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PROPERTIES OF CERTAIN ANALYTIC FUNCTIONS

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ABSTRACT

The object of the present paper is to derive some properties of certain analytic functions in the open unit disk. Our results are the generalizations of theorems given by M. Nunokawa and Hoshino (Kokyuroku 821 (1993), 4 - 7).

I. INTRODUCTION

Let A_n be the class of functions of the form

$$(1.1)\qquad f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \qquad (n \in \mathbb{N} = \{1,2,3,\dots\})$$

which are analytic in the open unit disk $\mathbb{U} = \{z: |z| < 1\}$. A function $f(z)$ in A_n is said to be in the class $S_n^*(\alpha)$ if it satisfies

$$(1.2)\qquad \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in \mathbb{U})$$

for some α $(0 \leqq \alpha < 1)$. Then a function $f(z) \in S_n^*(\alpha)$ is said to be starlike of order α in $\mathbb{U}$.

To derive our results, we need the following lemma due to Miller and Mocanu [1].

LEMMA. Let $w(z) = w_n z^n + w_{n+1} z^{n+1} + \dots$ be analytic in $\mathbb{U}$ with $w_n \neq 0$ and $n \geqq 1$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point z_0, then

$$z_0 w'(z_0) = k w(z_0),$$

where k is real and $k \geqq n \geqq 1$.

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2. Main Theorem

Our main theorem is contained in

THEOREM. Let $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \dots$ be analytic in U with $p_n \neq 0$. If $p(z)$ satisfies

$$\text{(2.1)} \qquad \operatorname{Re}\left\{p(z) + \alpha\frac{zp'(z)}{p(z)}\right\} > \frac{1 - \alpha n}{2} \qquad (z \in U)$$

for some α $(\alpha > 0)$, or if

$$\text{(2.2)} \qquad \operatorname{Re}\left\{p(z) + \alpha\frac{zp'(z)}{p(z)}\right\} < \frac{1 - \alpha n}{2} \qquad (z \in U)$$

for some α $(\alpha < -1)$, then

$$\text{(2.3)} \qquad \left|\frac{p(z) - 1}{p(z)}\right| < 1 \qquad (z \in U),$$

or

$$\text{(2.4)} \qquad \operatorname{Re}(p(z)) > \frac{1}{2} \qquad (z \in U).$$

PROOF. In view of the result by Nunokawa and Hoshino [2], we see that our conditions (2.1) and (2.2) imply $p(z) \neq 0$ for $z \in U$. We define the function $w(z)$ by

$$\text{(2.5)} \qquad w(z) = \frac{p(z) - 1}{p(z)},$$

or

$$\text{(2.6)} \qquad p(z) = \frac{1}{1 - w(z)}.$$

Then $w(z) = w_n z^n + w_{n+1} z^{n+1} + \dots$ is analytic in U with $w_n \neq 0$. It follows from (2.6) that

$$\text{(2.7)} \qquad \frac{zp'(z)}{p(z)} = \frac{zw'(z)}{1 - w(z)}$$

Suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = 1 \qquad (w(z_0) \neq 1).$$

Then, by Lemma, we have

$$z_0 w'(z_0) = k w(z_0) \qquad (k \geqq n \geqq 1).$$

Letting $w(z_0) = e^{i\theta}$, we see that

$$\text{(2.8)} \qquad \mathrm{Re}\left\{p(z_0) + \alpha \frac{z_0 p'(z_0)}{p(z_0)}\right\} = \mathrm{Re}\left\{\frac{1}{1 - w(z_0)} + \alpha \frac{z_0 w'(z_0)}{1 - w(z_0)}\right\}$$

$$= \mathrm{Re}\left\{\frac{1 + \alpha k e^{i\theta}}{1 - e^{i\theta}}\right\}$$

$$= \frac{1 - \alpha k}{2},$$

so that

$$\text{(2.9)} \qquad \frac{1 - \alpha k}{2} \leqq \frac{1 - \alpha n}{2} \qquad (\alpha > 0)$$

and

$$\text{(2.10)} \qquad \frac{1 - \alpha k}{2} \geqq \frac{1 - \alpha n}{2} \qquad (\alpha < -1).$$

This contradicts our conditions of the theorem. Therefore, $|w(z)| < 1$ for all $z \in U$. This shows that

$$\text{(2.11)} \qquad \left|\frac{p(z) - 1}{p(z)}\right| < 1 \qquad (z \in U),$$

or

$$\text{(2.12)} \qquad \mathrm{Re}(p(z)) > \frac{1}{2} \qquad (z \in U).$$

This completes the proof of Theorem.

If we take $\alpha = 1/n$ in Theorem, then we have

COROLLARY 1. Let $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \dots$ be analytic in U with $p_n \neq 0$. If $p(z)$ satisfies

$$(2.13)\qquad \mathrm{Re}\left\{p(z) + \frac{zp'(z)}{np(z)}\right\} > 0 \qquad (z \in U),$$

then

$$(2.14)\qquad \left|\frac{p(z) - 1}{p(z)}\right| < 1 \qquad (z \in U),$$

or

$$(2.15)\qquad \mathrm{Re}(p(z)) > \frac{1}{2} \qquad (z \in U).$$

Further, making $p(z) = zf'(z)/f(z)$ in Theorem, we have

COROLLARY 2. If $f(z) \in A_n$ with $a_{n+1} \neq 0$ and if

$$(2.16)\quad \mathrm{Re}\left\{(1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \frac{1 - \alpha n}{2} \qquad (z \in U)$$

for some $\alpha (\alpha > 0)$, or if

$$(2.17)\quad \mathrm{Re}\left\{(1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} < \frac{1 - \alpha n}{2} \qquad (z \in U)$$

for some $\alpha (\alpha < -1)$, then $f(z) \in S_n^*(1/2)$ and

$$(2.18)\qquad \left|\frac{zf'(z) - f(z)}{zf'(z)}\right| < 1 \qquad (z \in U).$$

Finally, letting $p(z) = f'(z)$ in Theorem, we have

COROLLARY 3. If $f(z) \in A_n$ with $a_{n+1} \neq 0$ and if

$$(2.19)\qquad \mathrm{Re}\left\{f'(z) + \alpha\frac{zf''(z)}{f'(z)}\right\} > \frac{1 - \alpha n}{2} \qquad (z \in U)$$

for some $\alpha (\alpha > 0)$, or if

$$(2.20)\qquad \mathrm{Re}\left\{f'(z) + \alpha\frac{zf''(z)}{f'(z)}\right\} < \frac{1 - \alpha n}{2} \qquad (z \in U)$$

for some $\alpha (\alpha < -1)$, then

$$(2.21)\qquad \left|\frac{f'(z) - 1}{f'(z)}\right| < 1 \qquad (z \in U),$$

or

$$\mathrm{Re}(f'(z)) > \frac{1}{2} \qquad (z \in \mathbb{U}). \tag{2.22}$$

REMARK. If we take $n = 1$ in our results, then we have the corresponding results by Nunokawa and Hoshino [2].

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