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Radial pressure distribution for flow in a circular pipe

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1. Objective of the research.

The objective of this research is to computationally and theoretically solve the problem of Osborne Reynolds and thus to get a physical theory of the transition from laminar to turbulent flow in a circular pipe.

2. Problem of Osborne Reynolds.

First of all we have to define the problem of O. Reynolds. Reynolds found that there were two different kinds of critical values for the velocity in a pipe, (1) the one at which steady motion changed into eddies - color bands experiment, (2) the other at which eddies changed into steady motion - resistance experiment. Accordingly, the problem must be to theoretically get two different critical Reynolds numbers which coincide with the results of Reynolds' experiments in 1883.

(1) From the experiments in glass pipes by means of color bands with three different pipes of (a) 2.68, (b) 1.527, and (c) 0.7886 cm in diameter, Reynolds observed the three critical Reynolds numbers: (a) 12,600, (b) 13,300, and (c) 12,800, respectively. Let Re represent the Reynolds number based on the diameter of the pipe: $\mathrm{Re} = \mathrm{D}u_0\rho/\mu$, where D is the diameter, u_0 is the mean axial velocity, ρ is the density, and μ is the viscosity.

(2) From the experiments to determine the critical velocity by means of resistance in the pipes, Reynolds measured the two real critical Reynolds numbers: (d) 2,028 and (e) 2,077. The results are summarized in Table 1.

Table 1: Reynolds' critical Re

	Pipe diameter	Pipe length	Critical Re	Experiment
a	2.68 cm	137.16 cm	12,600	Color bands
b	1.527	137.16	13,300	Color bands
c	0.7886	137.16	12,800	Color bands
d	0.615	-	2,028	Resistance
e	1.27	-	2,077	Resistance

3. Three kinds of critical Reynolds numbers.

There are three vital Reynolds numbers: a transitional Reynolds number, a critical Reynolds number, and the minimum critical Reynolds number. The first is a Reynolds number at which a transition takes place, and it has a wide range of values for the same inlet and experimental apparatus since transition always occurs above a critical value of Re. The minimum value of transitional Reynolds numbers is defined as the critical Reynolds number. It depends very strongly on the conditions prevailing at the inlet of the pipe, such as the shape of the bellmouth, as well as in the approach to the inlet. As far as is known, there is no upper limit for the critical Reynolds number. Ekman reached values of up to 51,000 with Reynolds' apparatus. If we use another experimental apparatus, we may observe another critical Re. The minimum value of the critical Reynolds number is defined as the minimum critical Reynolds number, and corresponds to Reynolds' "real critical value, 2028 or 2077." Schiller showed experimentally that it is 2320.

4. Entrance region and entrance length.

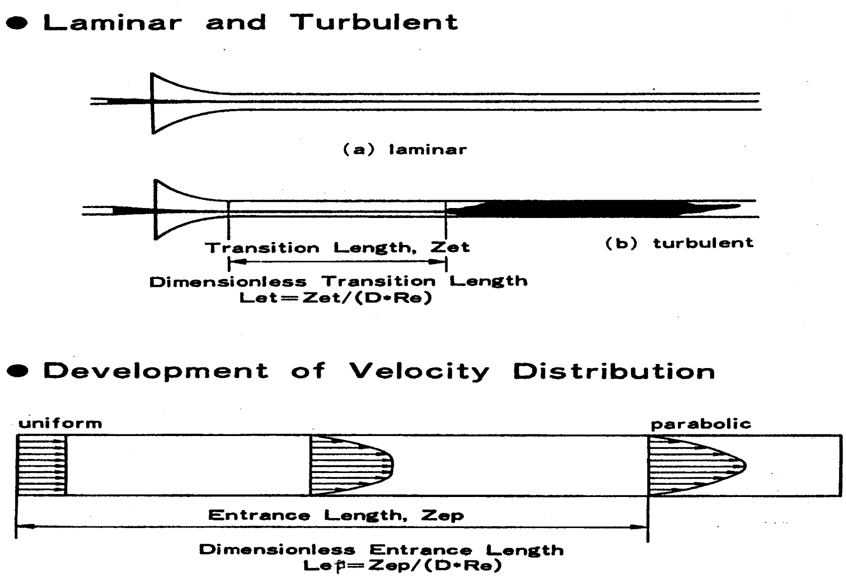

Fig. 1 Entrance length and transition length

The volume transported remains the same for each cross section. As the velocities of layers near the wall of a pipe are retarded, the velocities of inner parts near the central axis must be increased until finally the equiliblium relation between the pressure drop and the shear stress adjusts itself. Accordingly, the initially uniform velocity distribution in the inlet or entry is gradually transformed into a parabolic, Poiseuille, distribution further downstream by the action of viscous forces at the wall. In the entrance region, it is necessary to have a larger pressure drop per unit than is required in the fully developed region since a part of this drop is utilized for accelerating the inside flow and consequently for increasing the kinetic energy of the flow.

The axial pipe region is conveniently divided into three parts: bellmouth, entrance region, and fully developed region. The entrance length, zep, is the distance between the inner end of a bellmouth, i.e. the beginning of a straight circular pipe, and the point at which the velocity profile grows to the fully developed parabolic distribution. The dimensionless entrance length Lep is defined by dividing the entrance length by the product of the diameter and the Reynolds number, as shown in Fig. 1: Lep = zep/(D*Re). Note that Lep is independent of the Reynolds number and approximately from 0.06 to 0.1 as shown in Table 2. Its value depends on how fully the flow develops. Here we take the value of 0.085 as the entrance length of 99.6% developed flow.

5. Transition length.

The transition length, zet, is the distance between the beginning of the straight pipe and the point at which the transition from laminar to turbulent flow occurs. For comparison of the entrance length and the transition length, the same dimensionless unit is strongly desirable. The dimensionless transition length, Let, is also defined in the same way as Lep: Let = zet/(D*Re). Kanda and Oshima observed that Let decreases as Re increases.

Reynolds observed that under no circumstances did the disturbance occur nearer to the bellmouth than about 30 diameters in any of the pipes when he carried out the color bands experiments. In his case of (a) Re = 12,600, the dimensionless transition length is about 0.00238 (Let = 30/12600 = 0.00238). This value is much smaller than the entrance lengths. Table 2 shows the entrance lengths calculated and the transition lengths experienced by several researchers. We can easily conclude, from Table 2, that the transition always occurs in the entrance region, especially near the inlet of the pipe, for Re of more than critical values.

Table 2: Entrance and transition lengths

Researcher	Year	Lep	Developed ratio (%)	Let
Boussinesq	1891	0.065	99	-
Schiller	1922	0.0288		-
Langhaar	1942	0.057	99	-
Shar and London	1978	0.056		-
Mohanty and Asthana	1978	0.075	99.9	-
Kanda and Oshima	1988	0.055	99	-
Kanda and Oshima	1988	0.085	99.6	-
Reynolds	1883	-	-	0.00238
Kanda and Oshima	1988	-	-	0.00234
Kanda and Oshima	1988	-	-	0.00847

6. Reseach methodology.

Consequently from §5, we can consider the occurence of the transition according to the following line of thought:

(1) The transition must take place in the entrance region.

(2) A smooth numerical solution of the Navier-Stokes equations exists regardless of the Reynolds number.

(3) Using the quantities of the above laminar solution, we will try to find a paradox. In this way we can find a couple of parameters; one will be constant and independent of the

Reynolds number like Lep, and the other will vary with the Reynolds number like Let. If the lines of the parameters cross at a certain point, that point may lead to the critical value. Moreover, the two parameters will exist only in the entrance region and disappear in the fully developed region.

(4) In this paper, we will investigate a radial or normal pressure distribution as the force which accelerates the flow in the central core.

7. Dimensionless variables.

We begin with a simple situation in the two dimensional cylindrical coordinates and in the steady state. Let r and z be the radial and axial coordinates, respectively. We use the following dimensionless variables: u and v are the axial and radial components of velocities, p is the pressure, t is the time, and ψ is the stream function, ω is the vorticity.

$$r' = \frac{r}{D}, \quad z' = \frac{z}{D}, \quad z^* = \frac{z}{DRe}, \quad u' = \frac{u}{u_0}, \quad v' = \frac{v}{u_0}, \quad t' = \frac{u_0}{D}t,$$

$$p' = \frac{p}{\frac{1}{2}\rho u_0^2}, \quad \psi' = \frac{\psi}{D^2 u_0}, \quad \omega' = \frac{D}{u_0}\omega$$

We then drop the primes so that the expressions are simplified; z is used for calculation and z^* for presentation in figures.

8. Radial pressure distribution derived from Navier-Stokes equations.

What force accelerates the fluid particles in the central core? First, we should consider whether the radial pressure distribution is positive or negative at the wall. The Navier-Stokes equation for the radial component is written,

$$\frac{\partial p}{\partial r} = \frac{2}{Re}\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}\right) - 2\left(v\frac{\partial v}{\partial r} + u\frac{\partial v}{\partial z}\right) \tag{1}$$

$$\frac{1}{\delta} \qquad\qquad \frac{1}{\delta} \qquad 1 \qquad \delta \qquad \delta \qquad\qquad \delta \qquad \delta$$

where, δ is the boundary layer thickness.

In the fully developed region, the radial pressure distribution is constant throughout any cross section since the radial velocity v is always zero and each term of the previous equation becomes zero.

In the entrance region, a boundary layer similar to that on a flat plate initially forms. The boundary layer grows along the wall and finally fills the pipe. The magnitudes of order of each term will be given by the boundary layer theory. The order of u, p, ∂z, and r is 1, and the order of v and ∂r is δ. The axial and radial velocities, u and v, vanish at the wall. Since δ is small, the radial pressure distribution at the wall is written,

$$\frac{\partial p}{\partial r} = \frac{2}{Re}\frac{\partial^2 v}{\partial r^2} \tag{2}$$

Next we take the relation between the axial pressure distribution and the vorticity on the wall. The vorticity on the wall reduces to

$$\omega = \frac{\partial v}{\partial z} - \frac{\partial u}{\partial r} = -\frac{\partial u}{\partial r} \tag{3}$$

$$\delta \qquad \frac{1}{\delta}$$

By differentiating the vorticity with respect to z,

$$\frac{\partial \omega}{\partial z} = -\frac{\partial^2 u}{\partial r \partial z} \tag{4}$$

By differentiating the continuity equation with respect to r,

$$\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial r} + \frac{v}{r} + \frac{\partial u}{\partial z}\right) = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 u}{\partial r \partial z} = 0 \tag{5}$$

$$\frac{1}{\delta} \qquad 1 \qquad \delta \qquad \frac{1}{\delta}$$

By neglecting the small terms of order δ and 1 in Eq. (5), and from Eq. (4),

$$\frac{\partial^2 v}{\partial r^2} = -\frac{\partial^2 u}{\partial r \partial z} = \frac{\partial \omega}{\partial z} \tag{6}$$

Consequently, from Eqs. (2) and (6), the axial pressure distribution is expressed,

$$\frac{\partial p}{\partial r} = \frac{2}{Re}\frac{\partial^2 v}{\partial r^2} = \frac{2}{Re}\frac{\partial \omega}{\partial z} \tag{7}$$

The fluid next to the wall is forced to remain at rest, so that the velocity rises from zero at the wall to a value of the central core. The velocity gradient is very large. The vorticity on the wall is always positive from Eq. (3) since the derivative of the axial velocity with respect to r is negative. And, the axial velocity near the wall retards downstream until it becomes constant in the fully developed region. Accordingly, the vorticity at the wall decreases monotonically in the entrance region and becomes constant in the fully developed region. This means that the derivative of ω with respect to z, Eq. (4), is negative in the entrance region. Therefore, we must conclude that the radial pressure distribution is negative at the wall and the pressure at the wall is lower than that in the central core.

9. Basic Equations for calculation.

Stream-function, vorticity, pressure, and velocity are calculated with time in cylindrical coordinates. We consider the two-dimensional unsteady flow of an incompressible Newtonian fluid with constant viscosity and density, disregarding gravity and external forces. The set-up is a smooth, straight, circular pipe without a bellmouth or trumpet. The stream-function vorticity equation is

$$\frac{\partial\omega}{\partial t} - \frac{1}{r}\frac{\partial\psi}{\partial z}\frac{\partial\omega}{\partial z} + \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial\omega}{\partial z} + \frac{\omega}{r^2}\frac{\partial\psi}{\partial z} = \frac{1}{Re}\left[\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}(r\omega)\right\} + \frac{\partial^2\omega}{\partial z^2}\right] \tag{8}$$

and the Poisson equation is

$$-\omega = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{\psi}{r}\right) \tag{9}$$

The axial and radial velocities, u and v, are calculated from the derivatives of the stream function by the general method:

$$u = \frac{1}{r}\frac{\partial\psi}{\partial r}, \qquad v = -\frac{1}{r}\frac{\partial\psi}{\partial z} \tag{10}$$

The pressure p is written in the Poisson form,

$$\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial p}{\partial z}\right) = -2\left[r\left\{\left(\frac{\partial v}{\partial r}\right)^2 + 2\frac{\partial u}{\partial r}\frac{\partial v}{\partial z} + \left(\frac{\partial u}{\partial z}\right)^2\right\} + \frac{v^2}{r}\right] \tag{11}$$

10. Calculation method.

The finite differences for both the stream-function vorticity and the pressure drop are calculated by the Gauss-Seidel iterative method. The computational scheme uses the Forward-Time, Centered-Space (FTCS) method and the rectangular mesh system. I0 and J0 in Figs. 2 and 3 are the numbers of axial and radial mesh points, respectively. $\triangle z$:$\triangle r$ is the aspect ratio of axial and radial space increments.

11. Calculation results for radial pressure distribution.

Figs. 2-a - d show the radial pressure distribution for Re = 100 and Figs. 3-a - d show that for Re = 2,000, where the radial distance $r = 0$ is the center of a pipe and $r = 0.5$ is the wall surface of the pipe. The initial and boundary conditions for pressure at inlet $z = 0$ are constantly zero. The major results are:

(1) At the first axial space increment of the mesh system, $z^* = \triangle z^*$, in Figs. 2-a and 3-a, the pressures near the wall are higher than those in the central core, respectively. The pressure in Fig. 2-a increases and thus exceeds the initial value 0 near the wall. This is why the inlet condition strongly affects the pressure distribution nearest the inlet, and the initial kinetic energy is supposed to be converted into hydrostatic pressure.

(2) After the second axial space increment, $z^* = 2\triangle z^*$, the pressures in the central core are higher than those near the wall as previously estimated in §8.

12. Conclusions and discussions.

(1) The transition must take place in the entrance region.

(2) The pressure at the wall is lower than that in the central core in the entrance region. Through Bernoulli's equation we know that a low velocity implies a high pressure and vice versa. However, in fact, Bernoulli's equation cannot be applied to the viscous entrance region and to boundary layers.

(3) The radial pressure gradient is not the force which accelerates the fluid particles in the central core.

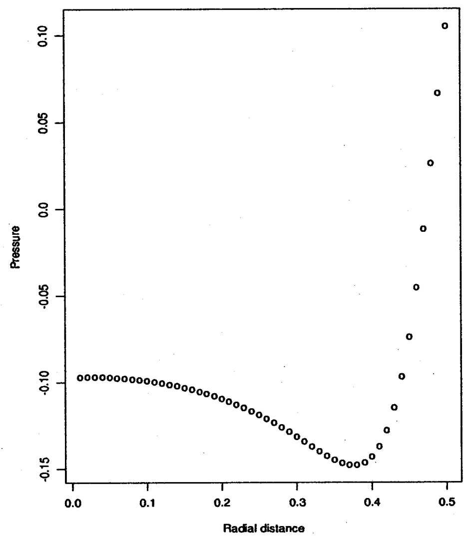

Fig. 2-a, $z^* = \triangle z^* = 0.0005$

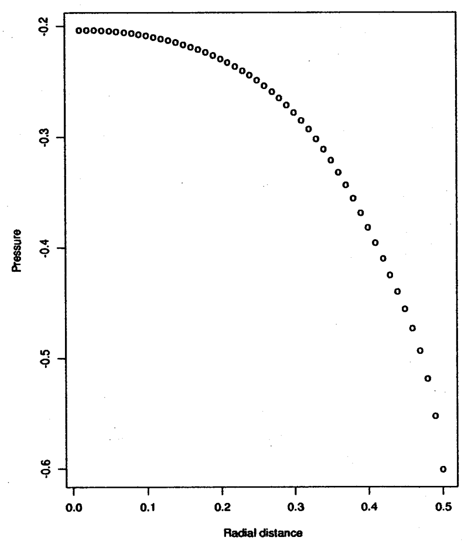

Fig. 2-b, $z^* = 2\triangle z^* = 0.001$

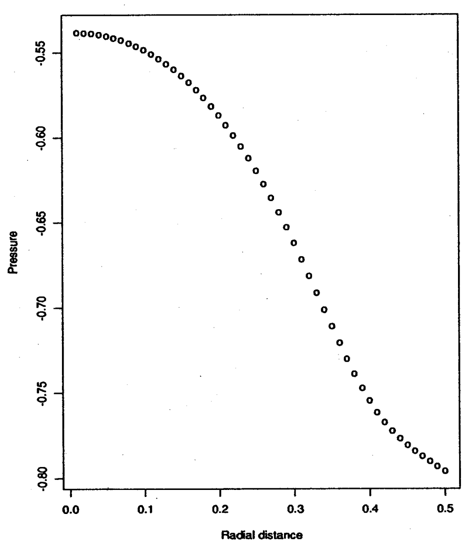

Fig. 2-c, $z^* = 5\triangle z^* = 0.0025$

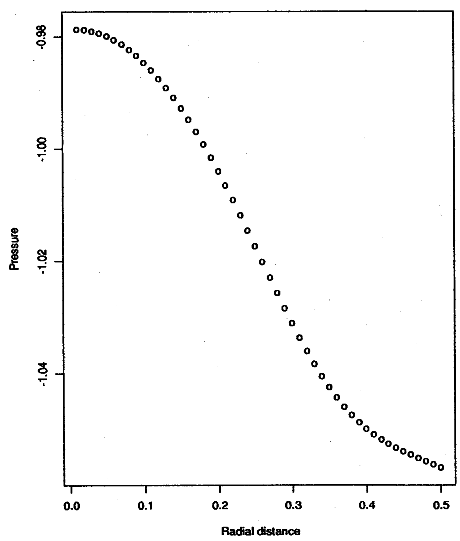

Fig. 2-d, $z^* = 10\triangle z^* = 0.005$

Fig. 2 Radial Pressure Distribution
Re = 100, I0 = 100, J0 = 50, $\triangle$z : $\triangle$r = 5 : 1

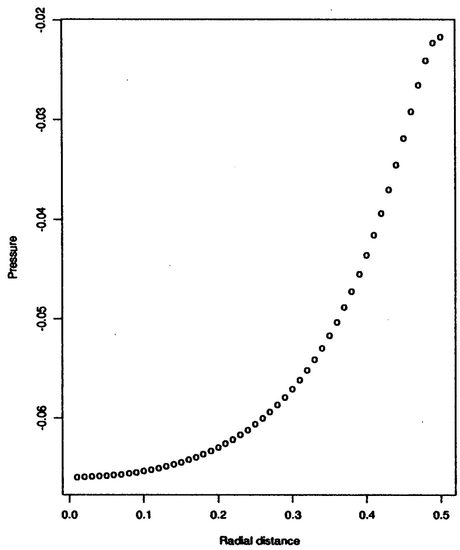

Fig. 3-a, $z^* = \triangle z^* = 0.0001$

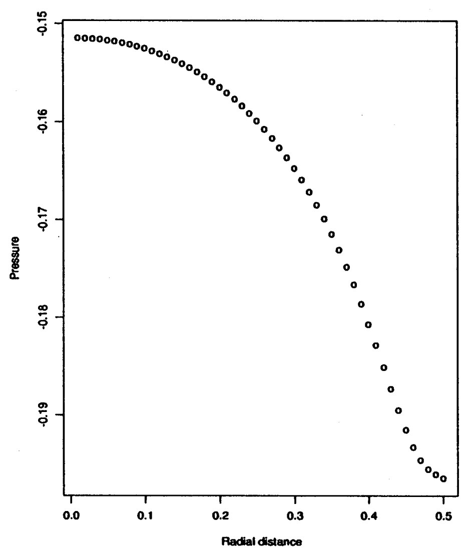

Fig. 3-b, $z^* = 2\triangle z^* = 0.0002$

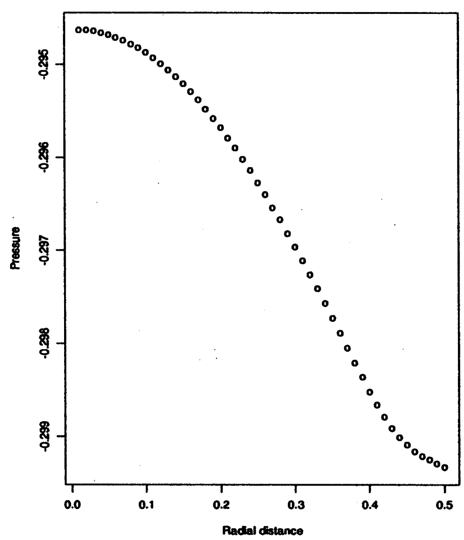

Fig. 3-c, $z^* = 5\triangle z^* = 0.0005$

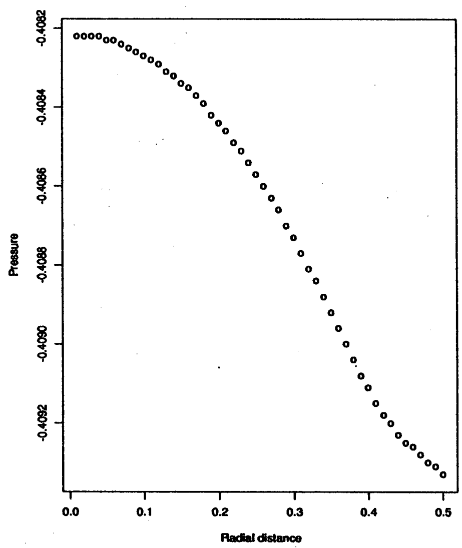

Fig. 3-d, $z^* = 10\triangle z^* = 0.001$

Fig. 3 Radial Pressure Distribution
Re = 2000, I0 = 150, J0 = 50, $\triangle$z : $\triangle$r = 20 : 1