = Q, \tag{3.12}$$

C_j being as in (3.8). Seemingly, we have still an over-determined system. However, making (3.11) $\times$ $(\rho c^2 + p)$ $-$ (3.12) $\times$ $(c^2 - p')$ and using (3.10), we get a single equation for Q:

$$2(c^2 - y)p' Q_y = (\rho c^2 + p)Q_\rho - (c^2 - p')Q. \tag{3.13}$$

On the other hand, it follows from (3.10) that there should exist a function $G = G(\rho)$ of ρ only such that

$$H = Q(\rho, y) + G(\rho).$$

Substitution of this into (3.11), together with (3.13), then yields

$$\rho G_\rho = \frac{c^2 - y}{\rho c^2 + p} Q - \frac{c^2 - y}{c^2} Q_\rho,$$

or putting $q = (c^2 - y)Q$,

$$G_\rho = \frac{1}{\rho c^2 + p} q - \frac{1}{c^2} q_\rho. \tag{3.14}$$

Since the left hand side of (3.14) is a function of ρ only, q must be of the form

$$q = e^{\phi(\rho)}[g(\rho) + h(y)], \tag{3.15}$$

where $\phi(\rho)$ is as in (2.5) while g and h are arbitrary functions. Substituting (3.15) into (3.13) and separating the variables, we have

$$\frac{\rho c^2 + p}{p'} \frac{dg}{d\rho} - g = 2(c^2 - y)\frac{dh}{dy} + h = constant,$$

which can be easily solved as

$$\begin{aligned} q &= e^{\phi(\rho)}[K_1(c^2 - y)^{1/2} + K_2' e^{\psi(\rho)}] \\ &= K_1(c^2 - y)^{1/2} e^{\phi(\rho)} + K_2(\rho c^2 + p), \end{aligned} \tag{3.16}$$

where K_j's are integration constants and

$$\begin{aligned} \psi(\rho) &= \int_{\bar\rho}^{\rho} \frac{p'(\rho)}{\rho c^2 + p(\rho)} d\rho \\ &= -\phi(\rho) + \log \frac{\rho c^2 + p(\rho)}{\bar\rho c^2 + p(\bar\rho)}, \end{aligned} \tag{3.17}$$

$\bar{\rho}$ being as in (2.5). Now, (3.14) combined with (3.16) gives $G' = -K_2 p'/c^2$, so that

$$G = -\frac{K_2}{c^2} p + K_3, \tag{3.18}$$

K_3 being also an integration constant. In view of (3.16) and (3.18), we get

$$\eta = H = \frac{K_1}{(c^2 - v^2)^{1/2}} e^{\phi(\rho)} + K_2 \left(\frac{\rho c^2 + p}{c^2 - v^2} - \frac{p}{c^2} \right) + K_3, \tag{3.19}$$

$$Q = \frac{K_1}{(c^2 - v^2)^{1/2}} e^{\phi(\rho)} + \frac{K_2}{c^2 - v^2} (\rho c^2 + p). \tag{3.20}$$

For the later purpose, we wish to choose the constants K_j, $j = 1, 2, 3$, so that (3.19) converges, as $c \to \infty$, to the entropy function for the non-relativistic case,

$$\eta^{(\infty)} = \frac{1}{2} \rho v^2 + \rho \int_{\bar{\rho}}^{\rho} \frac{dp}{\rho} - p, \tag{3.21}$$

which can be obtained exactly in the same way as (3.19). In view of (3.17), $\Phi(\rho)$ of (2.6) equals $e^{-\psi(\rho)}$ so that it can be expanded for large c as

$$\Phi(\rho) = 1 - \frac{1}{c^2} \int_{\bar{p}}^{p} \frac{dp}{\rho} + O(c^{-4}),$$

for each fixed $\rho \in (\rho_*, \rho^*)$. Insert this into (3.19) to deduce

$$\begin{aligned} \eta = & \frac{\rho + p/c^2}{(1 - v^2/c^2)^{1/2}} \{ \frac{cK_1}{K} + \frac{K_2}{(1 - v^2/c^2)^{1/2}} \\ & - \frac{K_1}{cK} \int_{\bar{p}}^{p} \frac{dp}{\rho} - \frac{K_1}{K} O(c^{-3}) \} - \frac{K_2 p}{c^2} + K_3, \end{aligned}$$

where K is as in (2.5). Therefore, the right choice is found to be

$$K_1 = -cK, \quad K_2 = c^2, \quad K_3 = 0,$$

with which (3.19) becomes

$$\eta = -\frac{cK}{(c^2 - v^2)^{1/2}} e^{\phi(\rho)} + c^2 \left(\frac{\rho c^2 + p}{c^2 - v^2} - \frac{p}{c^2} \right). \tag{3.22}$$

The change of variables (2.4) was derived from (3.22) via the formula (3.4) or

$$u = ((D_z w)^T)^{-1}(D_z \eta)^T,$$

combined with (3.6). Since the matrix $A^0(u)$ is positive definite in Ω_u as was shown in the preceding section, the entropy function (3.22) is strictly convex there.

4 Non-relativistic limit

In order to study the limit $c \to \infty$, we consider $c \geq c_0$ with a fixed c_0 sufficiently large and assume, without loss of generality, that (1.3) is satisfied for all $c \geq c_0$ with the same constants ρ_* and ρ^*. For the sake of simplicity, we discuss only the case $\rho^* < \infty$. The case $\rho^* = \infty$ can be treated similarly. Given $\delta > 0$ sufficiently small, define

$$\Omega_z(\delta, c_0) = \{\rho_* + \delta \leq \rho \leq \rho^* - \delta, \quad v^2 \leq (1-\delta)c_0^2\}. \tag{4.1}$$

Firstly, note that (2.4) is a diffeomorphism from the domain (4.1) onto

$$\begin{aligned} \Omega_u(\delta, c_0, c) = \ & \{u_0 < c^2, \ (1 - \frac{u_0}{c^2})^2 - \frac{u^2}{c_0^2(1-\delta)} \geq 0, \\ & \Phi(\rho^* - \delta)^2 \leq (1 - \frac{u_0}{c^2})^2 - \frac{u^2}{c^2} \leq \Phi(\rho_* + \delta)^2\}, \end{aligned} \tag{4.2}$$

cf. (2.9) and (2.10). Secondly, the matrices $A^\alpha(u)$ and all of their derivatives are uniformly bounded in the domain (4.2). Moreover, κ_0 and κ in (2.13) are bounded away from zero uniformly there, as seen from

$$\kappa_0 = \frac{\rho}{v^2 + p'} + O(c^{-2}),$$
$$\kappa = \rho + O(c^{-2}).$$

This means that the Friedlichs-Kato-Lax theory applies for (2.2) uniformly for all $c \geq c_0$. Go back to (1.1), which is possible due to the diffeomorphism (2.4), to conclude

Theorem 4.1. *Let $s \geq 3$. For any fixed M_0, $c_0 > 0$ sufficiently large and $\delta_0 > 0$ sufficiently small, there exist positive constants M and T such that for any initial $z_0 = (\rho_0, v_{01}, v_{02}, v_{03}) \in H^s_{ul}$ satisfying*

$$||z_0||_{H^s_{ul}} \leq M_0, \quad z_0(x) \in \Omega_z(\delta_0, c_0) \text{ for any } x \in \mathbf{R}^3,$$

and for any $c \geq c_0$, *the Cauchy problem (1.1),(1.2) and (1.4) possesses a unique solution* $z = (\rho, v_1, v_2, v_3)$ *belonging to the class (1.5),(1.6) and satisfying*

$$\|z(t)\|_{H^s_{ul}} \leq M, \quad z(t,x) \in \Omega_z(\delta_0/2, c_0) \text{ for any } x \in \mathbf{R}^3,$$

for almost all $t \in [0, T]$.

Let us show that the solutions z thus obtained converge as $c \to \infty$ to the solution of the non-relativistic Euler equation,

$$\tag{4.3} \begin{cases} \dfrac{\partial \rho}{\partial t} + \displaystyle\sum_{k=1}^{3} \frac{\partial}{\partial x_k}(\rho v_k) = 0 \\ \dfrac{\partial}{\partial t}(\rho v_i) + \displaystyle\sum_{k=1}^{3} \frac{\partial}{\partial x_k}(\rho v_i v_k + p\delta_{ik}) = 0, \quad i = 1,2,3, \end{cases}$$

with the same intial z_0.

The symmetrizing variables for (4.3) associated with the entropy function (3.21) are given by

$$\tag{4.4} \begin{aligned} u_0^{(\infty)} &= -\frac{1}{2}v^2 + \int_{\bar{p}}^{p} \frac{dp}{\rho}, \\ u_j^{(\infty)} &= v_j, \quad j = 1,2,3, \end{aligned}$$

and the resulting system is

$$\tag{4.5} A^{(\infty)0}(u^{(\infty)})u_t^{(\infty)} + \sum_{\ell=1}^{3} A^{(\infty)\ell}(u^{(\infty)})u_{x_\ell}^{(\infty)} = 0,$$

with

$$\begin{aligned} a_{00}^{(\infty)0} &= \frac{\rho}{p'}, \quad a_{i0}^{(\infty)0} = a_{0i}^{(\infty)0} = \frac{\rho}{p'}v_i, \\ a_{ij}^{(\infty)0} &= \frac{\rho}{p'}v_i v_j + \rho\delta_{ij}, \end{aligned}$$

for the matrix elements of $A^{(\infty)0}$ and so on. Between the transformations (2.4) and (4.4), it holds that

$$\begin{aligned} &u(z) = u^{(\infty)}(z) + O(c^{-2}), \\ &A^\alpha(u(z)) = A^{(\infty)\alpha}(u^{(\infty)}(z)) + O(c^{-2}), \quad \alpha = 0,1,2,3, \end{aligned}$$

uniformly for $c \geq c_0$ and $z \in \Omega_z(\delta_0/2, c_0)$, which implies, together with the uniform properties stated before Theorem 4.1 and by the arguments in [6], the uniform convergence of the solutions u of (2.2) to the solution of (4.5). Again we can go back to (1.1) and conclude

Theorem 4.2. *Let $s \geq 3$. Then, as $c \to \infty$, the solution z of (1.1),(1.2) and(1.4) given in Theorem 4.1 converges to the solution $z^{(\infty)}$ to (4.3) with the same initial data, uniformly on the time interval $[0, T]$ with T specified in Theorem 4.1, strongly in $H_{ul}^{s-\epsilon}$ for any $\epsilon > 0$.*

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