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A new formula for calculating Stark units over real quadratic number fields

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1. Let F be a real quadratic number field. For an integral ideal f of F, let $I(f)$ denote the group of fractional ideals of F generated by all prime ideals of F which do not divide f. Two ideals $a, b \in I(f)$ belong to the same ray class C mod f iff $ab^{-1} = (\alpha)$ is a principal ideal with a generator $\alpha \in 1+fb^{-1}$ satisfying the sign condition $\alpha' > 0$ (as usual, we denote by α' the image of α under the nontrivial automorphism of $F/\mathbb{Q}$). The ray class C gives rise to the partial zeta function

$$\zeta(C,s) = \sum_{a \in C} N(a)^{-s} \ , \ Re(s) > 1 \ ,$$

where a runs over all integral representatives of C. According to a well known conjecture of Stark [St], the derivative of $\zeta(C,s)$ at s=0 is the logarithm of a unit (also called Stark unit) in an abelian extension of F. In this paper, we report on a new formula for calculating the number $\zeta'(C,0)$. In comparison to the classical formula of Shintani [Sh] which expresses $\zeta'(C,0)$ in terms of the logarithm of the double gamma function, our formula is based on the function $\Lambda(u,v;w)$ defined by (3.1). This function has not been considered in the literature yet, but it deserves a closer examination.

2. As a preparation, we study the double series

$$S = \sum_{m,n}{}' \frac{\text{sign}(wm+n)}{mn} \mathbf{e}(mu+nv) \ , \ \ \mathbf{e}(x) = \exp(2\pi i x) \tag{2.1}$$

where w is a nonzero real number, and u, v are nonintegral real numbers, while (m,n) runs over all lattice points in $\mathbb{Z}^2$ with $m \neq 0$ and $n \neq 0$ (indicated, as usual, by a prime on the summation sign). The series converges only conditionally, so we need to explain first how to attach a value to it. Using the known estimate

$$\sum_{0<m<t} \frac{\mathbf{e}(mu)}{m} = -\log(1-\mathbf{e}(u)) + O(\tfrac{1}{t})$$

valid for a fixed $u \in \mathbb{R}\backslash\mathbb{Z}$ and $t \to \infty$, it is easy to see that the limit

$$S(u,v;w) \ = \lim_{A,B,C,D\to+\infty} \Big(\sum_{\substack{-A<m<B \\ -C<n<D}}{}' \frac{\text{sign}(wm+n)}{mn} \mathbf{e}(mu+nv) \Big) \ , \tag{2.2}$$

does exist for all real w (including w=0). More generally, if $X \subseteq \mathbb{R}^2$ is any bounded neighbourhood of the origin in $\mathbb{R}^2$, then the limit

$$\lim_{t\to+\infty} \Big(\sum_{(m,n) \in \mathbb{Z} \cap tX}{}' \frac{\text{sign}(wm+n)}{mn} \mathbf{e}(mu+nv) \Big)$$

does exist and equals $S(u,v;w)$ provided the boundary of $\overline{X}$ is a piecewise smooth curve which intersects the coordinate axes in $\mathbb{R}^2$ transversaly. For a proof of this statement we refer the reader to

[S2] where the case $X = \{(x,y) \in \mathbb{R}^2 : \ |Q(x,y)| < 1\}$ with a binary form Q was considered in detail. Since all these methods of summation lead to the same result, we will not specify them explicitly, but tacitly assume from now on that any of these methods is used to define the value of S. There is however one further natural method to limit S which deserves special attention. This method arises from the observation that $\mathrm{sign}(wm+n)$ does not change its value for all (m,n) on a ray through the origin. Let

$$P = \{(p,q) \in \mathbb{Z}^2\backslash\{0\} : p > 0\,, \gcd(p,q) = 1\}$$

be the set of all lattice points in the right half plane which are visible from the origin. Then we can write every $(m,n) \in \mathbb{Z}^2\backslash\{0\}$ as $(m,n) = r(p,q)$ with $(p,q) \in P$ and $r = \pm\gcd(m,n)$. Summing over r first, we get

$$\begin{aligned} S &= \sum_{(p,q)\in P}{}' \frac{\mathrm{sign}(pw+q)}{pq} \sum_{r\in\mathbb{Z}}{}' \frac{\mathrm{e}(r(pu+qv))}{r|r|} \\ &= \sum_{(p,q)\in P}{}' \frac{\mathrm{sign}(pw+q)}{pq} \lambda(pu+qv) \end{aligned}$$

with

$$\lambda(x) = \sum_{r\in\mathbb{Z}}{}' \frac{\mathrm{e}(rx)}{r|r|}\,.$$

The series over $(p,q) \in P$ is still conditionally convergent, but the sequence of partial sums with $|p|,|q| < t$ for $t \to \infty$ converges again to $S(u,v,w)$. With this ordering of p,q, we can write

$$S(u,v;w) = -\sum_{q/p<w} \frac{\lambda(pu-qv)}{pq} + \sum_{w<q/p} \frac{\lambda(pu-qv)}{pq}$$

since both partial series converge individually. From this representation we deduce that $S(u,v;w)$ is, as a function of w, discontinuous at all rational w, but continuous at all irrational w. Indeeed, if $w = \alpha/\beta$ with relatively prime α, β and $\beta > 0$, then

$$S(u,v;w+0) - S(u,v;w) = S(u,v;w) - S(u,v;w-0) = -\frac{\lambda(\beta u - \alpha v)}{\alpha\beta}$$

which is zero iff $\beta u - \alpha v \in \frac{1}{2}\mathbb{Z}$. On the other hand, since the sequence of partial sums given by (2.2) converges uniformly in (u,v) on every compact subset of the intervall $(0,1)\times(0,1)$, it follows that $S(u,v;w)$ is a continuous function of (u,v) on $(\mathbb{R}\backslash\mathbb{Z})^2$ for every fixed w. Assuming $w > 1$, we conclude by the same argument that the difference

$$S(u,v;w) - S(u,v;1) = -4i \sum_{\substack{0<m,n \\ m<n<wm}} \frac{\sin 2\pi(mu-nv)}{mn}$$

is continuous in u and v as long as u and v are not both integral. In other words, this difference is a continuous function on the punctured torus $T^2\backslash\{0\}$, $T = \mathbb{R}/\mathbb{Z}$.

3. From the arithmetical point of view, the definition of $S(u,v;w)$ is not complete yet because of the missing terms with $m=0$ resp. $n=0$ in (2.2). In order to compensate for this deficiency, we add two correction terms and introduce the function

$$\Lambda(u,v;w)=\frac{i}{4\pi}\Big(w\,\lambda(v)+\frac{1}{|w|}\lambda(u)+S(u,v;w)\Big)$$

$$=\frac{i}{4\pi}\left\{w\sum_n{}'\frac{\mathbf{e}(nv)}{n|n|}+\frac{1}{|w|}\sum_m{}'\frac{\mathbf{e}(mu)}{m|m|}+\sum_{m,n}{}'\frac{\operatorname{sign}(wm+n)}{mn}\mathbf{e}(mu+nv)\right\}. \qquad (3.1)$$

The two additional terms are essentially special values of the dilogarithm function Li_2,

$$\lambda(v)=2i\,\mathrm{Im}\,(\mathrm{Li}_2(\mathbf{e}(v)))\ ,\ \mathrm{Li}_2(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^2}\ ,\ |z|\le 1\ .$$

They are natural in view of the partial fraction decomposition

$$\frac{1}{mn}=\frac{1}{m(mw+n)}+\frac{w}{n(mw+n)}$$

(valid for $mn(mw+n)\neq 0$) which leads to the representation of $\Lambda(u,v;w)$ by two double series:

$$\frac{i}{4\pi}\left\{\sum_m{}'\frac{\mathbf{e}(mu)}{m}\sum_n{}'\frac{\mathbf{e}(nv)}{|mw+n|}+w\sum_n{}'\frac{\mathbf{e}(nv)}{n}\sum_m{}'\frac{\mathbf{e}(mu)}{|mw+n|}\right\}. \qquad (3.2)$$

The correction term $\lambda(u)/|w|$ is included here as the contribution of the terms with $n=0$ in the first double series, while $w\lambda(v)$ is the contribution of the terms with $m=0$ in the second double series. We emphasize that the above representation is only a formal one since each of the two double series diverges for generic w. Nevertheless it is of interest because of its similarity to the double series arising from the second Kronecker limit formula [Si].

We list some of the obvious properties of $\Lambda(u,v;w)$. First,

$$\Lambda(u,v;w)=\Lambda(v,u;w^{-1})\ \text{ for } w>0\ ,$$
$$\Lambda(-u,-v;w)=-\Lambda(u,v;w)\ ,$$
$$\Lambda(u,-v;-w)=\Lambda(u,v;w)\ ,$$

that is, Λ is odd in (u,v), while the last equation allows us to assume from now on that w is positive. The following distribution relation follows immediately from the definition of $\Lambda(u,v;w)$ and is valid for any two nonzero integers a,c.

Lemma 1:
$$\sum_{k(a)}\sum_{l(c)}\Lambda\left(\frac{u+k}{a},\frac{v+l}{c};w\right)=\operatorname{sign}(a)\,\Lambda\left(u,v;\frac{aw}{c}\right).$$

Our next result about $\Lambda(u,v;w)$ provides a link with the periodic Bernoulli functions $P_k(x)$ defined by the Fourier expansion

$$P_k(x)=-\frac{k!}{(2\pi i)^k}\sum_{n\in\mathbb{Z}}{}'\frac{\mathbf{e}(nx)}{n^k}\ ,\ k=1,2,3,\ldots \qquad (3.3)$$

They coincide with the Bernoulli polynomials $B_k(x)$ on the intervall $0<x<1$. In particular,

$$P_1(x)=x-\frac{1}{2}\ ,\ P_2(x)=x^2-x+\frac{1}{6}\quad\text{for}\quad 0<x<1.$$

Lemma 2: For $w\in\mathbb{Q}$, $w\neq 0$, and real $u,v\notin\mathbb{Z}$,

$$\Lambda(u,v;w)=PV\int_{-\infty}^{\infty}\frac{dt}{t}\left[\frac{w}{2}P_2(t+v)+\frac{1}{2|w|}P_2(t+u)-P_1(wt+u)P_1(t+v)\right],$$

where PV denotes the Cauchy principal value at $t=0$.

Proof: Ignoring questions of convergence and calculating formally, this identity follows easily from the well known integral

$$\operatorname{sign}(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dt}{t}\sin(2\pi xt) = \frac{1}{\pi i} PV\int_{-\infty}^{\infty} \frac{dt}{t}\,\mathbf{e}(xt)$$

by expressing every sign in (3.1) by this integral and then interchanging the order of summation and integration in the resulting expresssion. However, a direct justification for this interchange of limits does not seem to be easy, thus we proceed in a different way by calculating both sides of the Lemma independently. We note first that the integral in Lemma 2 satisfies the same distribution relation as $\Lambda(w,u;v)$. This follows from the distribution properties of the Bernoulli functions,

$$\sum_{l(n)} P_k(\tfrac{x+l}{n}) = n^{1-k}P_k(x)\ , \quad n = 1,2,3,\ldots. \tag{3.4}$$

Therefore it is enough to consider the case w=1. We show now that in this case, both sides are equal to

$$\Lambda(u,v;1) = P_1(u-v)\,\log\left|\frac{1-\mathbf{e}(u)}{1-\mathbf{e}(v)}\right|. \tag{3.5}$$

First, we notice that the representation (3.2) is valid for $w = 1$. This gives

$$\Lambda(u,v;1) = \frac{i}{4\pi}\left\{\sum_m{}' \frac{\mathbf{e}(m(u-v))}{m}\sum_p{}' \frac{\mathbf{e}(pv)}{|p|} + \sum_n{}' \frac{\mathbf{e}(n(v-u))}{n}\sum_p{}' \frac{\mathbf{e}(pu)}{|p|}\right\}$$

where $p = m{+}n$ runs now over all nonzero integers *independently* of m and n. Since

$$\sum_p{}' \frac{\mathbf{e}(pu)}{|p|} = -2\log|1-\mathbf{e}(u)|\ ,\ u \in \mathbb{R}\backslash\mathbb{Z}\ ,$$

(3.5) follows. To complete the proof of Lemma 2, it remains to evaluate the integral

$$PV\int_{-\infty}^{\infty} \frac{dt}{t}\left[\tfrac{1}{2}P_2(t{+}v) + \tfrac{1}{2}P_2(t{+}u) - P_1(t{+}u)P_1(t{+}v)\right].$$

To this end, we start with the trivial identity

$$(P_1(x) + P_1(y) + P_1(z))^2 = \tfrac{1}{4}$$

valid for all nonintegral real numbers x,y,z such that $x{+}y{+}z = 0$. Expanding this and using the relation

$$P_1(x)^2 = P_2(x) + \frac{1}{12}\ ,\quad x \notin \mathbb{Z}\ ,$$

we obtain the addition formula for the Bernoulli functions,

$$P_1(x)P_1(y) + P_1(y)P_1(z) + P_1(z)P_1(x) + \tfrac{1}{2}P_2(x) + \tfrac{1}{2}P_2(y) + \tfrac{1}{2}P_2(z) = 0\ . \tag{3.6}$$

Letting

$$x = t+u\ ,\ y = -t-v\ ,\ z = v-u\ ,$$

and integrating (3.6) with respect to dt/t, we get

$$PV\int_{-\infty}^{\infty} \frac{dt}{t}\left[\tfrac{1}{2}P_2(t{+}v) + \tfrac{1}{2}P_2(t{+}u) - P_1(t{+}u)P_1(t{+}v)\right]$$
$$= P_1(u{-}v)\ PV\int_{-\infty}^{\infty} \frac{dt}{t}\left[P_1(t{+}u) - P_1(t{+}v)\right].$$

For the calculation of the last integral, we can assume $0 < u < v < 1$. Then on the interval

$-v < t < 1-v$, the integer part function $[t+u]$ (Gauss bracket) equals -1 iff $-v < t < -u$ and vanishes otherwice. Therefore,

$$
\begin{aligned}
& PV\int_{-\infty}^{\infty} \frac{dt}{t}\left[P_1(t+u) - P_1(t+v)\right] \\
&= \sum_{n=-\infty}^{\infty} PV\int_{-v}^{1-v} \frac{dt}{n+t}\left[P_1(t+u) - P_1(t+v)\right] \\
&= \sum_{n=-\infty}^{\infty} PV\int_{-v}^{1-v} \frac{dt}{n+t}\,(u - v - [t+u]) \\
&= -\lim_{t\to\infty} \sum_{|n|<t} \int_{-v}^{1-v} \frac{dt}{n+t}\,[t+u] \\
&= \lim_{t\to\infty} \sum_{|n|<t} \int_{-v}^{-u} \frac{dt}{n+t} \\
&= \lim_{t\to\infty} \sum_{|n|<t} \log\left|\frac{n-u}{n-v}\right| \\
&= \log\left|\frac{1-\mathbf{e}(u)}{1-\mathbf{e}(v)}\right|
\end{aligned}
$$

using Euler's product decomposition of the sine function. This finishes the proof of Lemma 2. As a corollary to the above calculation, we in particular obtain the relation

$$
PV\int_{-\infty}^{\infty} \frac{dt}{t}\, P_1(t+u) = \log|1-\mathbf{e}(u)| \quad , \quad u\in\mathbb{R}\backslash\mathbb{Z} \tag{3.7}
$$

up to an additive constant. To see that this constant is in fact zero, it suffices to show that the left side vanishes for $u{=}1/6$. But this follows from the duplication formula

$$
P_1(2t+\tfrac{1}{3}) = P_1(t+\tfrac{1}{6}) + P_1(t-\tfrac{1}{3}) ,
$$

which is a special case of the distribution relations (3.4). Conversely, if (3.7) is already known, then the above calculation leads to a new proof of Euler's product expansion for the sine function.

Question: Does Lemma 2 hold for all real w ?

A positive answer to this question would in particular imply that the integral on the right converges for all nonzero real w, but we do not even know whether this simpler statement is true. The difficulty is to estimate the integral

$$
\int_{1}^{r} \frac{dt}{t} P_1(wt+u)P_1(t+v)
$$

for $r\to\infty$. Applying integration by parts, we see that it is enough to estimate

$$
\int_0^t P_1(wx+u)P_1(x+v)dx = \int_0^1 Q(t,x)P_1(x+v)dx \quad , \quad Q(t,x) = \sum_{k=0}^{t-1} P_1(wk+wx+u) .
$$

There are estimates of $Q(t,x)$ due to Hardy-Littlewood, Hecke, Ostrowski and others. Combining their results with the well known theorem of Roth, gives the estimate $Q(t,x) = O(t^{\varepsilon})$ for every $\varepsilon > 0$ in the case where w is an algebraic number. This proves the convergence of the integral on the right side of

Lemma 2 at least in the case of algebraic w, but the general case remains open.

As a first corollary to (3.5), we conclude that

$$\Lambda(u,v;w) - P_1(u)\log|1-\mathbf{e}(v)| - P_1(v)\log|1-\mathbf{e}(u)| \quad , \quad w>0$$

is a continuous function on the punctured torus $T^2\backslash\{0\}$ since the same is true for $\Lambda(u,v;w)-\Lambda(u,v;1)$. This expression displays therefore the logarithmic singularities of $\Lambda(u,v;w)$ as u or v (but not both) approach an integer. Next, combining Lemma 1 with (3.5) and (3.4), we obtain the following theorem.

Theorem 1: For positive integers a,c, and nonintegral u,v,

$$\begin{aligned}\Lambda(u,v;\tfrac{a}{c}) = &\sum_{k(a)} P_1(c\tfrac{u+k}{a}-v)\log|1-\mathbf{e}(\tfrac{u+k}{a})| \\ &+\sum_{l(c)} P_1(a\tfrac{v+l}{c}-u)\log|1-\mathbf{e}(\tfrac{v+l}{c})| .\end{aligned}$$

Remark: This relation can be regarded as a generalization of the reciprocity law for the classical Dedekind-Rademacher sums $S(a,c;u,v)$, defined by

$$S(a,c;u,v) \;=\; \sum_{k(c)} P_1(a\tfrac{u+k}{c}+v)\,P_1(\tfrac{u+k}{c}) .$$

Indeed, for positive relatively prime integers a,c, these sums satisfy the reciprocity law

$$S(a,c;u,v)+S(c,a;v,u) = P_1(u)P_1(v)+\tfrac{a}{2c}P_2(u)+\tfrac{1}{2ac}P_2(au+cv)+\tfrac{c}{2a}P_2(v)$$

provided $\frac{1}{4}$ is subtracted from the right side if u,v are both integral. The right side is here much easier to calculate than the left side, a fact which leads, as it is well known, to a polynomial time algorithm for calculating the Dedekind sum $S(a,c;u,v)$. Unfortunately, the situation in Theorem 1 is just the opposite. The quantity $\Lambda(u,v;\frac{a}{c})$ is much more difficult to calculate than the two finite sums on the right. In fact, the most efficient way to calculate $\Lambda(u,v;\frac{a}{c})$ we know off, is to calculate the two sums on the right side in Theorem 1.

Corollary. $$\exp(\Lambda(u,v;\tfrac{a}{c})) \;=\; \prod_{\substack{x \bmod 1\\ ax\equiv u}} \;\prod_{\substack{y \bmod 1\\ cy\equiv v}} \left|\frac{1-\mathbf{e}(x)}{1-\mathbf{e}(y)}\right|^{P_1(x-y)} \quad \text{for } u,v\in\mathbb{R}\backslash\mathbb{Z}.$$

In particular, $\exp(\Lambda(u,v;\frac{a}{c}))$ is an algebraic number for $u,v\in\mathbb{Q}\backslash\mathbb{Z}$. In fact, since $1-\mathbf{e}(\frac{p}{q})$ is a unit if q is not a power of a single prime (assuming (p,q)=1), the number $\exp(\Lambda(u,v;\frac{a}{c}))$ is a unit if none of the denominators of u/a and v/c is a power of a single prime. It is tempting to think of these units as the image of some Stickelberger elements applied to a fixed cyclotomic unit.

4. In view of these algebraic properties, it is very remarkable that the special values $\exp(\Lambda(u,v;w))$ where w is a quadratic irrationality, are also related to units in abelian extensions of the corresponding real quadratic number field. This fact is only a conjecture at present, but it follows from the well known conjecture of Stark [St], as we will show later. We first state the simplest and most attractive case of this conjecture.

Conjecture. Let $\varepsilon > 1$ be a unit in a real quadratic field F and $u \in \frac{1}{N}\mathbb{Z}\backslash\mathbb{Z}$, $N = N(\varepsilon-1)$. Then the number $\exp(2\Lambda(u, N(\varepsilon)u; \varepsilon))$ is a unit in an abelian extension of F.

It should be noted that for given ε, there are only finitely many u entering the conjecture. This is a significant difference to the case of rational w which partly explains why this conjecture is so inaccessible. We give two simple numerical examples where the conjecture is known to be true.

$$\exp(2\Lambda(\tfrac{1}{4},\tfrac{3}{4};\eta^3)) = \eta + \sqrt{\eta} \ , \quad \eta = \frac{1+\sqrt{5}}{2} \ , \tag{4.1}$$

$$\exp(2\Lambda(\tfrac{1}{3},\tfrac{1}{3};\varepsilon)) = \frac{\varepsilon - \sqrt{\varepsilon-1}}{\varepsilon + \sqrt{\varepsilon-1}} \ , \ \varepsilon = \frac{5+\sqrt{21}}{2} \ .$$

To give an example where the truth of the conjecture is not known, consider the polynomial

$$P(x) = x^4 - (4+3\sqrt{5})x^3 + 9\frac{3+\sqrt{5}}{2}x^2 - (4+3\sqrt{5})x + 1 \ .$$

If the conjecture is true, then the numbers $\exp(2\Lambda(\frac{k}{5},\frac{k}{5};\eta^4))$ with k=1,2,3,4 and η as in (4.1), are the four distinct roots of $P(x)$.

In order to state the general conjecture, we first need to define the values of $\Lambda(u,v;w)$ in the case where u or v (but not both) are integral. We define $\Lambda(u,v;w)$ in such a case by the limit

$$\lim_{(x,y)\to(u,v)} \Big(\Lambda(x,y;w) - \Lambda(x,y;\mathrm{sign} w)\Big) \tag{4.2}$$

as $(x,y) \in (\mathbb{R}\backslash\mathbb{Z})^2$ approaches (u,v). This is not a completely unreasonable definition. For instance, it is easily seen that with a small modification, Theorem 1 remains valid for all u, v which are not both integral. In particular, the numbers $\exp(\Lambda(u,v;\frac{a}{c}))$ are algebraic for all rational $(u,v) \notin \mathbb{Z}^2$.

A quadratic irrationality w is called reduced (in the narrow sense) if it satisfies the inequality $0 < w' < 1 < w$. A reduced quadratic irrationality w=w_0 determines a purely periodic sequence of reduced numbers w_k, $k \in \mathbb{Z}$, by the continued fraction expansion

$$w_{k+1} = \frac{1}{b_k - w_k} \ , \quad b_k = [w_k] + 1 \tag{4.3}$$

where $[w_k]$ denotes the integer part of w_k. All members of this sequense have the same discriminant, and it is known [Za] that there is a 1 to 1 correspondence between the set of narrow (ring-) ideal classes of discriminant D and the set of sequences of reduced numbers of discriminant D. Now let u, v be two rational numbers and w be a reduced number. The triple (u, v, w) defines a sequence of rational numbers u_k, $k \in \mathbb{Z}$, by

$$u_{-1} = v \ , \ u_0 = u \ , \ u_{k+1} = b_k u_k - u_{k-1} \ . \tag{4.4}$$

Then it is easy to see that the sequence $(w_k, u_k \bmod 1)$ is again periodic. Moreover, the set of all such sequences with a fixed w corresponds bijectively to the set of all narrow ray classes contained in the narrow ideal class of $\mathbb{Z}w+\mathbb{Z}$. Let r be the length of a minimal period.

Conjecture. $\exp(2\sum_{k(r)}\Lambda(u_{k-1},u_k;w_k))$ is a unit in an abelian extension of $\mathbb{Q}(w)$.

In the special case where w is a totally positive unit ε, this conjecture reduces to the one stated before since $w_k=\varepsilon$, $b_k=\mathrm{tr}(\varepsilon)=2-N$ with $N=\mathrm{N}(\varepsilon-1)$ for all k, and hence $u_k\equiv u(1)$ if $u=v\in\frac{1}{N}\mathbb{Z}$. To justify our definition of $\Lambda(u,v;w)$ in the case $u\in\mathbb{Z}$ or $v\in\mathbb{Z}$, we note that if $u_k\in\mathbb{Z}$, then $u_{k+1}\equiv -u_{k-1}(1)$, so formally $\Lambda(u_{k-1},u_k;1)+\Lambda(u_k,v_{k+1},1)=0$ which means that any contribution of $\Lambda(x,y;1)$ in (4.2) to $\Lambda(u,v;w)$ cancels out in the sum over the period. There is a homological explanation for this phenomenon. As we show later, the above sum over the period represents the value of a cocycle on a cycle. But such a value depends only on the (co)homology class of the (co)cycle, which means that any modification of the (co)cycle by a (co)boundary will not affect the final result.

The example below gives two further representations of the unit $\eta+\sqrt{\eta}$ in (4.1):

$$\begin{aligned}\tfrac{1}{2}\log(\eta+\sqrt{\eta}) &= \Lambda(\tfrac{3}{4},\tfrac{3}{4};\eta^2)+\Lambda(\tfrac{3}{4},\tfrac{2}{4};\eta^2)+\Lambda(\tfrac{2}{4},\tfrac{3}{4};\eta^2)\\ &= \Lambda(\tfrac{1}{4},\tfrac{0}{4};\eta^2)+\Lambda(\tfrac{0}{4},\tfrac{3}{4};\eta^2)+\Lambda(\tfrac{3}{4},\tfrac{1}{4};\eta^2)\ ,\ \eta=\frac{1+\sqrt{5}}{2}\,.\end{aligned}$$

Here $w_k=\eta^2$ and $b_k=3$ for all k. It is easily seen that in this case all periods of rational u_k with a denominator of 4 are given by $\pm(\frac{1}{4},\frac{1}{4},\frac{2}{4})$ and $\pm(\frac{1}{4},\frac{0}{4},\frac{3}{4})$. Since $\Lambda(u,v;w)$ is odd in (u,v), this means that the above example covers essentially all cases of the conjecture where $w_k=(3+\sqrt{5})/2$ and $4u_k\in\mathbb{Z}$. The next example does not require any comment.

$$\Lambda(\tfrac{1}{6},\tfrac{2}{6};2+\sqrt{3})+\Lambda(\tfrac{2}{6},\tfrac{1}{6};2+\sqrt{3})=\tfrac{1}{2}\log\Big(1+\sqrt{3}-\sqrt{3+2\sqrt{3}}\Big)\,.$$

Thinking about this example, one can hardly avoid the question about the arithmetic nature of every individual term in the sum on the left. Before attempting any experiments in this direction, it would be necessery, however, to calculate the values of $\Lambda(u,v;w)$ to a high degree of precision (hundreds of digits of accuracy). In general, this is a difficult problem, but in the special case we are interested in (w a quadratic irrationality and u,v rational), we were often able to calculate $\Lambda(u,v;w)$ to a modest accuracy in the following way. The continued fraction expansion of w produces a sequence of rational numbers p_n/q_n converging to w. Since Λ is continuous at w, the sequence $\Lambda(u,v;p_n/q_n)$ converges to $\Lambda(u,v;w)$. Using Theorem 1, we can calculate the first few members of this sequence (the calculational cost being directly proportional to the height of p_n/q_n). Assuming Λ is smooth in the variable w, we can speed up the convergence by approximating Λ with a Lagrange polynomial (constructed from the first few $\Lambda(u,v,p_n/q_n)$) and then extrapolate to the limit $p_n/q_n\to w$. In this way, we were able to calculate Λ in all of the above examples to over 40 digits of accuracy, but it would be difficult to achieve a significantly higher accuracy using this method.

In order to explain the connection with Stark's conjecture, we return now to the partial zeta function $\zeta(C,s)$ of the introduction and choose a $\mathbb{Z}$-basis (α,β) for the fractional ideal $b(\sqrt{D}f)^{-1}$, D the discriminant of the real quadratic field F, such that $w=\alpha/\beta$ is a reduced number. For $u=\mathrm{tr}(\alpha)$,

$v = \mathrm{tr}(\beta)$, let u_k, w_k $(k \in \mathbb{Z})$ be the corresponding periodic sequence defined by (4.3) and (4.4). Then, summing over a minimal period, we have:

Theorem 2.
$$\zeta'(C,0) = -\sum_{k(r)} \Lambda(u_{k-1}, u_k; w_k) .$$

A proof of this theorem will be given in [S1].

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Rerences

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