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EIGENVALUE PROBLEM OF EVOLUTION OPERATORS AND DISSIPATION IN CONSERVATIVE SYSTEMS

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§1. Introduction

For certain conservative systems, it has been shown that the exact dissipative modes with phenomenological counterparts are expressed by generalized eigenfunctions of the time evolution operator of distribution functions or wave functions, and the corresponding relaxation rates by the imaginary parts of the generalized eigenvalues. It then turned out that the mathematical basis of the generalized eigenvalue problem can be provided by a dual pair of functional spaces, such as seen in the theory of the rigged Hilbert space, or Gel'fand triple. Here, we will review such an approach using two examples. We begin with some historical remarks about the use of generalized vectors in the problem of dissipation.

Perhaps, the first examples of conservative systems with dissipative time evolution which attracted many researchers' attention are quantum systems involving metastable states/particles. In his pioneering work on the α-decay, Gamow [1] obtained the lifetime of the α-particle trapped by the nucleus from the imaginary part of a complex eigenvalue of the Hamiltonian. The corresponding eigenfunction turned out to be exponentially growing and do not fit with the conventional Hilbert-space framework of quantum mechanics. For the Lee-Friedrichs model, Nakanishi [2] explicitly constructed an eigenfunction of the Hamiltonian with a complex eigenvalue with the aid of a delta function concentrated on a complex number. In 70s, several approaches have been proposed to deal with such eigenfunctions of self-adjoint operators with complex eigenvalues. Combes et al. [3,4] proposed a method known as the complex scaling, where the Hamiltonian is mapped via a similarity transformation to some operator which admits complex eigenvalues. The transformation brings the exponentially growing Gamow's eigenfunction to a square integrable eigenfunction of the transformed Hamiltonian. A similar, but slightly different method is the spectral deformtion method of Sudarshan et al. [5,6], where all physical quantities such as wavefunctions and operators, as functions of the energy variable, are analytically continued to a certain contour in the complex energy plane. These approaches avoid the explicit use of generalized vectors. Bohm et al. [7,8] recognized the possible description of unstable states in terms of the rigged Hilbert space (Gel'fand triple), which was introduced by Gel'fand et al. [9] in connection with the spectral theory of self-adjoint operators and was used to justify the Dirac's formalism of quantum mechanics [10].

The mixing systems in ergod theory have some similarity to the quantum mechanical systems with unstable states. In the former, the expectation value of a given observable with respect to any initial distribution asymptotically tends to the one with respect to the ergodic measure. This convergence could be regarded as a relaxation of the initial distribution to the 'equilibrium' one. For two classes of mixing systems, namely axiom-A systems and expanding maps, Pollicott [11] and Ruelle [12] have shown that the rates of the relaxations can be characterized by the complex poles of the power spectra of the correlation functions, known as Pollicott-Ruelle resonances. Moreover, those complex poles are eigenvalues of the generator of motion for the distribution functions, of which eigenfunctions are represented by generalized functions.

The possibility and the generality of such a characterization of the relaxation rates in terms of the (generalized) eigenvalue problem of the evolution operator have been emphasized and discussed by Prigogine and coworkers [13-17] over the last thirty years in the field of non-equilibrium statistical mechanics. One of the purposes of non-equilibrium statistical mechanics is to explain dissipation on the macroscopic level in terms of conservative microscopic dynamics, and a lot of approaches have been proposed so far [18]. One of the popular approaches is the projection operator method, where 'irrelevant' degrees of freedom are projected out and the closed equation for the 'relevant' part of the distribution function is derived from the Liouville-von Neumann equation. This equation is dissipative, but is not Markovian. Prigogine and his coworkers [14] tried to relate the solution of this equation to Markovian evolutions and reached a formulation where the time evolution of the distribution function is expressed as a superposition of exponentially decaying terms. Later, this method is applied to the Lee-Friedrichs model [16] and turned out to give the same results obtained by Sudarshan et al. [5]. Until very recent, the method has been developed rather formally and did not have enough mathematical justification. Then, it has been recognized that the method may be justified in terms of generalized eigenvalue problem such as seen in the theory of rigged Hilbert spaces [15,17]. The novelty of this approach is that the relaxation is described at the level of the microscopic phase-space dynamics without any modification of the laws of motion.

We review the (generalized) eigenvalue problem associated with the decay in a quantum mechanical model (the Lee-Friedrichs model) in §2, and the diffusive relaxation in a conservative map (the multibaker map) in §3. Sec. 4 is devoted to some concluding remarks.

§2. Dissipative eigenvalue problem of evolution operator I —— decay modes in the Lee-Friedrichs model ——

§§2.1 *A phenomenological approach to the decay of metastable states*

Excited states of atoms and molecules as well as states in a potential well, possessing enough energy to escape, are unstable. Phenomenologically, we know that the population P of occupying such a state decays exponentially in time:

$$\frac{dP}{dt} = -2\gamma P \ , \tag{2.1}$$

where $1/(2\gamma)$ is the lifetime of the state in question. Eq.(2.1) admits a decaying solution

$$P \propto \exp(-2\gamma t) \ . \tag{2.2}$$

Needless to say, the decay (2.2) is an irreversible process.

§§2.2 *Lee-Friedrichs model, a quantum mechanical model of a metastable state*

Friedrichs [19] discussed a slovable quantum mechanical model of resonance scattering, which is referred to as the Lee-Friedrichs model since it is equivalent to the one-particle sector of the Lee model [20]. The eigenvector of the Hamiltonian, of which eigenvalue is the complex pole of the S-matrix, was constructed by Nakanishi [2] with the aid of a delta function concentrated on a complex number. He also gave an argument about the test function space for such distributions. A complete set of left and right eigenvectors including Nakanishi's eigenvector as a part was obtained by Sudarshan, Chiu and Gorini [5] by the approach without the explicit use of distributions. The same result was derived by de Haan [16] by using a method Prigogine et al. [13,14] developed in the field of non-equilibrium statistical mechanics. In order to justfy those eigenstates of the Hamiltonian with complex eigenvalues, Antoniou and Prigogine [15,17] used the rigged Hilbert space of Hardy class functions constructed by Bohm and Gadella [8]. Here we review their result.

For the Lee-Friedrichs model, we consider a Hilbert space expressed as a direct sum of the set of complex numbers and the space of square integrable functions on the positive real axis (which we denote $\mathbf{R}^+$):

$$\mathcal{H} = \left\{ \psi = \begin{pmatrix} \psi_0 \\ \psi(\omega) \end{pmatrix} \middle| \psi_0 \in \mathbf{C} \,, \psi(\omega) \in L^2(\mathbf{R}^+) \right\} , \tag{2.3}$$

equipped with the inner product

$$\langle \psi, \phi \rangle \equiv \psi_0^* \phi_0 + \int_0^\infty d\omega \psi(\omega)^* \phi(\omega) \ . \tag{2.4}$$

And the Hamiltonian is defined as

$$H\psi \equiv \begin{bmatrix} \Omega_0 \psi_0 + \lambda \int_0^\infty d\omega V(\omega)\psi(\omega) \\ \omega\psi(\omega) + \lambda V(\omega)\psi_0 \end{bmatrix} , \tag{2.5}$$

where λ is the coupling constant, $\Omega_0 > 0$ and the real-valued function $V(\omega)$ is at least square integrable on $\mathbf{R}^+$ (its precise specification will be given later by eq.(2.11)).

As a self-adjoint operator on the Hilbert space $\mathcal{H}$, the Hamiltonian H has only a real spectrum and, thus, the time evolution operator $U_t = \exp(-iHt)$ generated by H is unitary. Moreover, for sufficiently small coupling constant λ, the spectrum of H just consists of a continuous part, which coincides with the positive real axis $\mathbf{R}^+$. Hence, the discrete eigenstate of the unperturbed Hamiltonian $H|_{\lambda=0}$ corresponding to the component ψ_0 of $\psi \in \mathcal{H}$ disappears by the interaction, but appears as a resonance in the scattering [19].

Survival probability P_s of the unperturbed discrete eigenstate corresponds to the population P of the decaying state in the phenomenological approach and is given by

$$P_s(t) \equiv |\langle \phi_0, U_t \phi_0 \rangle|^2 \ , \tag{2.6}$$

where $\phi_0 = \binom{1}{0} \in \mathcal{H}$. It is known that the survival probability approximately obeys exponential decay law $\propto \exp(-2\gamma t)$ for not too small or not too large t [21]. Also, it can be shown that the decay law of $P_s(t)$ deviates from the exponential one for very short time and for very long time, because of the Hermiticity of the Hamiltonian and the boundedness of the spectrum of H from below respectively [22].

§§*2.3 Genelarized eigenvalue problem and decaying modes*

Here we introduce subspaces of the Hilbert space $\mathcal{H}$ along the argument of Bohm and Gadella [8], and consider the eigenvalue problem in their conjugate spaces.

We begin with some definitions.

Definition 2.1 (Hardy class functions)

A complex valued function $f(x)$ on the set of real numbers $\mathbf{R}$ is a Hardy class function on the upper half complex plane $\mathbf{C}^+$ if

1. There exists a complex valued function $f(z)$ analytic for Im$z > 0$ and the value $f(x)$ is given as the boundary value of the function $f(z)$:

$$f(x) = \lim_{y \searrow 0} f(x + iy) \ . \tag{2.7a}$$

2. For each fixed $y_0 > 0$, the function $f(x + iy_0)$ is square integrable as a function of x and satisfies

$$\sup_{y>0} \int_{-\infty}^{\infty} dx |f(x+iy)|^2 < +\infty \ . \tag{2.7b}$$

Let $H^2_+(\mathbf{R})$ be a set of all Hardy class functions on $\mathbf{C}^+$. Then, $H^2_+(\mathbf{R})$ is a closed subspace of the Hilbert space $L^2(\mathbf{R})$ of square integrable functions on $\mathbf{R}$ and, hence, is itself a Hilbert space. By changing $\mathbf{C}^+$ to the lower half complex plane $\mathbf{C}^-$, one can also define the space $H^2_-(\mathbf{R})$ of Hardy class functions on $\mathbf{C}^-$. Note that if $f \in H^2_+(\mathbf{R})$, then $f^* \in H^2_-(\mathbf{R})$ since the analytic continuation of f^* is $\tilde{f}(z) = f^*(z^*)$.

Thanks to the theorem of van Winter [8, p.49], any Hardy class function can be reconstructed from its value on the positive real axis $\mathbf{R}^+$. Therefore, if we define operators $\theta_\pm : H^2_\pm(\mathbf{R}) \to L^2(\mathbf{R}^+)$ as the restriction of functions on $\mathbf{R}$ to the positive real axis $\mathbf{R}^+$: $\theta_\pm f(x) = f(x)$ for $x \geq 0$, then they are one-to-one.

Let $\mathcal{S}$ be the space of rapidly decreasing functions defined on $\mathbf{R}$:

$$\mathcal{S} \equiv \{s(x)|s(x)\text{: infinitely differentiable complex} - \text{valued function on } \mathbf{R}$$
$$\lim_{x\to\pm\infty} x^n s^{(m)}(x) = 0 \quad (\text{for } n, m = 0, 1, 2, \cdots)\} \;, \tag{2.8}$$

where the topology is generated, in the standard way, by a countable set of seminorms:

$$\|s\|^2_m = \int_{-\infty}^{\infty} dx |\hat{N}^m s(x)|^2 \quad (m = 0, 1, 2, \cdots), \tag{2.9}$$

with $\hat{N} = x^2 + \frac{d^2}{dx^2} + 1$. And we define delta functions concentrated on complex numbers [2, 8 III of Chap. III, 9 vol.2]*)

Definition 2.2 (Delta function concentrated on a complex number)

1. Test function spaces†)
   Necessary test-function spaces are given by $\theta_\pm\Big(\mathcal{S} \cap H^2_\pm(\mathbf{R})\Big)$ equipped with a countable set of seminorms: $\{\|\theta^{-1}_\pm f\|_m\}$ $(m = 0, 1, \cdots)$. These are nuclear Fréchet spaces and are dense in the Hilbert space $L^2(\mathbf{R}^+)$ [8, Chap.III].
2. Delta function concentrated on z
   Let Im$z > 0$ (Im$z < 0$), and define a delta function δ_z concentrated on z by

$$\delta_z(f) = \int_0^\infty d\omega \delta_z(\omega) f(\omega) \equiv f(z) = \frac{\overset{+}{(-)}1}{2\pi i} \int_{-\infty}^{\infty} dx \frac{\theta^{-1}_\pm f(x)}{x - z} \tag{2.10}$$

   where $f \in \theta_+(\mathcal{S} \cap H^2_+(\mathbf{R}))$ (respectively $f \in \theta_-(\mathcal{S} \cap H^2_-(\mathbf{R}))$). Eq.(2.10) is well-defined as an element of the topological dual of $\theta_+(\mathcal{S} \cap H^2_+(\mathbf{R}))$ (respectively of $\theta_-(\mathcal{S} \cap H^2_-(\mathbf{R}))$).

*) The test function space used in Ref.9, vol. 2 is different from the present one.
†) In [8], these spaces are denoted as $D_\pm$ (i.e., $= \theta_\pm(\mathcal{S} \cap H^2_\pm(\mathbf{R}))$).

Proof of the well-definedness: Let Im$z > 0$, then for $f \in \theta_+(\mathcal{S} \cap H^2_+(\mathbf{R}))$, (2.10) is well-defined. Also, from the last expression of (2.10), we have

$$|\delta_z(f)| \le \frac{1}{2\pi}\sqrt{\int_{-\infty}^{\infty} dx \frac{1}{|x-z|^2}}\, \|\theta_+^{-1} f\|_0 = \frac{1}{\sqrt{4\pi \mathrm{Im} z}} \|\theta_+^{-1} f\|_0 ,$$

which implies the continuity of $f \mapsto \delta_z(f)$ and, thus, δ_z belongs to the topological dual of $\theta_+(\mathcal{S} \cap H^2_+(\mathbf{R}))$. The proof for the case of Im$z < 0$ is the same. *Q.E.D.*

Here we specify the interaction $V(\omega)$ of the Lee-Friedrichs model (2.5):

$$V(\Omega_0) \neq 0 , \qquad V(\omega) \in \theta_+(\mathcal{S} \cap H^2_+(\mathbf{R})) \cap \theta_-(\mathcal{S} \cap H^2_-(\mathbf{R})) . \tag{2.11}$$

The second assumption of (2.11) may give a nontrivial $V(\omega)$ as the set in the right hand side is different from $\{0\}$ [8 eq.(3-41) in p.60, 23].

With these preparations, we define two subspaces $\Phi_\pm \subset \mathcal{H}$ as

$$\Phi_\pm \equiv \left\{ \psi = \begin{pmatrix} \psi_0 \\ \psi(\omega) \end{pmatrix} \middle| \psi(\omega) \in \theta_\pm(\mathcal{S} \cap H^2_\pm(\mathbf{R})) \right\} , \tag{2.12}$$

where the topology is generated by a countable set of seminorms:

$$\|\psi\|_{\Phi_\pm,-1} = |\psi_0| , \qquad \|\psi\|^2_{\Phi_\pm,m} = \sqrt{\int_{-\infty}^{\infty} dx |\hat{N}^m \theta_\pm^{-1} \psi(x)|^2} \quad (m = 0, 1, \cdots) . \tag{2.13}$$

Then, we have

Prop. 2.1

i) The space $\Phi_\pm$ is complete with respect to the topology generated by the countable set of seminorms $\|\cdot\|_{\Phi_\pm,m}$ ($m = -1, 0, 1, \cdots$) and is nuclear, i.e., $\Phi_\pm$ is a nuclear Fréchet space.

ii) The subspace $\Phi_\pm$ is dense in the Hilbert space $\mathcal{H}$ and its topology is stronger than the Hilbert space topology. Thus we have an inclusion: $\Phi_\pm \subset \mathcal{H} \subset \Phi_\pm^\dagger$, where $\Phi_\pm^\dagger$ is the conjugate space to $\Phi_\pm$, i.e., the linear space of continuous anti-linear functionals over $\Phi_\pm$*).

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*) $\ell \in \Phi_\pm^\dagger$ means that $\ell : \Phi_\pm \mapsto \mathbf{C}$ is continuous and $\ell(\alpha\psi + \beta\phi) = \alpha^* \ell(\psi) + \beta \ell^*(\phi)$ for every $\alpha, \beta \in \mathbf{C}$ and $\psi, \phi \in \Phi_\pm$.

iii) For the Hamiltonian H, we have $H\Phi_\pm \subset \Phi_\pm$ and H is continuous there.

Proof. i) & ii) They are the restatements of the corresponding properties of $\theta_\pm(\mathcal{S} \cap H^2_\pm(\mathbf{R}))$ [8, Chap.III].

iii) The Hamiltonian H is decomposed as $H = H_0 + \lambda\hat{V}$:

$$H_0\psi \equiv \begin{bmatrix} \Omega_0\psi_0 \\ \omega\psi(\omega) \end{bmatrix} , \qquad \hat{V}\psi \equiv \begin{bmatrix} \int_0^\infty d\omega V(\omega)\psi(\omega) \\ V(\omega)\psi_0 \end{bmatrix} .$$

The continuity and the invariance of $\Phi_\pm$ with respect to H_0 follow from the corresponding properties of the multiplication operator $\hat{\omega} : f(\omega) \mapsto \omega f(\omega)$ in the spaces $\theta_\pm(\mathcal{S} \cap H^2_\pm(\mathbf{R}))$ [8 Chap.III]. The assumption (2.11) on the interaction $V(\omega)$ guarantees $\hat{V}\Phi_\pm \subset \Phi_\pm$. The continuity of $\hat{V}$ follows from the inequalities:

$$\|\hat{V}\psi\|_{\Phi_\pm,-1} \le \|\theta_\mp^{-1}V\|_0 \; \|\psi\|_{\Phi_\pm,0} \;, \qquad \|\hat{V}\psi\|_{\Phi_\pm,m} = \|\theta_\pm^{-1}V\|_m \; \|\psi\|_{\Phi_\pm,-1} \,.$$

Q.E.D.

From this proposition, the Hamiltonian $H = H^\dagger$ can be continuously extended to the conjugate spaces $\Phi_\pm^\dagger$. Those extensions admit eigenvectors with complex eigenvalues, which control the decay of the metastable state:

Prop. 2.2

i) Let $\eta^+(z)$ be a function analytic on the complex plane except the negative real axis: $\mathbf{C}\backslash\mathbf{R}^-$, which coincides with

$$\eta(z) = z - \Omega_0 + \lambda^2 \int_0^\infty d\omega \frac{V_\omega^2}{\omega - z} \,, \tag{2.14}$$

on the upper half complex plane $z \in \mathbf{C}^+$. Then for sufficiently small coupling constant λ, $\eta^+(z) = 0$ has a unique zero $z = z_R$ with $\mathrm{Im} z_R < 0$ in the domain $\{z \in \mathbf{C} : \mathrm{Im} z > 0, \mathrm{Im} z < -c_1 \text{ or } \mathrm{Re} z > c_2\}$ $(\ni \Omega_0)$ with positive constants c_1, c_2. For this value of z_R, we define anti-linear functionals

$$f_1(\phi) \equiv \frac{1}{\sqrt{\eta^{+\prime}(z_R)}} \left\{ \phi_0^* + \lambda \int_0^\infty d\omega V(\omega) \Big\{ \frac{1}{z_R - \omega} - 2\pi i \delta_{z_R}(\omega) \Big\} \phi^*(\omega) \right\} , \tag{2.15}$$

where $\phi \in \Phi_-$, and

$$\tilde{f}_1(\psi) \equiv \frac{1}{\left[\sqrt{\eta^{+\prime}(z_R)}\right]^*}\left\{\psi_0^* + \lambda \int_0^\infty d\omega V(\omega)\left\{\frac{1}{z_R^* - \omega} + 2\pi i \delta_{z_R^*}(\omega)\right\}\psi^*(\omega)\right\} , \quad (2.16)$$

where $\psi \in \Phi_+$. Then $f_1 \in \Phi_-^\dagger$ and $\tilde{f}_1 \in \Phi_+^\dagger$, and they are (generalized) eigenvectors of the Hamiltonian H with complex eigenvalues

$$H f_1(\psi) \equiv f_1(H^\dagger \phi) = z_R f_1(\phi) , \quad (\text{for } \phi \in \Phi_-) \quad (2.17a)$$

$$H \tilde{f}_1(\psi) \equiv \tilde{f}_1(H^\dagger \psi) = z_R^* \tilde{f}_1(\psi) , \quad (\text{for } \psi \in \Phi_+) . \quad (2.17b)$$

Note that the anti-linear functionals f_1 and $\tilde{f}_1$ precisely correspond to the phenomenological decay modes (2.2).

ii) For any $\psi \in \Phi_-$ and $\phi \in \Phi_+$, we have a decomposition for $t \geq 0$:

$$\langle \phi, e^{-iHt}\psi \rangle = f_1(\phi)\tilde{f}_1^*(\psi)e^{-iz_R t} + w_t(\phi, \psi) , \quad (2.18)$$

where $w_t(\phi, \psi)$ is a sesquilinear form.

Proof: i) As $V(\Omega_0) \neq 0$, there is a positive $c_2 (< \Omega_0)$ such that $V(\omega) \neq 0$ for $|\omega - \Omega_0| \leq \Omega_0 - c_2$. We then choose the coupling constant λ so that $\lambda^2 \|\theta_-^{-1} V\|_0^2 < c_2(\Omega_0 - c_2)$. We have, for Im$z > 0$, Im$\eta^+(z) = $ Im$z\{1 + \lambda^2 \int_0^\infty d\omega \; V(\omega)^2 |\omega - z|^{-2}\} > 0$ and, for $\omega \in \mathbf{R}$ and $|\omega - \Omega_0| \leq \Omega_0 - c_2$, Im$\eta^+(\omega) = \pi\lambda^2 V(\omega)^2 \neq 0$, i.e., for these values of z, $\eta^+(z) \neq 0$.

The other part of the analyticity of $\eta^+(\omega)$ will be proved with the aid of Rouché's theorem [24]. By rotating the integration contour in the defining equation of η from $(0, \infty)$ to $(0, -\infty)$, we obtain the estimation:

$$\left|\eta^+(z) - \left(z - \Omega_0\right)\right| \leq \frac{\lambda^2}{d(z, \mathbf{R}^-)} \|\theta_-^{-1} V\|_0^2 , \quad (2.19)$$

with $d(z, \mathbf{R}^-)$ the distance between z and the negative real axis $\mathbf{R}^-$. Now, for $a \geq \Omega_0$, consider a circle $\Gamma_a \equiv \{z \in \mathbf{C} : |z - a| = a - c_2\}$ which encloses $z = \Omega_0$. As easily seen from (2.19) and the inequality for λ, we have $\left|\eta^+(z) - \left(z - \Omega_0\right)\right| < |z - \Omega_0|$ on $z \in \Gamma_a$. Then, by Rouché's theorem, $\eta^+(z)$ and $z - \Omega_0$ have the same number of zeros inside the circle Γ_a, obviously which is one. Hence, $\eta^+(z)$ has a unique zero $z = z_R$ in the domain $\cup_{a \geq \Omega_0}\{z : |z - a| < a - c_2\} = \{z : \text{Re}z > c_2\}$. Moreover, Im$z_R < 0$, because of $|z_R - \Omega_0| \leq$

$\Omega_0 - c_2$ and the previous observation on $\eta^+(z)$. By applying the same arguments to the family of circles $\{z : |z - ib| = b - c_1\}$ $(b \geq c_1 = \sqrt{(\sqrt{\Omega_0^4 + 4c_2^2(\Omega_0 - c_2)^2} - \Omega_0^2)/2})$, we find $\eta^+(z) \neq 0$ for $z \in \{z : \mathrm{Im}z < -c_1\}$.

As $\phi(\omega) \in \theta_+[\mathcal{S} \cap H_+^2(\mathbf{R}^+)]$ implies $\phi^*(\omega) \in \theta_-[\mathcal{S} \cap H_-^2(\mathbf{R}^+)]$, f_1 is well-defined and

$$|f_1(\phi)| \leq \frac{1}{\sqrt{|\eta^{+\prime}(z_R)|}} \left\{ \|\phi\|_{\Phi_+,-1} + \lambda \left(\sqrt{\int_0^\infty d\omega \frac{V_\omega^2}{|z_R^* - \omega|^2}} + \frac{\sqrt{\pi}|V_{z_R^*}|}{\sqrt{|\mathrm{Im} z_R|}} \right) \|\phi\|_{\Phi_+,0} \right\} ,$$

which implies $f_1 \in \Phi_+^\dagger$.

Also, by a straightforward clculation, we have (2.17a):

$$f_1(H^\dagger \phi) = f_1(H\phi) = z_R f_1(\phi) + \frac{1}{\sqrt{\eta^{+\prime}(z_R)}} \left(-z_R + \Omega_0 + \lambda^2 \int_0^\infty d\omega \frac{V(\omega)^2}{z_R - \omega} - 2\pi i \lambda^2 V_{z_R}^2 \right) \phi_0^*$$
$$= z_R f_1(\phi) - \frac{1}{\sqrt{\eta^{+\prime}(z_R)}} \eta^+(z_R) \phi_0^* = z_R f_1(\phi) \; .$$

The proofs of $\tilde{f}_1 \in \Phi_-^\dagger$ and (2.17b) are the same.

ii) Here we follow the argument of Bohm and Gadella [8]. Let

$$\langle \phi, \varphi_\omega^\pm \rangle \equiv \phi^*(\omega) + \frac{V(\omega)}{\eta^\pm(\omega)} \left[\phi_0^* + \int_0^\infty d\omega' \frac{V(\omega')}{\omega - \omega' \pm i0} \phi^*(\omega') \right] ,$$

where $\varphi_\omega^\pm$ are outgoing $(+)$ and incoming $(-)$ scattering states [19], and

$$\eta^\pm(\omega) = \lim_{\epsilon \searrow 0} \eta(\omega \pm i\epsilon) , \qquad \int_0^\infty d\omega' \frac{V(\omega')}{\omega - \omega' \pm i0} \phi^*(\omega') = \lim_{\epsilon \searrow 0} \int_0^\infty d\omega' \frac{V(\omega')}{\omega - \omega' \pm i\epsilon} \phi^*(\omega') .$$

Then as easily seen, provided $\phi \in \Phi_+$ and $\psi \in \Phi_-$, $\langle \phi, \varphi_\omega^- \rangle$ and $\langle \psi, \varphi_\omega^+ \rangle^*$ have analytic continuation to the lower half ω-complex plane. Moreover, straightforward calculations give

$$\langle \phi, \psi \rangle = \int_0^\infty d\omega \langle \phi, \varphi_\omega^+ \rangle \langle \psi, \varphi_\omega^+ \rangle^* ,$$
$$\langle \phi, \varphi_\omega^+ \rangle = \frac{\eta^-(\omega)}{\eta^+(\omega)} \langle \phi, \varphi_\omega^- \rangle , \qquad \langle e^{-iHt}\psi, \varphi_\omega^+ \rangle^* = e^{-i\omega t} \langle \psi, \varphi_\omega^+ \rangle^* ,$$

where $\eta^-(\omega)/\eta^+(\omega)$ is the S-matrix element [19]. Hence, we have

$$\langle \phi, e^{-iHt}\psi \rangle = \int_0^\infty d\omega \langle \phi, \varphi_\omega^+ \rangle e^{-i\omega t} \langle \psi, \varphi_\omega^+ \rangle^* = \int_0^\infty d\omega \frac{\eta^-(\omega)}{\eta^+(\omega)} e^{-i\omega t} \langle \phi, \varphi_\omega^- \rangle \langle \psi, \varphi_\omega^+ \rangle^* .$$

From i) above and the fact that $e^{-i\omega t}$ $(t > 0)$ is analytic in $\mathbf{C}^-$ and vanishes at infinity, the integrand has a mermorphic extension to the domain $\mathbf{C}^- \backslash \{z \in \mathbf{C} : \mathrm{Im}z \geq -c_1$ and $\mathrm{Re}z <$

$c_2\}$ with a simple pole at $z = z_R$. Thus, by deforming the integration contour from $(0, \infty)$ to $\Gamma = \{z \in \mathbf{R}, 0 < z \le c_1\} \cup \{z = c_1 - iy : 0 \le y < +\infty\}$, we have the desired result:

$$\begin{aligned}\langle \phi, e^{-iHt}\psi\rangle = & -2\pi i \frac{\eta^-(z_R)}{\eta^{+\prime}(z_R)} e^{-iz_R t} \langle \phi, \varphi_\omega^- \rangle\Big|_{\omega=z_R} \langle \psi, \varphi_\omega^+ \rangle^* \Big|_{\omega=z_R} \\ & + \int_\Gamma d\omega \frac{\eta^-(\omega)}{\eta^+(\omega)} e^{-i\omega t} \langle \phi, \varphi_\omega^- \rangle \langle \psi, \varphi_\omega^+ \rangle^* = f_1(\phi) \tilde{f}_1^*(\psi) e^{-iz_R t} + w_t(\phi, \psi) ,\end{aligned}$$

where we have used

$$\langle \phi, \varphi_\omega^- \rangle\Big|_{\omega=z_R} = \frac{V(z_R)}{\eta^-(z_R)} \sqrt{\eta^{+\prime}(z_R)} f_1(\phi) , \qquad \langle \psi, \varphi_\omega^+ \rangle^* \Big|_{\omega=z_R} = \frac{V(z_R)}{\eta^-(z_R)} \sqrt{\eta^{+\prime}(z_R)} \tilde{f}_1(\psi)^* ,$$

and introduced a sesquilinear form:

$$w_t(\phi, \psi) = \int_\Gamma d\omega \frac{\eta^-(\omega)}{\eta^+(\omega)} e^{-i\omega t} \langle \phi, \varphi_\omega^- \rangle \langle \psi, \varphi_\omega^+ \rangle^* .$$

Q.E.D.

Remark 2.1 As firstly shown by Antoniou [15,17, see also 8], the spaces of test vectors are 'asymmetric' with respect to the time evolution.

Prop. 2.3

Let $t > 0$, then $e^{iHt}\Phi_+ \subset \Phi_+$, $e^{-iHt}\Phi_+ \not\subset \Phi_+$, and $e^{-iHt}\Phi_- \subset \Phi_-$, $e^{iHt}\Phi_- \not\subset \Phi_-$.

§3. Dissipative eigenvalue problem of evolution operator II —— diffusive relaxation in the multibaker map ——

§§*3.1 A phenomenological model of diffusion*

Matters like ink in water have a tendency to uniformly distribute over the container. This phenomenon is diffusion, which is a typical irreversible process. As is well known, a simple probabilisitic model of diffusion is provided by the random walk. In the simplest one-dimensional case, a 'particle' moves on a one-dimensional lattice in such a way that, at each time, the 'particle' on a certain site jumps to its adjacent sites with the same probability 1/2. Then the probability $\Pi_t(n)$ of finding the 'particle' at site n ($n = 0, \pm 1, \pm 2, \cdots$) and at time t ($t = 0, 1, 2, \cdots$) obeys

$$\Pi_t(n) = \frac{1}{2}\{\Pi_{t-1}(n+1) + \Pi_{t-1}(n-1)\} . \tag{3.1}$$

Note that the probability $\Pi_t(n)$ can be regarded as the concentration of the diffusing matter. As easily seen, (3.1) admits eigenfunctions $\exp(\pm iqn)$ $(-\pi < q \leq \pi)$ decaying at the rate of $\ln \cos q$:

$$\Pi_t(n) \propto (\cos q)^t \exp(\pm iqn) , \tag{3.2}$$

which expresses the uniformization process of the distribution.

§§3.2 Multi-baker map, a reversible dynamical model

The multibaker map is a reversible dynamical model of the random walk and is defined on a periodic array of countably many unit squares, where each square corresponds to each site in the random walk. The transition of a 'particle' from one square to its neighbors is controlled by a baker map instead of a random process. A 4-adic multibaker map has been proposed by Gaspard [25] and the properties of diffusion and nonequilibrium states have been rigorously studied with the aid of zeta functions and of the "thermodynamic formalism". Multibaker maps admit the Lebesgue measure as an invariant measure and the relaxation of the deviations from this equilibrium state is described by the corresponding Frobenius–Perron operator. The spectral properties of the Frobenius–Perron operator are recently studied by Gaspard [26], Hasegawa and Driebe [27] and Tasaki, Hakmi and Antoniou [28]. The logarithms of its eigenvalues give the decay rates of the correlation functions, which are known as Pollicott–Ruelle resonances [11,12]. Here we discuss about the generalized eigenfunctions associated with Pollicott-Ruelle resonances.

The multibaker map discussed here is defined on a one-dimensional array of unit squares:

$$B(n,x,y) = \begin{cases} \left(n-1,\ 2x,\ \frac{y}{2}\right) , & 0 \leq x < \frac{1}{2} , \\ \left(n+1,\ 2x-1,\ \frac{y+1}{2}\right) , & \frac{1}{2} \leq x < 1 , \end{cases} \tag{3.3}$$

where an integer n labels the unit squares and a pair (x,y) of real numbers $(0 \leq x < 1, 0 \leq y < 1)$ stands for the coordinates in each unit square. This map is conservative so that it admits the Lebesgue measure, $dxdy$, as an invariant measure. The multibaker map B is uniformly hyperbolic with a stretching factor 2 and, thus, possesses a positive Lyapunov exponent equal to $\log 2$.

The evolution operator of the distribution functions (i.e., the Frobenius–Perron operator) U is then given by

$$U\rho(n,x,y) \equiv \rho\big(B^{-1}(n,x,y)\big) = \begin{cases} \rho\Big(n+1,\ \frac{x}{2},\ 2y\Big) \ , & 0 \le y < \frac{1}{2} \ , \\ \rho\Big(n-1,\ \frac{x+1}{2},\ 2y-1\Big) \ , & \frac{1}{2} \le y < 1 \ . \end{cases} \tag{3.4}$$

The evolution operator U is unitary in the Hilbert space $\widetilde{\mathcal{H}}$ of square integrable functions:

$$\widetilde{\mathcal{H}} \equiv \Big\{ f(n,x,y) \Big| \sum_{n=-\infty}^{\infty} \int_{[0,1)^2} dxdy |f(n,x,y)|^2 < +\infty \Big\} \ , \tag{3.5}$$

which is equipped with the inner product: $\langle f,g \rangle \equiv \sum_n \int_{[0,1]^2} dxdy f^*(n,x,y) g(n,x,y)$. Therefore, the spectrum of U on the Hilbert space $\widetilde{\mathcal{H}}$ is a unit circle: $\{z : |z| = 1\}$.

Because of the periodicity of the system, it is convenient to introduce the Fourier representation:

$$\mathcal{F}\rho(q,x,y) \equiv \sum_{n=-\infty}^{+\infty} e^{-inq} \rho(n,x,y) \ , \tag{3.6}$$

where $-\pi < q \le \pi$. Then, the expectation value of the complex conjugate of a given observable A at time t with respect to the initial distribution ρ can be rewritten as

$$\begin{aligned} \langle A^* \rangle_t \equiv \langle A, U^t \rho \rangle &= \sum_{n=-\infty}^{+\infty} \int_{[0,1)^2} dxdy A^*(n,x,y) U^t \rho(n,x,y) \\ &= \int_{-\pi}^{\pi} \frac{dq}{2\pi} \int_{[0,1)^2} dxdy \mathcal{F}A(q,x,y)^* U_q^t \mathcal{F}\rho(q,x,y) \ , \end{aligned} \tag{3.7}$$

where U_q is the Fourier component of the evolution operator given by

$$U_q \mathcal{F}\rho(q,x,y) \equiv \begin{cases} e^{iq} \mathcal{F}\rho\Big(q,\ \frac{x}{2},\ 2y\Big) \ , & 0 \le y < \frac{1}{2} \ , \\ e^{-iq} \mathcal{F}\rho\Big(q,\ \frac{x+1}{2},\ 2y-1\Big) \ , & \frac{1}{2} \le y < 1 \ . \end{cases} \tag{3.8}$$

§§3.3 Genelarized eigenvalue problem and decaying modes

Here we introduce two subspaces of the Hilbert space $\widetilde{\mathcal{H}}$ and consider the eigenvalue problem in their conjugate spaces.

First we define $C_x^2 \subset \widetilde{\mathcal{H}}$ as

$$C_x^2 \equiv \Big\{ f(n,x,y) \;\Big|\; \text{i) for almost every } y,\ \mathcal{F}f(q,x,y) \text{ is two times continuously differentiable in } x \text{ and continuous in } q\ ,$$
$$\text{ii) for fixed } x \text{ and } q, \mathcal{F}f(q,x,y) \text{ is square integrable in } y\ ,$$
$$\text{iii)} \int_0^1 dy \sup_{-\pi\le q\le\pi} \sup_{0\le x\le 1} |\partial_x^j \mathcal{F}f(q,x,y)|^2 < +\infty \qquad (j=0,1,2) \quad \Big\} . \tag{3.9}$$

The space is endowed with the norm

$$\|f\|_{C_x} \equiv \sum_{j=0}^{2} \sqrt{\int_0^1 dy \sup_{-\pi\le q\le\pi} \sup_{0\le x\le 1} |\partial_x^j \mathcal{F}f(q,x,y)|^2}\ . \tag{3.10}$$

The other subspace C_y^2 of twice y-differentiable functions is given by interchanging x and y in the definition of C_x^2 , and is endowed with a norm $\|f\|_{C_y} \equiv \sum_{j=0}^{2} \sqrt{\int_0^1 dx \sup_q \sup_y |\partial_y^j \mathcal{F}f|^2}$. For these spaces, we have

Prop. 3.1

i) The space C_x^2 is a Banach space with respect to the norm $\|\cdot\|_{C_x}$. And thus, it is not nuclear.

ii) The subspace C_x^2 is dense in the Hilbert space $\widetilde{\mathcal{H}}$ and its norm topology is stronger than the Hilbert space topology. Thus we have an inclusion: $C_x^2 \subset \widetilde{\mathcal{H}} \subset C_x^{2\dagger}$, where $C_x^{2\dagger}$ is the space of continuous antilinear functionals on (i.e., the conjugate space to) C_x^2.

iii) The space C_x^2 is invariant with respect to the evolution operator U: $UC_x^2 \subset C_x^2$ and is bounded there: $\|Uf\|_{C_x} \le \|f\|_{C_x}$. But, it is not invariant with respect to $U^\dagger$.

The above statements i) and ii) are valid for C_y^2 and, instead of iii), $U^\dagger C_y^2 \subset C_y^2$ holds where $U^\dagger$ is the adjoint operator of U.

Proof. i) We show that C_x^2 is complete. Let $\{f_n\} \subset C_x^2$ be a Cauchy sequence. Then,

$$\sqrt{\int_0^1 dy \sup_q \sup_x |\partial_x^s(\mathcal{F}f_n(q,x,y) - \mathcal{F}f_m(q,x,y))|^2} \le \|f_n - f_m\|_{C_x} \to 0 \quad (n,m\to\infty)\ . \tag{3.11}$$

In the same way as the proof of the completeness of L^p [29 p.192], one can find a subsequence $\{f_{n_j}\}$ of $\{f_n\}$ which satisfies, for $s = 0, 1$, or 2,

$$\sup_q \sup_x |\partial_x^s(\mathcal{F}f_{n_j}(q,x,y) - \mathcal{F}f_{n_k}(q,x,y))| \to 0 \quad (j,k \to \infty;\ \text{for almost every } y)\ . \tag{3.12}$$

Since the space of functions twice differentiable in x and continuous in q defined on a compact subset $(q,x) \in [-\pi,\pi] \times [0,1]$ of $\mathbf{R}^2$ is complete with respect to the norm $\|f\| = \sum_{s=0}^2 \sup_q \sup_x |\partial_x^s f(q,x)|$, (3.12) implies that, for each y, the sequence $\{\mathcal{F}f_{n_j}(q,x,y)\}$ converges uniformly to a function $g(q,x,y)$ which is two times continuously differentiable in x and continuous in q.

Now we show that $\mathcal{F}^{-1}g \in C_x^2$ and $\|f_n - \mathcal{F}^{-1}g\|_{C_x} \to 0$ for $n \to \infty$. For an arbitrary $\epsilon > 0$, we choose $N \in \mathbf{N}$ such that for $n, m \geq N$, $\|f_n - f_m\|_{C_x} < \epsilon$. Then, for any $m > N$,

$$\begin{aligned}
&\sqrt{\int_0^1 dy |g(q,x,y) - \mathcal{F}f_m(q,x,y)|^2} \\
&\leq \sum_{s=0}^2 \sqrt{\int_0^1 dy \sup_q \sup_x |\partial_x^s(g(q,x,y) - \mathcal{F}f_m(q,x,y))|^2} \\
&= \sum_{s=0}^2 \sqrt{\int_0^1 dy \sup_q \sup_x \liminf_{j\to\infty} |\partial_x^s(\mathcal{F}f_{n_j}(q,x,y) - \mathcal{F}f_m(q,x,y))|^2} \\
&= \sum_{s=0}^2 \sqrt{\int_0^1 dy \liminf_{j\to\infty} \sup_q \sup_x |\partial_x^s(\mathcal{F}f_{n_j}(q,x,y) - \mathcal{F}f_m(q,x,y))|^2} \\
&\leq \sum_{s=0}^2 \sqrt{\liminf_{j\to\infty} \int_0^1 dy \sup_q \sup_x |\partial_x^s(\mathcal{F}f_{n_j}(q,x,y) - \mathcal{F}f_m(q,x,y))|^2} \\
&= \liminf_{j\to\infty} \|f_{n_j} - f_m\|_{C_x} \leq \epsilon\ ,
\end{aligned} \tag{3.13}$$

where we have used the uniformity in x and q of the convergence of $\{\mathcal{F}f_{n_j}\}$ in the second equality, Fatou's lemma [29, p.172] in the second inequality and the fact $\|f_{n_j} - f_m\|_{C_x} < \epsilon$ for all j satisfying $n_j > N$, in the third inequality.

Inequality (3.13) shows the square integrability of g for fixed x, q and, as g is continuous in x and q, its square integrability with respect to x, y and q. The latter implies the well-definedness of $\mathcal{F}^{-1}g$. As stated above, $\mathcal{F}\mathcal{F}^{-1}g = g$ has the desired smoothness in x and q. Moreover, (3.13) shows $\|\mathcal{F}^{-1}g - f_m\|_{C_x} \leq \epsilon$ for $m > N$, which means that $\mathcal{F}^{-1}g \in C_x^2$ and $\{f_n\}$ converges to $\mathcal{F}^{-1}g$ in C_x^2-norm. Therefore, C_x^2 is a Banach space. As C_x^2 is of infinite dimension, it is not nuclear [30, ch. 50-12].

ii) Because of the denseness of the space of continuous functions in the Hilbert space of square integrable functions [29 p.197, Theorem 13.21] and the Stone-Weierstrass theorem of polynomial approximation of continuous functions [29 p.95, Theorem 7.30], the space

$$\mathcal{P} \equiv \{ f \in \widetilde{\mathcal{H}} | \text{ for each } l, f(l,x,y) \text{ is a polynomial of } x \text{ and } y, \\ \text{and for } l < -M \text{ or } l > N, \ f(l,x,y) = 0 \ . \qquad \}$$

is dense in the Hilbert space $\widetilde{\mathcal{H}}$. As C_x^2 contains the set $\mathcal{P}$, it is also dense in $\widetilde{\mathcal{H}}$. Moreover, as a result of the unitarity of the Fourier transformation $\mathcal{F}$, $\langle f, f \rangle = \int \frac{dq}{2\pi} dx dy |\mathcal{F}f(q,x,y)|^2 \leq \|f\|_{C_x}{}^2$ for $f \in C_x^2$ and, thus, the topology of C_x^2 is stronger than that of $\widetilde{\mathcal{H}}$.

iii) The twice continuous differentiability of $\mathcal{F}Uf(q,x,y)$ in x and the continuity in q immediately follow from $\mathcal{F}Uf(q,x,y) = U_q \mathcal{F}f(q,x,y)$ and the definition (3.8) of U_q. As easily seen from (3.8), we also have

$$\sqrt{\int_0^1 dy |\mathcal{F}Uf(q,x,y)|^2} \leq \sum_{j=0}^{2} \sqrt{\int_0^1 dy \sup_q \sup_x |\partial_x^j \mathcal{F}Uf(q,x,y)|^2}$$
$$= \sum_{j=0}^{2} \sqrt{\frac{1}{2^j} \int_0^1 dy \sup_q \sup_x |U_q \partial_x^j \mathcal{F}f(q,x,y)|^2} \leq \sum_{j=0}^{2} \sqrt{\frac{1}{2^j} \int_0^1 dy \sup_q \sup_x |\partial_x^j \mathcal{F}f(q,x,y)|^2} \leq \|f\|_{C_x}$$

which implies the square integrability of $\mathcal{F}Uf(q,x,y)$ with respect to y for fixed x, q as well as $Uf \in C_x^2$. Thus, $UC_x^2 \subset C_x^2$. The above inequality also shows $\|Uf\|_{C_x} \leq \|f\|_{C_x}$. $U^\dagger C_x^2 \not\subset C_x^2$ holds since, in general, $U^\dagger$ introduces a discontinuity at $x = 1/2$.
The proof for C_y^2 is the same as above. *Q.E.D.*

From this proposition, the adjoint $U^\dagger$ of the evolution operator can be continuously extended to the conjugate space $C_x^{2\dagger}$ and U to $C_y^{2\dagger}$. These extensions admit decaying eigenfunctions, which control the decay of expectation values. More precisely, we have the following proposition, which is the main result of this section

Prop. 3.2

i) Suppose the observable $A \in C_y^2$ and the initial distribution $\rho \in C_x^2$. Then there exist antilinear functionals F_{0q}, F_{1q}^a, $F_{1q}^b \in C_y^{2\dagger}$ and $\tilde{F}_{0q}$, $\tilde{F}_{1q}^a$, $\tilde{F}_{1q}^b \in C_x^{2\dagger}$, which are principal vectors of the extensions (i.e., generalized principal vectors) of U and $U^\dagger$ respectively:

$$\begin{cases} UF_{0q}(A) \equiv F_{0q}(U^\dagger A) = \cos q \ F_{0q}(A) \ , \\ UF^a_{1q}(A) = \frac{\cos q}{2} F^a_{1q}(A) + \frac{1}{16\cos q} F^b_{1q}(A) \ , \\ UF^b_{1q}(A) = \frac{\cos q}{2} F^b_{1q}(A) \ , \end{cases} \tag{3.14}$$

$$\begin{cases} U^\dagger \tilde{F}_{0q}(\rho) \equiv \tilde{F}_{0q}(U\rho) = \cos q \ \tilde{F}_{0q}(\rho) \ , \\ U^\dagger \tilde{F}^a_{1q}(\rho) = \frac{\cos q}{2} \tilde{F}^a_{1q}(\rho) \ , \\ U^\dagger \tilde{F}^b_{1q}(\rho) = \frac{\cos q}{2} \tilde{F}^b_{1q}(\rho) + \frac{1}{16\cos q} \tilde{F}^a_{1q}(\rho) \ . \end{cases} \tag{3.15}$$

The concrete forms of the functionals are given below. Note that the functionals F_{0q} and $\tilde{F}_{0q}$ precisely correspond to the phenomenological decay modes (3.2).

ii) The time evolution of the expectation value of A at time t is given by

$$\langle A, U^t \rho \rangle = \int_{|\cos q|>1/4} \frac{dq}{2\pi} \cos^t q \ F_{0q}(A) \tilde{F}_{0q}(\rho)^*$$
$$+ \int_{|\cos q|>1/2} \frac{dq}{2\pi} \left(\frac{\cos q}{2}\right)^t \left[F^a_{1q}(A)\tilde{F}^a_{1q}(\rho)^* + F^b_{1q}(A)\tilde{F}^b_{1q}(\rho)^* + \frac{t}{8\cos^2 q} F^b_{1q}(A) \tilde{F}^a_{1q}(\rho)^* \right]$$
$$+ W_t(A,\rho) \ . \tag{3.16a}$$

In (3.16a), $W_t(A,\rho)$ is a sesquilinear form satisfying

$$|W_t(A,\rho)| \le \frac{1}{4^t}\{K_3 t^3 + K_2 t^2 + K_1 t + K_0\} \ , \tag{3.16b}$$

where K_j's are positive constants depending on A and ρ.

Before going to the proof of the above proposition, we describe the antilinear functionals F_{0q}, F^a_{1q}, F^b_{1q} and $\tilde{F}_{0q}$, $\tilde{F}^a_{1q}$, $\tilde{F}^b_{1q}$. For that purpose, first we introduce continuous functions G_q, G^1_q (where $|\cos q| > 1/2$) and $\bar{G}_q$ (where $1/2 \ge |\cos q| > 1/4$), which are defined respectively as unique solutions of functional equations of de Rham type [31,32,33]:

$$G_q(x) = \begin{cases} \frac{e^{iq}}{2\cos q} G_q(2x) \ , & 0 \le x \le 1/2 \\ \frac{e^{-iq}}{2\cos q} G_q(2x-1) + \frac{e^{iq}}{2\cos q} \ , & 1/2 \le x \le 1 \end{cases} \tag{3.17a}$$

$$G^1_q(x) = \begin{cases} \frac{e^{iq}}{2\cos q} G^1_q(2x) + \frac{G_q(x)}{8\cos^2 q} - \frac{e^{-iq}}{2\cos q} \int_0^x dx' G_q(x') \ , & 0 \le x \le 1/2 \\ \frac{e^{-iq}}{2\cos q} G^1_q(2x-1) + \frac{G_q(x)-G_q(1/2)}{8\cos^2 q} - \frac{e^{iq}}{2\cos q} \int_{1/2}^x dx' [1 - G_q(x')] \ , & 1/2 \le x \le 1 \end{cases} \tag{3.17b}$$

and

$$\bar{G}_q(x) = \begin{cases} \frac{e^{iq}}{4\cos q}\bar{G}_q(2x) , & 0 \le x \le 1/2 \\ \frac{e^{-iq}}{4\cos q}\bar{G}_q(2x-1) + \frac{e^{iq}}{2\cos q}x - \frac{1}{8\cos^2 q} . & 1/2 \le x \le 1 \end{cases} \tag{3.17c}$$

The uniqueness and the continuity of the solutions of the above equations can be shown easily [32,33]: Firstly, we observe that the right hand sides of eqs.(3.17a-c), by regarding G_q, G_q^1 and $\bar{G}_q$ respectively as operands, define operators in the space of bounded functions, which are contractive and preserve continuity. Then, these opeartors admit unique continuous fixed points, which are, respectively, G_q, G_q^1 and $\bar{G}_q$. These functions are not necessarily regular. Indeed, the function G_q is of infinite variation and has a fractal graph [33].

In terms of these functions, antilinear functionals F_{0q}, F_{1q}^a, and F_{1q}^b are given by

$$F_{0q}(A) \equiv \begin{cases} \int_{[0,1)^2} dx dG_q(y) \mathcal{F}A(q,x,y)^* & \text{(for } |\cos q|>1/2) \\ \int_0^1 dx \mathcal{F}A(q,x,1)^* - \int_{[0,1)^2} dx d\bar{G}_q(y) \partial_y \mathcal{F}A(q,x,y)^* & \text{(for } 1/2 \ge |\cos q|>1/4) \end{cases} , \tag{3.18a}$$

$$F_{1q}^a(A) \equiv \int_{[0,1)^2} dx dG_q(y) \Big(x - \frac{e^{-iq}}{2\cos q}\Big) \mathcal{F}A(q,x,y)^* - \int_{[0,1)^2} dx dG_q^1(y) \partial_y \mathcal{F}A(q,x,y)^* , \tag{3.18b}$$

$$F_{1q}^b(A) \equiv \int_{[0,1)^2} dx dG_q(y) \partial_y \mathcal{F}A(q,x,y)^* , \tag{3.18c}$$

where the integrals with respect to y are the Riemann-Stieltjes ones, which are well-defined [34] since the functions G_q, G_q^1 and $\bar{G}_q$ are continuous and the integrands are of finite variation with respect to y as a result of $A \in C_y^2$. Similarly, we have

$$\tilde{F}_{0q}(\rho) \equiv \begin{cases} \int_{[0,1)^2} dG_q^*(x) dy \mathcal{F}\rho(q,x,y)^* & \text{(for } |\cos q|>1/2) \\ \int_0^1 dy \mathcal{F}\rho(q,1,y)^* - \int_{[0,1)^2} d\bar{G}_q^*(x) dy \partial_x \mathcal{F}\rho(q,x,y)^* & \text{(for } 1/2 \ge |\cos q|>1/4) \end{cases} , \tag{3.19a}$$

$$\tilde{F}_{1q}^a(\rho) \equiv \int_{[0,1)^2} dG_q^*(x) dy \partial_x \mathcal{F}\rho(q,x,y)^* , \tag{3.19b}$$

$$\tilde{F}_{1q}^b(\rho) \equiv \int_{[0,1)^2} dG_q^*(x) dy \Big(y - \frac{e^{iq}}{2\cos q}\Big) \mathcal{F}\rho(q,x,y)^* - \int_{[0,1)^2} dG_q^{1*}(x) dy \partial_x \mathcal{F}\rho(q,x,y)^* , \tag{3.19c}$$

where the integrals with respect to x are again the Riemann-Stieltjes ones.

Proof of Proposition 3.2 i) We consider the functional F^b_{1q}. As $\mathcal{F}A(q,x,y)$ is continuous in q, the value $F^b_{1q}(A)$ is well-defined for each q. With the aid of integration by parts [34] and $G_q(1)=1$,

$$F^b_{1q}(A) = \int_0^1 dx\partial_y\mathcal{F}A(q,x,1)^* - \int_{[0,1)^2} dxdyG_q(y)\partial_y^2\mathcal{F}A(q,x,y)^* ,$$

which leads to

$$|F^b_{1q}(A)| \le \{1+\sup_{0\le y\le 1}|G_q(y)|\} \ \|A\|_{C_y} ,$$

and, thus, to $F^b_{1q}\in C_y^{2\dagger}$.

The last equation of (3.14) can be shown easily: Indeed, because $\partial_y\mathcal{F}U^\dagger A(q,x,y) = (1/2)U_q^\dagger\partial_y\mathcal{F}A(q,x,y)$, $e^{iq}dG_q(2y) = 2\cos q dG_q(y)$ and $e^{-iq}dG_q(2y-1) = 2\cos q dG_q(y)$ (cf. eq.(3.17a)), we obtain the desired relation:

$$\begin{aligned} F^b_{1q}(U^\dagger A) &= \frac{1}{2}\int_{[0,1)^2} dxdG_q(y)\Big\{e^{iq}\bar\theta(x)\hat A_y(q,2x,\tfrac{y}{2})^* + e^{-iq}[1-\bar\theta(x)]\hat A_y(q,2x-1,\tfrac{y+1}{2})^*\Big\} \\ &= \frac{1}{4}\int_0^1 dx\int_0^{1/2} e^{iq}dG_q(2y)\hat A_y(q,x,y)^* + \frac{1}{4}\int_0^1 dx\int_{1/2}^1 e^{-iq}dG_q(2y-1)\hat A_y(q,x,y)^* \\ &= \frac{\cos q}{2}\int_{[0,1)^2} dxdG_q(y)\hat A_y(q,x,y)^* = \frac{\cos q}{2}F^b_{1q}(A) , \end{aligned}$$

where $\hat A_y$ is the abbreviation of $\partial_y\mathcal{F}A$ and $\bar\theta$ is the function defined as $\bar\theta(x) = 1$ for $x\le 1/2$ and $=0$ otherwise.

The proofs for the other antilinear functionals are the same as above.

ii) The key idea of deriving (3.16) is to convert the weak convergence of $U^t\rho$ into the uniform one. This is realized by considering an integrated distribution function: $\mathcal{G}_t = \int_0^y dy'\int_0^{y'} dy''U^t\rho(n,x,y'')$, which converges uniformly for $t\to\infty$. The deviation $\mathcal{G}_t - \mathcal{G}_\infty$ consists of terms which decay exponentially at different rates and provide generalized eigenfunctions. The details of the calculations will be discussed elsewhere [35]. *Q.E.D.*

Remark 3.1: The expression (3.16a) may be regarded as a special case of the Pollicott-Ruelle theorem [11,12], which is valid for axiom A systems. But, the setting of the functional spaces is different (in the former, the spaces of Hölder continuous functions are used).

Remark 3.2: At first sight, it seems that the decay property as expressed by (3.16a) is the operator property of U restricted to the subspace C_x^2. However, it is not the case and (3.16a) is the property of a triple (C_y^2, C_x^2, U). Indeed, for the operator U restricted to C_x^2, we have

Prop. 3.3

The spectral set $\sigma(U|_{C_x^2})$ of U restricted to the space C_x^2 satisfies

$$\{z : 1/4 < |z| < 1\} \subset \sigma(U|_{C_x^2}) \subset \{z : |z| \leq 1\} . \tag{3.20}$$

Proof From Prop. 3.1, $\|U\|_{C_x} \leq 1$ and, then, the spectral radius formula [36, Theorem VI.6] gives (spectral radius of $U|_{C_x^2}$) $= \lim_{n\to\infty} \|U^n\|_{C_x}{}^{1/n} \leq 1$, which implies the second inclusion. To show the first inclusion, we use the lemma

Lemma

Let $U : X \to X$ be a bounded operator on a Banach space X and suppose, for $\lambda \in \mathbf{C}$, there exists an element $y^*(\neq 0)$ of the conjugate space $X^\dagger$ such that $y^*(Ux) = \lambda y^*(x)$ holds for any $x \in X$. Then λ is the spectrum of U: $\lambda \in \sigma(U)$.

To show the first inclusion, we construct $h_z \in C_x^{2\dagger}$ satifying $h_z(U\rho) = zh_z(\rho)$ ($\rho \in C_x^2$) for an arbitrary $z \in \{z : 1/4 < |z| < 1\}$. We set

$$h_z(\rho) = \sum_{n=1}^{\infty} z^n \int_{[0,1]^2} dxdy h_0(x) \left[U_q^{-n} \mathcal{F}\rho(q,x,y)\right]^* + \sum_{n=0}^{\infty} \left(\frac{1}{4z}\right)^n \int_{[0,1]^2} dxdy h_0(x) \left[J_x U_q^n \partial_x^2 \mathcal{F}\rho(q,x,y)\right]^* , \tag{3.21}$$

where $h_0(x)$ is a continuous function to be determined and the operator J_x is defined by $J_x f(q,x,y) \equiv \int_0^x dx' \int_0^{x'} dx'' f(q,x'',y)$. Each term of (3.21) is well-defined for $\rho \in C_x^2$ and, for any $z \in \{z : 1/4 < |z| < 1\}$, we have $h_z \in C_x^{2\dagger}$ because (3.21) converges absolutely and

$$|h_z(\rho)| \leq \sqrt{\int_0^1 dx |h_0(x)|^2} \ \left[\frac{|z|}{1-|z|} + \frac{2\sqrt{2}|z|}{4|z|-1}\right] \ \|\rho\|_{C_x}.$$

Now we consider $h_z(U\rho)$. Because of $\mathcal{F}U\rho(q,x,y) = U_q\mathcal{F}\rho(q,x,y)$, we have

$$h_z(U\rho) - zh_z(\rho) = \int_0^1 dy\Big\{\mathcal{F}\rho(q,0,y)^* \int_0^1 dx h_0(x) + \partial_x\mathcal{F}\rho(q,0,y)^* \int_0^1 dx x h_0(x)\Big\} .$$

Hence, provided

$$\int_0^1 dx h_0(x) = 0 , \qquad \int_0^1 dx x h_0(x) = 0 ,$$

(e.g., $h_0(x) = x^2 - x + 1/6$), h_z becomes the desired element of the conjugate space: $h_z(U\rho) = zh_z(\rho)$. *Q.E.D.*

§4. Conclusion

We have rigorously constructed decay modes, which have phenomenological counter parts, as generalized eigenfunctionals of the evolution operator for the Lee-Friedrichs model and the multibaker map. The generalized eigenfunctionals are defined as eigenvectors of the extension of the evolution operator to the conjugate space of a certain 'test function' space. This result suggests the possibility of a purely dynamical characterization of relaxation prossesses for (a certain class of) conservative systems *without any approximation.*

One of the problems related to the problem of dissipation in conservative systems is the problem of understanding macroscopic irreversibility based on a reversible microscopic laws of dynamics. The Lee-Friedrichs model and the multibaker map are not only conservative, but also reversible. In the previous sections, we have constructed decaying eigenmodes which represent irreversible changes and, thus, our results give some insights to the solution of this problem. Here, we point out one of such aspects resulting from the asymmetric time evolution of test vector spaces. As we have seen, the 'test vector' space $\tilde{\Phi}_+$ for the right eigenvectors is invariant under the adjoint $U_t^\dagger$ of the forward time evolution $t > 0$, but not invariant under the adjoint $U_t^\dagger = U_{|t|}$ of the backward time evolution $t < 0$ (cf. Prop. 2.3 for the Lee-Friedrichs model and iii) of Prop. 3.1 for the multibaker map). As a result, only the forward evolution operator U_t $(t > 0)$ can be defined on the conjugate space $\tilde{\Phi}_+^\dagger$. In other words, the family $\{U_t\}$ of time evolution operators, which is a group (i.e., reversible) on the Hilbert space, becomes a forward semi-group (i.e., irreversible) on the extended space $\tilde{\Phi}_+^\dagger$. Similarly, on the conjugate space $\tilde{\Phi}_-^\dagger$ of 'test vectors' for the left eigenvectors, the family $\{U_t\}$ becomes a backward semi-group since only the backward evolution U_t $(t < 0)$ can be defined there.

There is an objection [37] to the use of formalisms like rigged Hilbert spaces to describe resonances since such formalisms may allow all complex numbers to be eigenvalues of a given operator [see also Prop. 3.3]. However, at least in the models described here, there are appropriate generalized eigenvectors which have phenomenological counter parts and correctly describe the dissipative phenomena. So we believe that the approach explained in this article will provide an interesting perspective to the problem of dissipation and irreversibility in conservative/reversible systems.

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