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Quantum Stochastic Calculus と物理

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1 Introduction

We will show how the formulation of Non-Equilibrium Thermo Field Dynamics (NETFD) [1]-[8], a *canonical formalism* of quantum systems in far-from-equilibrium state, can be generalized further to take into account *quantum stochastic calculus*.

It will be done by extending the quantum stochastic calculus provided by mathematicians, i.e., Hudson, Parthasarathy and Lindsay [9]-[13] to the one in the thermal space within NETFD.

Then, the framework of NETFD has come to provide us with a unified viewpoint covering whole the aspects in non-equilibrium statistical mechanics, i.e. the Boltzmann, the Fokker-Planck, the Langevin and the stochastic Liouville equations (see Fig. 1).

In the extension of the quantum stochastic calculus, we will see a possible realization of the concept, i.e. the *spontanious creation of dissipation*, proposed in [14, 15].

2 System of Classical Stochastic Differential Equations

Let us remember here the system of stochastic differential equations for classical systems.

The stochastic Liouville equation is given by [16]

$$\frac{\partial}{\partial t}f(u,t)=\Omega(u,t)f(u,t),\qquad \Omega(u,t)=-\frac{\partial}{\partial u}\dot{u}, \tag{1}$$

with the initial condition $f(u,0)=P(u,0)$. The flow $\dot{u}$ in the velocity space is given, for example, by

$$\dot{u}=-\gamma u+\frac{1}{m}R(t). \tag{2}$$

The random force $R(t)$ is specified by

$$\big\langle R(t) \big\rangle = 0, \qquad \big\langle R(t_1)R(t_2) \big\rangle = 2m\gamma T\delta(t_1 - t_2), \tag{3}$$

where T represents the temperature of environment. The Boltzmann constant has been put to equal to unity.

The Langevin equation of the system has the same structure as the flow equation (2), i.e.

$$\dot{u}(t) = -\gamma u(t) + \frac{1}{m}R(t). \tag{4}$$

In precise, the stochastic distribution function $f(u,t)$ means that

$$f(u,t) = f(u,t; \Omega(u,t), P(u,0)). \tag{5}$$

Averaging over all possibilities of $\Omega(u,t)$, we have the usual distribution function $P(u,t)$ as

$$P(u,t) = \big\langle f(u,t; \Omega(u,t), P(u,0)) \big\rangle, \tag{6}$$

which satisfies the Fokker-Planck equation

$$\frac{\partial}{\partial t}P(u,t) = \frac{\partial}{\partial u}\gamma \left(u + \frac{T}{m}\frac{\partial}{\partial u} \right) P(u,t). \tag{7}$$

We see that it has a stationary solution

$$P_{eq}(u) = C \exp\left(-\frac{m}{2T}u^2 \right). \tag{8}$$

Note that $f(u,t)$ satisfies the conservation of probability within the velocity space:

$$\int du\ f(u,t) = \text{constant} = 1, \tag{9}$$

which can be seen from (1). Note also that the Langevin equation (4) does *not* contain the diffusion term. This is a Stratonovich type stochastic differential equation [17]. One can proceed calculation as if the stochastic function $u(t)$ were an analytic one. The fluctuation-dissipation theorem of the second kind is introduced in order that the Langevin equation (4) is consistent with the Fokker-Planck equation (7).

3 Liouville Equation

The most basic characteristics of the Liouville equation

$$\frac{\partial}{\partial t}\rho(t) = -iL\rho(t), \tag{10}$$

may be summarized as follows:

D1. The hermiticy of the Liouville operator iL: $(iL)^\dagger = iL$.

D2. The conservation of probability (tr $\rho = 1$): tr $LX = 0$.

D3. The hermiticy of the density operator: $\rho^\dagger(t) = \rho(t)$.

The expectation value of an observable A is given by $\langle A\rangle_t = \mathrm{tr}\ A\rho(t)$. Substitution of the formal solution $\rho(t) = \mathrm{e}^{-iLt}\rho(0)$ gives us a Heisenberg operator

$$A(t) = \mathrm{e}^{iLt}A\mathrm{e}^{-iLt}, \tag{11}$$

through the process $\langle A\rangle_t = \mathrm{tr}\ A\mathrm{e}^{-iLt}\rho = \mathrm{tr}\ \mathrm{e}^{iLt}A\mathrm{e}^{-iLt}\rho = \mathrm{tr}\ A(t)\rho$. The Heisenberg operator satisfies the Heisenberg equation

$$\frac{dA(t)}{dt} = i[L,\ A(t)]. \tag{12}$$

4 Technical Basics of NETFD

The minimal technical tools needed for NETFD is the following.

Tool 1. Any operator A in NETFD is accompanied by its partner (tilde) operator $\tilde{A}$. The tilde conjugation $\sim$ is defined by $(A_1A_2)^\sim = \tilde{A}_1\tilde{A}_2$, $(c_1A_1 + c_2A_2)^\sim = c_1^*\tilde{A}_1 + c_2^*\tilde{A}_2$, $(\tilde{A})^\sim = A$, and $(A^\dagger)^\sim = \tilde{A}^\dagger$.

Tool 2. The tilde and non-tilde operators at an equal time are commutative: $[A,\ \tilde{B}] = 0$.

Tool 3. The tilde and non-tilde operators are mutually related through the bra-vacuum $\langle 1|$ as $\langle 1|A^\dagger = \langle 1|\tilde{A}$.

Corresponding to the characteristics of the Liouville operator given in the previous section, the Schrödinger equation ($\hbar = 1$) within NETFD

$$\frac{\partial}{\partial t}|0(t)\rangle = -i\hat{H}\ |0(t)\rangle, \tag{13}$$

has the following basic characters:

B1. The characteristics named *tildian*:

$$\left(i\hat{H}\right)^\sim = i\hat{H}. \tag{14}$$

The tildian hat-Hamiltonian is not necessarily hermitian operator.

B2. The hat-Hamiltonians have zero eigenvalue for the thermal bra-vacuum:

$$\langle 1|\hat{H} = 0. \tag{15}$$

This is the manifestation of the conservation of probability, i.e. $\langle 1|0(t)\rangle = 1$.

B3. The thermal vacuums $\langle 1|$ and $|0\rangle$ are ***tilde invariant***:

$$\langle 1|^{\sim} = \langle 1|, \quad |0\rangle^{\sim} = |0\rangle, \tag{16}$$

and are normalized as $\langle 1|0\rangle = 1$.

The Heisenberg equation within NETFD is given by

$$\frac{d}{dt}A = i[\hat{H},\ A]. \tag{17}$$

5 Quantum Fokker-Planck Equation

The hat-Hamiltonian for a bosonic semi-free field is bi-linear in a, $a^\dagger$ and their tilde-conjugates, and is invariant under the phase transformation $a \to a\mathrm{e}^{i\theta}$:

$$\hat{H} = \hat{H}_S - i\kappa\left[(1+2\bar{n})\left(a^\dagger a + \tilde{a}^\dagger \tilde{a}\right) - 2\,(1+\bar{n})\,a\tilde{a} - 2\bar{n}a^\dagger\tilde{a}^\dagger\right] - i2\kappa\bar{n}, \tag{18}$$

with

$$\hat{H}_S = \omega\left(a^\dagger a - \tilde{a}^\dagger \tilde{a}\right), \tag{19}$$

$$\bar{n} = \frac{1}{\mathrm{e}^{\omega/T}-1}. \tag{20}$$

T is the temperature of environment. The boson operators a etc. satisfy the canonical commutation relations:

$$[a,\ a^\dagger] = 1, \qquad [\tilde{a},\ \tilde{a}^\dagger] = 1. \tag{21}$$

The Fokker-Planck equation of the system is given by

$$\frac{\partial}{\partial t}|0(t)\rangle = -i\hat{H}|0(t)\rangle, \tag{22}$$

with an initial ket-thermal vacuum, $|0\rangle = |0(0)\rangle$, satisfying

$$a|0\rangle = f\tilde{a}^\dagger|0\rangle, \tag{23}$$

with $f = n/(1+n)$, $n = n(0)$. The Fokker-Planck equation (22) has the form of the Schrödinger equation. It was noticed first by Crawford [18] that the introduction of two

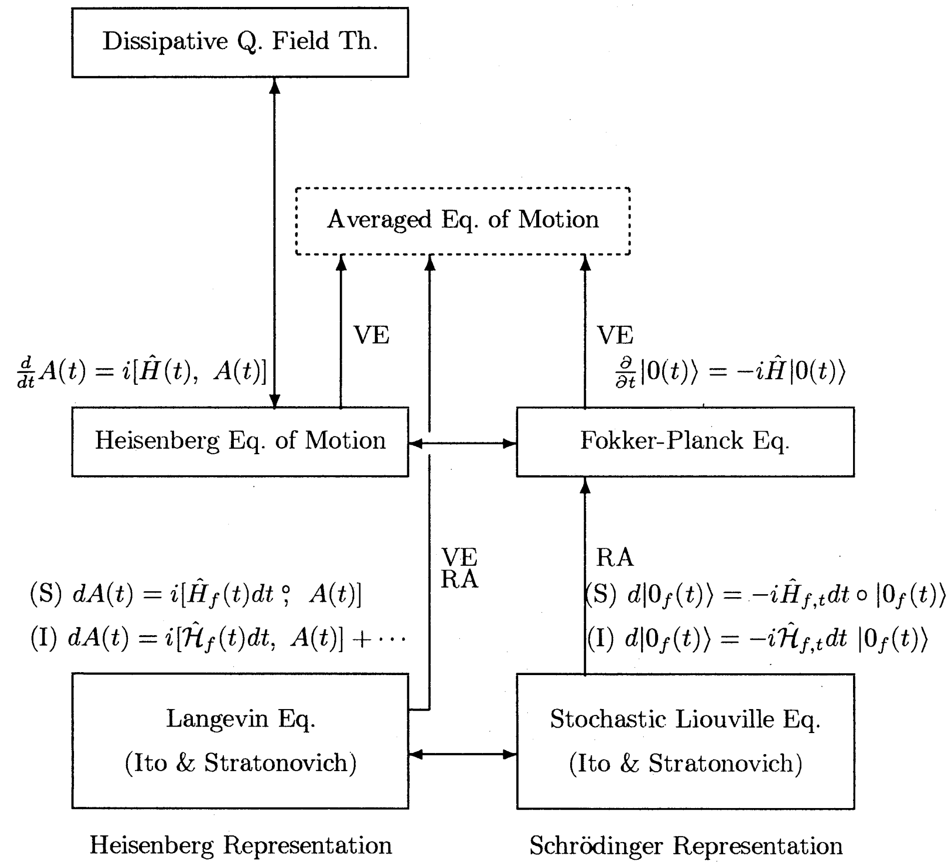

Figure 1: Structure of the Formalism. RA stands for the random average. VE stands for the vacuum expectation. (I) and (S) indicate Ito and Stratonovich types, respectively.

kinds of operators for each operator enables us to handle the Liouville equation as the Schrödinger equation.

With the help of the Fokker-Planck equation (22), we see that the one-particle distribution function $n(t) = \langle 1|a^\dagger a|0(t)\rangle$ satisfies the Boltzmann equation

$$\frac{d}{dt}n(t) = -2\kappa\,[n(t) - \bar{n}]\,. \tag{24}$$

6 Quantum Stochastic Calculus

In this section, we will review the quantum stochastic calculus introduced by Hudson and Parthasarathy [9]-[12].

6.1 Fock's Space

6.1.1 Fock's bases

We take the vectors:

$$|t_1, t_2, \cdots, t_n\rangle = \frac{1}{\sqrt{n!}} b^\dagger(t_1) b^\dagger(t_2) \cdots b^\dagger(t_n)|0\rangle, \tag{25}$$

as a set of bases for a Fock space. The argument t represents time. The vacuum $|0\rangle$ is defined by

$$b(t)|0\rangle = 0. \tag{26}$$

The annihilation and creation operators $b(t)$, $b^\dagger(t)$ satisfy the canonical commutation relation:

$$[b(t),\ b^\dagger(t')] = \delta(t - t'). \tag{27}$$

The bases form an orthnormal and complete set:

$$\langle t_1, \cdots, t_n | t'_1, \cdots, t'_m\rangle = \delta_{n,m} \frac{1}{n!} \sum_{(P)} \delta(t_1 - t'_1) \cdots \delta(t_n - t'_n), \tag{28}$$

$$\sum_{n=0}^{\infty} \left(\prod_{\ell=1}^{n} \int_0^\infty dt_\ell \right) |t_1, \cdots, t_n\rangle\langle t_1, \cdots, t_n| = I. \tag{29}$$

The Fock space $\Gamma(\mathcal{H})$ over a Hilbert space $\mathcal{H}$ is the infinite Hilbert space direct sum

$$\Gamma(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\hat{\otimes} n}, \tag{30}$$

where $\mathcal{H}^{\hat{\otimes} n=0} = \boldsymbol{C}$, and, for $n \geq 1$, $\mathcal{H}^{\hat{\otimes} n}$ is the symmetric subspace of the n-fold Hilbert space tensor product of $\mathcal{H}$ (the Wiener-Ito expansion).

For $|\psi\rangle \in \Gamma(\mathcal{H})$, we have

$$\begin{aligned} |\psi\rangle &= \sum_{n=0}^{\infty} \left(\prod_{\ell=1}^{n} \int_0^\infty dt_\ell \right) |t_1, \cdots, t_n\rangle \psi_n(t_1, \cdots, t_n) \\ &= (\psi_0,\ \psi_1,\ \psi_2,\ \cdots), \end{aligned} \tag{31}$$

where

$$\psi_n(t_1, \cdots, t_n) = \langle t_1, \cdots, t_n | \psi\rangle \in \mathcal{H}^{\hat{\oplus} n}. \tag{32}$$

This situation is similar to the one in quantum field theory when expanding a state in a Fock space in terms of the state vecotrs in the n-particle subspace. In that case, ψ_n is the wave-function of n-particle system in quantum mechanics.

6.1.2 Annihilation and Creation Operators

For each $f \in \mathcal{H}$, let us introduce the annihilation and creation operators, $b[f]$ and $b^\dagger[f]$, by

$$b[f] = \int_0^\infty dt\ f^*(t)b(t), \qquad b^\dagger[f] = \int_0^\infty dt\ f(t)b^\dagger(t), \tag{33}$$

which satisfy the commutation relation

$$[b[f],\ b^\dagger[g]] = \langle f,\ g\rangle, \tag{34}$$

where the inner product is defined by

$$\langle f,\ g\rangle = \int_0^\infty dt\ f^*(t)g(t). \tag{35}$$

6.1.3 Exponential Vector

Let us introduce the exponential vector $|e[f]\rangle \in \Gamma(\mathcal{H})$ for $f \in \mathcal{H}$ by

$$\begin{aligned} |e[f]\rangle &= \mathrm{e}^{b^\dagger[f]}|0\rangle \\ &= \left(1,\ \frac{1}{\sqrt{1!}}f,\ \frac{1}{\sqrt{2!}}f^{\otimes 2},\ \cdots\right). \end{aligned} \tag{36}$$

The vector satisfies

$$\langle e[f]|e[g]\rangle = \mathrm{e}^{\langle f,\ g\rangle}, \tag{37}$$

for $f,\ g \in \mathcal{H}$.

For $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, $f_1 \in \mathcal{H}_1$, $f_2 \in \mathcal{H}_2$, we see that

$$|e[f_1 + f_2]\rangle = |e[f_1]\rangle \otimes |e[f_2]\rangle. \tag{38}$$

The actions of the annihilation and creation operators on the exponential vector are given by

$$\begin{aligned} b[f]|e[g]\rangle &= \langle f,\ g\rangle|e[g]\rangle, \\ b^\dagger[f]|e[g]\rangle &= \frac{d}{d\epsilon}|e[g+\epsilon f]\rangle\Big|_{\epsilon=0}. \end{aligned} \tag{39}$$

Note that

$$\begin{aligned} b(t)|e[f]\rangle &= f(t)|e[f]\rangle, \\ b^\dagger(t)|e[f]\rangle &= \frac{\delta}{\delta f(t)}|e[f]\rangle. \end{aligned} \tag{40}$$

6.1.4 Weyl's Operator

The Weyl operator defined by

$$W[f] = e^{b^\dagger[f]-b[f]}, \tag{41}$$

for $f \in \mathcal{H}$ has the properties

$$W[f]|0\rangle = e^{-\|f\|^2/2}|e[f]\rangle, \tag{42}$$

$$W[f]|e[g]\rangle = e^{-\|f\|^2/2-\langle f,\, g\rangle}|e[f+g]\rangle, \tag{43}$$

and satisfies the Weyl relation:

$$W[f]W[g] = e^{-i\Im m\langle f,\, g\rangle}W[f+g]. \tag{44}$$

6.2 Quantum Brownian Motion

Introducing the operators

$$B(t) = b[\chi_{[0,t]}] = \int_0^t dt'\ b(t'), \tag{45}$$

$$B^\dagger(t) = b^\dagger[\chi_{[0,t]}] = \int_0^t dt'\ b^\dagger(t'), \tag{46}$$

for $t \geq 0$, we see that they satisfy

$$B(0) = 0, \tag{47}$$

$$[B(s),\ B^\dagger(t)] = \min(s,t). \tag{48}$$

Here, $\chi_{[0,t]}$ is the indicator function of the interval $[0,t]$.

With $B(t)$ and $B^\dagger(t)$, the equations in (33) read

$$b[f] = \int_0^\infty dB(t)\ f^*(t), \qquad b^\dagger[f] = \int_0^\infty dB^\dagger(t)\ f(t). \tag{49}$$

This is called the Wiener integral.

6.3 Ito's Stochastic Product

The action of the operator representing the quantum Brownian motion on the exponential vector is given by

$$\begin{aligned} B(t)|e[f]\rangle &= \int_0^t dt'\ b(t')|e[f]\rangle \\ &= \int_0^t dt'\ f(t')|e[f]\rangle, \end{aligned} \tag{50}$$

which becomes

$$dB(t)|e[f]\rangle = f(t)dt|e[f]\rangle, \tag{51}$$

in the differential form.[1]

When the integral

$$\langle e[f]|I(t)|e[g]\rangle = \int_0^t dt' \langle e[f]| \{f^*(t')F(t') + G(t')g(t') + H(t')\} |e[g]\rangle, \tag{53}$$

exists for $f,\ g \in L^2(\boldsymbol{R}_{\geq 0})$ with $t \geq 0$, we say that $I(t)$ is a stochastic integral:

$$I(t) = \int_0^t dt' \left\{dB^\dagger(t')F(t') + G(t')dB(t') + H(t')dt'\right\}. \tag{54}$$

This can be written in a differential form as

$$dI(t) = dB^\dagger(t)F(t) + G(t)dB(t) + H(t)dt, \qquad I(0) = 0. \tag{55}$$

For

$$dI_1(t) = dB^\dagger(t)F_1(t) + G_1(t)dB(t) + H_1(t)dt, \qquad I_1(0) = 0, \tag{56}$$

$$dI_2(t) = dB^\dagger(t)F_2(t) + G_2(t)dB(t) + H_2(t)dt, \qquad I_2(0) = 0, \tag{57}$$

we have the Ito's stochastic product [22]

$$\begin{aligned} d(I_1I_2) &= dB^\dagger\left(F_1I_2 + I_1F_2\right) + \left(G_1I_2 + I_1G_2\right)dB(t) + \left(H_1I_2 + I_1H_2 + G_1F_2\right)dt \\ &= dI_1I_2 + I_1dI_2 + dI_1dI_2. \end{aligned} \tag{58}$$

Here we used

$$dB(t)dB^\dagger(t) = dt, \tag{59}$$

which can be shown by the commutation relation

$$[dB(t),\ dB^\dagger(t)] = dt. \tag{60}$$

Note that the commutation relation is a consequence of (48).

In precise, the Ito formula (58) is proven in the representation of the exponential vectors. It can be shown that the Weyl relation (44) is the consequence of the Ito formula (58).

[1]Within the white noise analysis [19]-[21], one can use the operator equation

$$dB(t) = b(t)dt, \tag{52}$$

instead of (51).

7 Quantum Stochastic Calculus within NETFD

7.1 Thermal Space

Now, put the above materials in the Hilbert space into the thermal space within NETFD. The approach seems somewhat related to the one by [13].

The operators representing the quantum Brownian motion annihilate the vacuums $|0\rangle$ and $\langle 0|$:

$$dB(t)|0\rangle = 0, \qquad d\tilde{B}(t)|0\rangle = 0, \tag{61}$$

$$\langle 0|dB^{\dagger}(t) = 0, \qquad \langle 0|d\tilde{B}^{\dagger}(t) = 0. \tag{62}$$

Let us introduce a set of new operators by the relation

$$d\mathcal{B}(t)^{\mu} = \bar{B}^{\mu\nu} dB(t)^{\nu}, \tag{63}$$

with the Bogoliubov transformation difined by

$$\bar{B}^{\mu\nu} = \begin{pmatrix} 1+\bar{n} & -\bar{n} \\ -1 & 1 \end{pmatrix}, \tag{64}$$

where $\bar{n}$ is given in (20), and we introduced the thermal doublet: $dB(t)^{\mu=1} = dB(t)$, $dB(t)^{\mu=2} = d\tilde{B}^{\dagger}(t)$, $d\bar{B}(t)^{\mu=1} = dB^{\dagger}(t)$, $d\bar{B}(t)^{\mu=2} = -d\tilde{B}(t)$, and the similar doublet notations for $d\mathcal{B}(t)^{\mu}$ and $d\bar{\mathcal{B}}(t)^{\mu}$. The new operators annihilate the new vacuums $\langle|$ and $|\rangle$:

$$d\mathcal{B}(t)|\rangle = 0, \qquad d\tilde{\mathcal{B}}(t)|\rangle = 0, \tag{65}$$

$$\langle|d\mathcal{B}^{\dagger}(t) = 0, \qquad \langle|d\tilde{\mathcal{B}}^{\dagger}(t) = 0. \tag{66}$$

In the following, we will use the representation space constructed on the vacuums $\langle|$ and $|\rangle$. Then, we have, for example,

$$\langle|dB^{\dagger}(t)dB(t)|\rangle = \bar{n}dt, \tag{67}$$

$$\langle|dB(t)dB^{\dagger}(t)|\rangle = (\bar{n}+1)\,dt, \tag{68}$$

which was derived by inspecting $\langle|d\tilde{B}(t)dB(t)|\rangle$ with the help of the thermal state conditions (65) and (66).

For later convenience, let us introduce the random force operators by

$$dF(t) = \sqrt{2\kappa}dB(t), \qquad dF^{\dagger}(t) = \sqrt{2\kappa}dB^{\dagger}(t). \tag{69}$$

Then, we have $\left\langle dF(t)\right\rangle = \left\langle d\tilde{F}(t)\right\rangle = \left\langle dF^{\dagger}(t)\right\rangle = \left\langle d\tilde{F}^{\dagger}(t)\right\rangle = 0$, and

$$\left\langle dF^{\dagger}(t)dF(s)\right\rangle = 2\kappa\bar{n}\delta(t-s)dtds, \qquad \left\langle dF(t)dF^{\dagger}(s)\right\rangle = 2\kappa\,(\bar{n}+1)\,\delta(t-s)dtds, \tag{70}$$

and zero for other combinations (see (67) and (68)). Here we introduced an abbreviation $\langle\cdots\rangle = \langle|\cdots|\rangle$.

The thermal state condition (65) and (66) reads

$$\langle|dF^\dagger(t) = \langle|d\tilde{F}(t), \tag{71}$$

$$(1+\bar{n})dF(t)|\rangle = \bar{n}d\tilde{F}^\dagger(t)|\rangle. \tag{72}$$

The above Bogoliubov transformation introduces the effect of thermal dissipation which manifests itself in the correlation of the random forces (70), i.e. the appearance of $2\kappa\bar{n}$ which is related to the diffusion term in the Fokker-Planck generator (18). Therfore, this can be a realization of the concept named the ***spontanious creation of dissipation*** [14, 15].

7.2 Quantum Stochastic Liouville Equation

The hat-Hamiltonian for the *stochastic semi-free* field is bi-linear in a, $a^\dagger$, $dF(t)$, $dF^\dagger(t)$ and their tilde conjugates, and is invariant under the phase transformation $a \to a\mathrm{e}^{i\theta}$, and $dF(t) \to dF(t)\ \mathrm{e}^{i\theta}$. Here, a, $a^\dagger$ and their tilde conjugates are stochastic operators of a relevant system satisfying the canonical commutation relation[2]

$$[a,\ a^\dagger] = 1, \qquad [\tilde{a},\ \tilde{a}^\dagger] = 1, \tag{73}$$

whereas $dF(t)$, $dF^\dagger(t)$ and their conjugates are the operators representing the quantum Brownian motion.

The tilde and non-tilde operators are related with each other by the relation

$$\langle 1|a^\dagger = \langle 1|\tilde{a}, \tag{74}$$

where $\langle 1|$ is the thermal bra-vacuum of the relevant system.

Now, let us derive the infinitesimal stochastic time-evolution generator $\hat{\mathcal{H}}_{f,t}dt$ in the stochastic Liouville equation

$$d|0_f(t)\rangle = -i\hat{\mathcal{H}}_{f,t}dt|0_f(t)\rangle. \tag{75}$$

Introducing a generator $\hat{V}_{f,t}(t)$ by

$$|0_f(t)\rangle = \hat{V}_{f,t}(t)|0\rangle, \tag{76}$$

we see that it satisfies

$$d\hat{V}_f(t) = -i\hat{\mathcal{H}}_{f,t}dt\ \hat{V}_f(t), \tag{77}$$

[2]We use the same notation a etc. for the stochastic semi-free operators as those for the coarse grained semi-free operators. We expect that there will be no confusion between them.

with $\hat{V}_f(0) = 1$. Here, it is assumed that, at $t = 0$, the relevant system starts to contact with the irrelevant system representing the stochastic process described by the random force operators $dF(t)$, etc. defined in (70).[3]

We will employ the characteristics of the stochastic Liouville equation of classical systems to quantum cases, i.e., the stochastic distribution function satisfies the conservation of probability within the phase space of a relevant system (see (9)). This means in NETFD that $\langle 1|0_f(t)\rangle = 1$, leading to

$$\langle 1|\hat{\mathcal{H}}_{f,t}dt = 0. \tag{78}$$

Here, we emphasize again that the thermal bra-vacuum $\langle 1|$ is of the relevant system.

From the investigation in the pevious section (see (55)), we know that the required form of the hat-Hamiltonian should be

$$\hat{\mathcal{H}}_{f,t}dt = \hat{H}dt + d\hat{M}(t), \tag{79}$$

where the martingale $d\hat{M}(t)$ is the term containing the operaotrs representing the quantum Brownian motion, and $\hat{H}$ is given by (18), i.e. $\hat{H} = \hat{H}_S + i\hat{\Pi}$, $\hat{\Pi} = -\kappa\left(\alpha^{\natural}\alpha + \text{t.c.}\right) + 2\kappa\left[\bar{n} + \nu\right]\alpha^{\natural}\tilde{\alpha}^{\natural}$. We introduced a set of canonical stochastic operators[4] $\alpha = \mu a + \nu\tilde{a}^{\dagger}$, $\alpha^{\natural} = a^{\dagger} - \tilde{a}$, with $\mu + \nu = 1$, which satisfy the commutation relation $[\alpha,\ \alpha^{\natural}] = 1$.

The condition (78) reads

$$\langle 1|d\hat{M}(t) = 0, \tag{80}$$

which gives us for the case of semi-free field

$$d\hat{M}(t) = i\left[\alpha^{\natural}dW(t) + \text{t.c.}\right]. \tag{81}$$

The expression (79) of the hat-Hamiltonian with the martingale term (81) was derived in [8] by generalizing the Theorem 5.3 in [23].

The random force operators $dW(t), d\tilde{W}(t)$ are of the quantum stochastic Wiener process satisfying

$$\left\langle dW(t)\right\rangle = \left\langle d\tilde{W}(t)\right\rangle = 0, \qquad \left\langle dW(t)dW(s)\right\rangle = \left\langle d\tilde{W}(t)d\tilde{W}(s)\right\rangle = 0, \tag{82}$$

$$\left\langle dW(t)d\tilde{W}(s)\right\rangle = \left\langle d\tilde{W}(s)dW(t)\right\rangle = 2\kappa\left(\bar{n} + \nu\right)\delta(t-s)dtds, \tag{83}$$

where the random force operator $dW(t)$ is defined by

$$dW(t) = \mu dF(t) + \nu d\tilde{F}^{\dagger}(t). \tag{84}$$

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[3]Within the formalism, the random force operators $dF(t)$ and $dF^{\dagger}(t)$ are assumed to commute with any relevant system operator A in the Schrödinger representation: $[A,\ dF(t)] = [A,\ dF^{\dagger}(t)] = 0$ for $t \geq 0$.

[4]The expression of $\hat{\Pi}$ was given here by means of a set of canonical stochastic operators α, $\alpha^{\natural}$ and their tilde conjugates.

The *original* random force operators $dF(t)$ and $dF^{\dagger}(t)$ are of the *stationary* Gaussian white process, which is defined by (70).

The one-particle distribution function

$$n(t) = \langle\!\langle 1|a^{\dagger\dagger}(t)a(t)|0_f\rangle\!\rangle, \tag{85}$$

satisfies the Boltzmann equation (24). Here, $\langle\!\langle\cdots\rangle\!\rangle = \langle|\langle 1|\cdots|0\rangle|\rangle$, which means to take both random average and vacuum expectation.

7.3 Stochastic Semi-Free Operators

The stochastic semi-free operators are defined by

$$a(t) = \hat{V}_f^{-1}(t)a\hat{V}_f(t), \qquad \tilde{a}^{\dagger\dagger}(t) = \hat{V}_f^{-1}(t)\tilde{a}^{\dagger}\hat{V}_f(t), \tag{86}$$

where $\hat{V}_f(t)$ satisfies (77).

The semi-free operators (86) keep the equal-time canonical commutation relation:

$$[a(t),\ a^{\dagger\dagger}(t)] = 1, \qquad [\tilde{a}(t),\ \tilde{a}^{\dagger\dagger}(t)] = 1, \tag{87}$$

and satisfy **Tool 3**: $\langle 1|a^{\dagger\dagger}(t) = \langle 1|\tilde{a}(t)$. The tildian nature $\left(i\hat{\mathcal{H}}_{f,t}dt\right)^{\sim} = i\hat{\mathcal{H}}_{f,t}dt$, (see **B1**) is consistent with the definition (86) of the semi-free operators. Since the tildian hat-Hamiltonian $\hat{\mathcal{H}}_{f,t}dt$ is not necessarily hermite, we introduced the symbol $\dagger\dagger$ in order to distinguish it from the hermite conjugation $\dagger$. However, we will use $\dagger$ instead of $\dagger\dagger$, for simplicity, unless it is confusing.

The stochastic semi-free operators and the random force operators satisfy the orthogonality $\left\langle a(t)d\mathcal{F}^{\dagger}(t)\right\rangle = 0$, etc., where the random force operator $d\mathcal{F}^{\dagger}(t)$ in the *Heisenberg* representation[5] is defined by $d\mathcal{F}^{\dagger}(t) = \hat{V}_f^{-1}(t)dF^{\dagger}(t)\hat{V}_f(t)$.

7.4 Quantum Langevin Equation

The quantum Langevin equation of the Ito type is given by

$$\begin{aligned} dA(t) &= d\hat{V}_f^{-1}(t)A\hat{V}_f(t) + \hat{V}_f^{-1}(t)Ad\hat{V}_f(t) + d\hat{V}_f^{-1}(t)Ad\hat{V}_f(t) \\ &= i[\hat{\mathcal{H}}_f(t)dt,\ A(t)] \\ &\quad + \left\{\alpha^{\female}(t)[\tilde{\alpha}^{\female}(t),\ A(t)] + \tilde{\alpha}^{\female}(t)[\alpha^{\female}(t),\ A(t)]\right\} dW(t)d\tilde{W}(t) \\ &= i[\hat{H}_S(t),\ A(t)]dt + \kappa\left\{[\alpha^{\female}(t)\alpha(t),\ A(t)] + [\tilde{\alpha}^{\female}(t)\tilde{\alpha}(t),\ A(t)]\right\} dt \\ &\quad +2\kappa(\bar{n}+\nu)[\tilde{\alpha}^{\female}(t),\ [\alpha^{\female}(t),\ A(t)]]dt \\ &\quad -\left\{[\alpha^{\female}(t),\ A(t)]dW(t) + [\tilde{\alpha}^{\female}(t),\ A(t)]d\tilde{W}(t)\right\}, \end{aligned} \tag{88}$$

[5] It can be the interaction representation when one includes non-linear terms in the hat-Hamiltonian, and performs a perturbational calculation. As we are dealing with only the semi-free case in this paper, we call the representation as the Heisenberg one.

with

$$\hat{H}_S(t) = \hat{V}_f^{-1}(t)\hat{H}_S\hat{V}_f(t). \tag{89}$$

Here, $d\hat{V}_f^{-1}(t)$ is derived by inspecting $d\left(\hat{V}_f^{-1}(t)\hat{V}_f(t)\right) = 0$ within the Ito stochastic calculus. Since (88) is the time-evolution equation for any relevant stochastic operator $A(t)$, it is *Ito's formula* for quantum systems [22].

Putting a and $\tilde{a}^\dagger$ for A, we see that (88) reduces to

$$d\alpha(t) = i[\hat{H}_S(t)dt,\ \alpha(t)] - \kappa\alpha(t)dt + dW(t), \tag{90}$$

$$d\alpha^{\text{♀}}(t) = i[\hat{H}_S(t)dt,\ \alpha^{\text{♀}}(t)] + \kappa\alpha^{\text{♀}}(t)dt. \tag{91}$$

7.5 Equation of Motion for the Ket-Vectors

Applying the bra-vacuum $\langle 1|$ to the Ito type Langevin equation (88), we have

$$\begin{aligned} d\langle 1|A(t) = {} & i\langle 1|[H_S(t),\ A(t)]dt + \kappa\langle 1|A(t)\left[\alpha^{\text{♀}}\alpha + \text{t.c.}\right]dt + 2\kappa\bar{n}\langle 1|A(t)\alpha^{\text{♀}}(t)\tilde{\alpha}^{\text{♀}}(t)dt \\ & + \langle 1|A(t)\left[\alpha^{\text{♀}}(t)dW(t) + \text{t.c.}\right]. \end{aligned} \tag{92}$$

Applying further the random force bra-vacuum $\langle|$ to (92), we can derive the stochastic equation of motion of Ito type for the bra-vector state $\langle\!\langle 1|A(t)$ in the form

$$\begin{aligned} d\langle\!\langle 1|A(t) = {} & i\langle\!\langle 1|[H_S(t),\ A(t)]dt + \kappa\left\{\langle\!\langle 1|[a^\dagger(t),\ A(t)]a(t) + \langle\!\langle 1|a^\dagger(t)[A(t),\ a(t)]\right\}dt \\ & + 2\kappa\bar{n}\langle\!\langle 1|[a(t),\ [A(t),\ a^\dagger(t)]]dt \\ & + \langle\!\langle 1|[A(t),\ a^\dagger(t)]dF(t) + \langle\!\langle 1|[a(t),\ A(t)]dF^\dagger(t), \end{aligned} \tag{93}$$

where we used the property $\langle|dW(t) = \langle|dF(t)$ and $\langle|d\tilde{W}(t) = \langle|dF^\dagger(t)$. This equation of motion for the bra-vector state may be intimately related with the Langevin equations given by Gardiner and Collett [24].

Putting the random force ket-vacuum $|\rangle$ and the ket-vacuum $|0\rangle$ of the relevant system to (93), we obtain the equation of motion for the expectation value of an arbitrary operator $A(t)$ of the relevant system as

$$\begin{aligned} \frac{d}{dt}\langle\!\langle A(t)\rangle\!\rangle = {} & i\,\langle\!\langle [H_S(t),\ A(t)]\rangle\!\rangle + \kappa\left(\langle\!\langle [a^\dagger(t),\ A(t)]a(t)\rangle\!\rangle + \langle\!\langle a^\dagger(t)[A(t),\ a(t)]\rangle\!\rangle\right) \\ & + 2\kappa\bar{n}\langle\!\langle [a(t),\ [A(t),\ a(t)^\dagger]]\rangle\!\rangle. \end{aligned} \tag{94}$$

This is the exact equation of motion for systems with linear-dissipative coupling to reservoir, which can be also derived by means of Fokker-Planck equation (22). Note that (94) was derived for general $\hat{H}_S$ including non-linear interaction terms within the conventional treatment [25].

8 Discussions

We showed how it should be if one wants to construct a quantum stochastic calculus compatible with ***dissipative*** quantum systems in physics. It has become possible by means of the unified formulation of NETFD.

As was shown in this paper, whole the structure of the formulation was constructed in the manner being consistent with that of classical one: The stochastic Liouville equation satisfies the conservation of probability within the relevant system; The Stratonovich stochastic differential equation contains the relaxation generator but does not contain the diffusion generator, whereas the Ito equation does both generators.

If we start with a Schrödinger equation, as many mathematicians do, making it stochastic by introducing a martingale term under the assumption that the time-generation of wave-function is controled by a unitary operator[6] [27], we obtain

$$\hat{\mathcal{H}}_{f,t}dt = \hat{H}dt + i\left[\left(a^\dagger - \tilde{a}\right)dW(t) + \text{t.c.}\right] - i\left[\left(\mu a + \nu\tilde{a}^\dagger\right)dW^{\natural}(t) + \text{t.c.}\right]. \tag{95}$$

In addition to the random force operators $dW(t)$ and its tilde conjugate, we need to introduce

$$dW^{\natural}(t) = dF^\dagger(t) - d\tilde{F}(t), \tag{96}$$

and its tilde conjugate which annihilate the ket-vacuum $\langle|$:

$$\langle|dW^{\natural}(t) = 0, \qquad \langle|d\tilde{W}^{\natural}(t) = 0. \tag{97}$$

The new random force operators satisfy the correlations

$$\left\langle dW^{\natural}(t)\right\rangle = \left\langle d\tilde{W}^{\natural}(t)\right\rangle = 0, \qquad \left\langle dW^{\natural}(t)dW(s)\right\rangle = \left\langle d\tilde{W}^{\natural}(t)d\tilde{W}(s)\right\rangle = 0, \tag{98}$$

$$\left\langle dW(t)dW^{\natural}(s)\right\rangle = \left\langle d\tilde{W}(t)d\tilde{W}^{\natural}(s)\right\rangle = 2\kappa\delta(t-s)dt\,ds. \tag{99}$$

In this case, it turns out to be that since the hat-Hamiltonian $\hat{H}_{f,t}$ for the quantum stochastic Liouville equation of the ***Stratonovich type*** is hermitian:

$$\left(\hat{H}_{f,t}dt\right)^\dagger = \hat{H}_{f,t}dt, \tag{100}$$

the stochastic time-evolution generator $\hat{V}_f(t)$ satisfying

$$d\hat{V}_f(t) = -i\hat{H}_{f,t} \circ \hat{V}_f(t), \tag{101}$$

[6]This assumption is a manifestation of the conservation of the probability within the stochastic interpretation of wave-function proposed by N. Bohr, i.e. the Copenhaagen interpretaion of quantum mechanics, and is different from the condition (78) of the conservation of probability within the relevant system.

becomes unitary:

$$\hat{V}_f^\dagger(t) = \hat{V}_f^{-1}(t), \tag{102}$$

within the Stratonovich calculation. The symbol $\circ$ represents the Stratonovich's product [17].

Starting with this hermitian hat-Hamiltonian, we can proceed just the same way as in previous sections for the rest of the formulation to construct a unified framework of the stochastic differential equations. We see that the equation of motion for the ket-vector $\langle\!\langle 1|A(t)$ reduces to (93). Therefore, the averaged equation of motion reduces also to (94) in both cases [8]. Note that the above mentioned consistency with the classical stochastic calculus is violated for the approach with the generator (95). In connection with the relation between a microscopic viewpoint and a stochastic viewpoint, a further investigation will be required to reveal which approach should be appropriate.

The correspondence of the equation of motion for the ket-vector $\langle\!\langle 1|A(t)$, (93), to the Langevin equation derived by Gardiner and Collet [24], is an attractive future problem. For spin systems, there is a similar correspondence between the equation of motion for ket-vector and the Langevin equation derived by Shibata and Hashitsume [26]. An investigation related to these correspondences may give us a deeper insight for the derivation of the stochastic differential equations from a microscopic point of view.

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