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SOME APPLICATIONS OF FRACTIONAL CALCULUS OPERATORS TO CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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ABSTRACT. The object of the present paper is to prove various distortion theorms for the fractional calculus of functions in the class $P^*(n,\alpha,\beta)$ consisting of analytic and univalent functions with negative coefficients. Furthermore, distortion theorem for a fractional integral operator of functions in the class $P^*(n,\alpha,\beta)$ is shown.

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1. Introduction

Let S denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic and univalent in the unit disc $U=\{z: |z|<1\}$. For a function f(z) in S, we define

$$D^0 f(z) = f(z), \tag{1.2}$$

$$D^1 f(z) = Df(z) = zf'(z), \tag{1.3}$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \quad (n \in N = \{1,2....\}). \tag{1.4}$$

The differential operator D^n was introduced by Salagean [3]. With the help of the differential operator D^n, we say that a function f(z) belonging to S is in the class $S(n,\alpha,\beta)$ if and only if

$$\left| \frac{\dfrac{D^n f(z)}{z} - 1}{\dfrac{D^n f(z)}{z} + 1 - 2\alpha} \right| < \beta \quad (n \in N_0 = N \cup \{0\}) \tag{1.5}$$

for $0 \le \alpha < 1$, $0 < \beta \le 1$, and for all $z \in U$.

Let T denote the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \ge 0). \tag{1.6}$$

Further, we define the class $P^*(n,\alpha,\beta)$ by

$$P^*(n,\alpha,\beta) = S(n,\alpha,\beta) \cap T. \tag{1.7}$$

The class $P^*(n,\alpha,\beta)$ was studied by Aouf and Nunokawa [2].

We note that, by specializing the parameters n, α, and β, we obtain the following subclasses studied by various authors:

(i) $P^*(0,\alpha,\beta) = P^*(\alpha,\beta)$ (Srivastava and Owa [9]);

(ii) $P^*(1,\alpha,\beta) = P^*(\alpha,\beta)$ (Gupta and Jain [3]);

(iii) $P^*(0,\alpha,0) = P^{**}(\alpha)$ (Sarangi and Uralegaddi [7]);

(iv) $P^*(1,\alpha,1) = T^{**}(\alpha)$ (Sarangi and Uralegaddi [7] and Al-Amiri [1]).

In order to prove our results for functions belonging to the class $P^*(n,\alpha,\beta)$, we shall require the following lemma given by Aouf and Nunokawa [2].

Lemma 1. Let the function $f(z)$ be defined by (1.6). Then $f(z) \in P^*(n,\alpha,\beta)$ if and only if

$$\sum_{k=2}^{\infty}(1+\beta)k^n a_k \le 2\beta(1-\alpha). \tag{1.8}$$

The result is sharp.

2. Fractional Calculus

We begin with the statements of the following definitions of fractional calculus (that is, fractional derivatives and fractional integrals) which were defined by Owa ([4], [5]) and were used recently by Srivastava and Owa [8].

Definition 1. The fractional integral of order λ is defined, for a function $f(z)$, by

$$D_z^{-\lambda}f(z)=\frac{1}{\Gamma(\lambda)}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}}d\zeta \quad (\lambda > 0), \tag{2.1}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta>0$.

DEFINITION 2. The fractional derivative of order λ is defined, for a function f(z), by

$$D_z^{\lambda}f(z)=\frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z\frac{f(\zeta)}{(z-\zeta)^{\lambda}}d\zeta \quad (0\le\lambda<1), \tag{2.2}$$

where f(z) is constrained, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed, as in Definition 1.

DEFINITION 3. Under the hypotheses of Definition 2, the fractional derivative of order $n+\lambda$ is defined by

$$D_z^{n+\lambda}f(z)=\frac{d^n}{dz^n}D_z^{\lambda}f(z) \quad (0\le\lambda<1;\ n\in N_0). \tag{2.3}$$

THEOREM 1. Let the function f(z) defined by (1.6) be in the class $P^*(n,\alpha,\beta)$. Then we have

$$\left|D_z^{-\lambda}(D^i f(z))\right| \ge \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)}\left\{1-\frac{\beta(1-\alpha)}{2^{n-i-2}(1+\beta)(2+\lambda)}|z|\right\} \tag{2.4}$$

and

$$\left|D_z^{-\lambda}(D^i f(z))\right| \le \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)}\left\{1+\frac{\beta(1-\alpha)}{2^{n-i-2}(1+\beta)(2+\lambda)}|z|\right\} \tag{2.5}$$

for $\lambda>0$, $0\le i\le n$, and $z\in U$. The result is sharp.

PROOF. Note that $f(z)\in P^*(n,\alpha,\beta)$ if and only if $D^i f(z)\in P^*(n-i,\alpha,\beta)$, and that

$$D^i f(z) = z-\sum_{k=2}^{\infty}k^i a_k z^k. \tag{2.6}$$

Using Lemma 1, we know that

$$2^{n-i}(1+\beta)\sum_{k=2}^{\infty} k^{i}a_k \le \sum_{k=2}^{\infty}(1+\beta)k^{n}a_k \le 2\beta(1-\alpha), \tag{2.7}$$

that is, that

$$\sum_{k=2}^{\infty} k^{i}a_k \le \frac{\beta(1-\alpha)}{2^{n-i-1}(1+\beta)} . \tag{2.8}$$

Let

$$\begin{aligned} F(z) &= \Gamma(2+\lambda)z^{-\lambda}D_z^{-\lambda}(D^{i}f(z)) \\ &= z - \sum_{k=2}^{\infty}\frac{\Gamma(k+1)\Gamma(2+\lambda)}{\Gamma(k+1+\lambda)}\, k^{i}a_k z^{k} \\ &= z - \sum_{k=2}^{\infty}\psi(k)k^{i}a_k z^{k} , \end{aligned} \tag{2.9}$$

where

$$\psi(k) = \frac{\Gamma(k+1)\Gamma(2+\lambda)}{\Gamma(k+1+\lambda)} \quad (k \ge 2) . \tag{2.10}$$

Since

$$0 < \psi(k) \le \psi(2) = \frac{2}{2+\lambda} , \tag{2.11}$$

Therefore, by using (2.8) and (2.11), we can see that

$$\begin{aligned} |F(z)| &\ge |z| - \psi(2)|z|^{2}\sum_{k=2}^{\infty} k^{i}a_k \\ &\ge |z| - \frac{\beta(1-\alpha)}{2^{n-i-2}(1+\beta)(2+\lambda)}\ |z|^{2} \end{aligned} \tag{2.12}$$

and

$$|F(z)| \le |z| + \psi(2)|z|^{2}\sum_{k=2}^{\infty} k^{i}a_k$$

$$\leq |z| + \frac{\beta(1-\alpha)}{2^{n-i-2}(1+\beta)(2+\lambda)} |z|^2 \tag{2.13}$$

which prove the inequalities of Theorem 1. Further, equalities are attained for the function f(z) defined by

$$D_z^{-\lambda}(D^i f(z)) = \frac{z^{1+\lambda}}{\Gamma(2+\lambda)} \left\{ 1 - \frac{\beta(1-\alpha)}{2^{n-i-2}(1+\beta)(2+\lambda)} z \right\} \tag{2.14}$$

or

$$D^i f(z) = z - \frac{\beta(1-\alpha)}{2^{n-i-2}(1+\beta)} z^2. \tag{2.15}$$

Thus we complete the proof of Theorem 1.

Taking i = 0 in Theorem 1, we have

Corollary 1. Let the function f(z) defined by (1.6) be in the class $P^*(n,\alpha,\beta)$. Then we have

$$\left| D_z^{-\lambda} f(z) \right| \geq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left\{ 1 - \frac{\beta(1-\alpha)}{2^{n-2}(1+\beta)(2+\lambda)} |z| \right\} \tag{2.16}$$

and

$$\left| D_z^{-\lambda} f(z) \right| \leq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left\{ 1 + \frac{\beta(1-\alpha)}{2^{n-2}(1+\beta)(2+\lambda)} |z| \right\} \tag{2.17}$$

for $\lambda > 0$, $0 \leq i \leq n$, and $z \in U$. The result is sharp for the function f(z) given by

$$f(z) = z - \frac{\beta(1-\alpha)}{2^{n-2}(1+\beta)(2+\lambda)} z^2 . \tag{2.18}$$

Theorem 2. Let the function f(z) defined by (1.2) be in the class $P^*(n,\alpha,\beta)$. Then we have

$$\left|D_z^{\lambda}(D^i f(z))\right| \geq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)}\left\{1 - \frac{\beta(1-\alpha)}{2^{n-i-2}(1+\beta)(2-\lambda)}|z|\right\} \tag{2.19}$$

and

$$\left|D_z^{\lambda}(D^i f(z))\right| \leq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)}\left\{1 + \frac{\beta(1-\alpha)}{2^{n-i-2}(1+\beta)(2-\lambda)}|z|\right\} \tag{2.20}$$

for $0 \leq \lambda < 1$, $0 \leq i \leq n-1$, and $z \in U$. The result is sharp.

Proof. Let

$$\begin{aligned} G(z) &= \Gamma(2-\lambda) z^{\lambda} D_z^{\lambda}(D^i f(z)) \\ &= z - \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)} k^i a_k z^k \\ &= z - \sum_{k=2}^{\infty} \Phi(k) k^{i+1} a_k z^k, \end{aligned} \tag{2.21}$$

where

$$\Phi(k) = \frac{\Gamma(k)\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)} \quad (k \geq 2). \tag{2.22}$$

Noting

$$0 < \Phi(k) \leq \Phi(2) = \frac{1}{2-\lambda}. \tag{2.23}$$

Consequently, with the aid of (2.8) and (2.23), we have

$$\begin{aligned} |G(z)| &\geq |z| - \Phi(2)|z|^2 \sum_{k=2}^{\infty} k^{i+1} a_k \\ &\geq |z| - \frac{\beta(1-\alpha)}{2^{n-i-2}(1+\beta)(2-\lambda)} |z|^2 \end{aligned} \tag{2.24}$$

and

$$|G(z)| \leq |z| + \Phi(2)|z|^2 \sum_{k=1}^{\infty} k^{i+1} a_k$$

$$\leq |z| + \frac{\beta(1-\alpha)}{2^{n-i-2}(1+\beta)(2-\lambda)} |z|^2. \tag{2.25}$$

Thus (2.19) and (2.20) follows from (2.24) and (2.25), respectively. Futher, since the equalities in (2.19) and (2.20) are attained for the function

$$D_z^{\lambda}(D^i f(z)) = \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} \left\{1 - \frac{\beta(1-\alpha)}{2^{n-i-2}(1+\beta)(2-\lambda)} z\right\} \tag{2.26}$$

or for the function f(z) defined by (2.15). Thus we complete the proof of Theorem 2.

Taking i = 0 in Theorem 2, we have

Corollary 2. Let the function f(z) defined by (1.6) be in the class $P^*(n,\alpha,\beta)$. Then we have

$$\left|D_z^{\lambda} f(z)\right| \geq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)} \left\{ 1 - \frac{\beta(1-\alpha)}{2^{n-2}(1+\beta)(2-\lambda)} |z|\right\} \tag{2.27}$$

and

$$\left|D_z^{\lambda} f(z)\right| \leq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)} \left\{ 1 + \frac{\beta(1-\alpha)}{2^{n-2}(1+\beta)(2-\lambda)} |z|\right\} \tag{2.28}$$

for $0 \leq \lambda < 1$, and $z \in U$. The equalities in (2.27) and (2.28) are attained for the function f(z) given by (2.18).

3. Fractional Integral Operator

We need the following definition of fractional integral operator given by Srivastava, Saigo and Owa [10].

Definition 4. For real numbers $\eta > 0$, γ and δ, the fractional operator $I_{0,z}^{\eta,\gamma,\delta}$ is defined by

$$I_{0,z}^{\eta,\gamma,\delta} f(z)=\frac{z^{-\eta-\gamma}}{\Gamma(\eta)}\int_0^z (z-t)^{\eta-1}F(\eta+\gamma,-\delta;\eta;1-\frac{t}{z})f(t)dt \tag{3.1}$$

where f(z) is an analytic function in a simply connected region of the z-plane containing the origin with the order

$$f(z) = O(|z|^{\epsilon}), \quad z \longrightarrow 0,$$

where

$$\epsilon > \max(0, \gamma - \delta) - 1,$$

$$F(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k\ (b)_k}{(c)_k\ (1)_k} z^k , \tag{3.2}$$

where $(\nu)_k$ is the Pochhammer symbol defined by

$$(\nu)_k = \frac{\Gamma(\nu+k)}{\Gamma(\nu)} = \begin{cases} 1 & (k = 0), \\ \nu(\nu+1)\ldots(\nu+k-1) & (k \in N), \end{cases} \tag{3.3}$$

and the multiplicity of $(z-t)^{\eta-1}$ is removed by requiring $\log(z-t)$ to be real when $z-t > 0$.

Remark. For $\gamma = -\eta$, we note that

$$I_{0,z}^{\eta,-\eta,\delta} f(z)=D_z^{-\eta}f(z) .$$

In order to prove our result for the fractional integral operator, we have to recall here the following lemma due to Srivastava, Saigo and Owa [10].

Lemma 2. If $\eta>0$ and $k > \gamma-\delta-1$, then

$$I_{0,z}^{\eta,\gamma,\delta} z^k = \frac{\Gamma(k+1)\Gamma(k-\gamma+\delta+1)}{\Gamma(k-\gamma+1)\Gamma(k+\eta+\delta+1)} z^{k-\gamma} . \tag{3.4}$$

With the aid of Lemma 2, we prove

Theorem 3. Let $\eta > 0$, $\gamma < 2$, $\eta+\delta > -2$, $\gamma-\delta < 2$, $\gamma(\eta + \delta) \le 3\eta$. If the function $f(z)$ defined by (1.6) is in the class $P^*(n,\alpha,\beta)$, then

$$\left| I_{0,z}^{\eta,\gamma,\delta} f(z) \right| \ge \frac{\Gamma(2-\gamma+\delta)|z|^{1-\gamma}}{\Gamma(2-\gamma)\Gamma(2+\eta+\delta)} \left\{ 1 - \frac{\beta(1-\alpha)(2-\gamma+\delta)}{2^{n-2}(1+\beta)(2-\gamma)(2+\eta+\delta)} |z| \right\} \quad (3.5)$$

and

$$\left| I_{0,z}^{\eta,\gamma,\delta} f(z) \right| \le \frac{\Gamma(2-\gamma+\delta)|z|^{1-\gamma}}{\Gamma(2-\gamma)\Gamma(2+\eta+\delta)} \left\{ 1 + \frac{\beta(1-\alpha)(2-\gamma+\delta)}{2^{n-2}(1+\beta)(2-\gamma)(2+\eta+\delta)} |z| \right\} \quad (3.6)$$

for $z \in U_0$, where

$$U_0 = \begin{cases} U & (\gamma \le 1) \\ U-\{0\} & (\gamma > 1). \end{cases}$$

The equalities ik (3.5) and (3.6) are attained by the function $f(z)$ given by (2.18).

Proof. By using Lemma 2, we have

$$I_{0,z}^{\eta,\gamma,\delta} f(z) = \frac{\Gamma(2-\gamma+\delta)}{\Gamma(2-\gamma)\Gamma(2+\eta+\delta)} z^{1-\gamma}$$

$$- \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(k-\gamma+\delta+1)}{\Gamma(k-\gamma+1)\Gamma(k+\eta+\delta+1)} a_k z^{k-\gamma}. \quad (3.7)$$

Letting

$$H(z) = \frac{\Gamma(2-\gamma)\Gamma(2+\eta+\delta)}{\Gamma(2-\gamma+\delta)} z^{\gamma} I_{0,z}^{\eta,\gamma,\delta} f(z)$$

$$= z - \sum_{k=2}^{\infty} h(k) a_k z^k, \quad (3.8)$$

where

$$h(k) = \frac{(2-\gamma+\delta)_{k-1}(1)_k}{(2-\gamma)_{k-1}(2+\eta+\delta)_{k-1}} \quad (k \geq 2), \tag{3.9}$$

we can see that h(k) is non-increasing for integers k (k≥2), and we have

$$0 < h(k) \leq h(2) = \frac{2(2-\gamma+\delta)}{(2-\gamma)(2+\eta+\delta)} . \tag{3.10}$$

Therefore, by using (2.8) (with i= 0) and (3.10), we have

$$|H(z)| \geq |z| - h(2)|z|^2 \sum_{k=2}^{\infty} a_k$$

$$\geq |z| - \frac{\beta(1-\alpha)(2-\gamma+\delta)}{2^{n-2}(1+\beta)(2-\gamma)(2+\eta+\delta)}|z|^2 \tag{3.11}$$

and

$$|H(z)| \leq |z| + h(2)|z|^2 \sum_{k=2}^{\infty} a_k$$

$$\leq |z| + \frac{\beta(1-\alpha)(2-\gamma+\delta)}{2^{n-2}(1+\beta)(2-\gamma)(2+\eta+\delta)} |z|^2. \tag{3.12}$$

This completes the proof of Theorem 3.

Remark. Taking $\gamma = -\eta$ in Theorem 3, we get the result of Corollary 1.

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