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The Tate conjecture and the semisimplicity conjecture for t-modules*

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§0. Introduction.

Let l be a prime number. Let k be an algebraic number field and A an abelian variety over k of dimension d. Then the l-adic Tate module

$$V_l(A) \stackrel{\mathrm{def}}{=} (\varprojlim \mathrm{Ker}(l^n \cdot \mathrm{id} : A(\overline{k}) \to A(\overline{k}))) \underset{\mathbb{Z}_l}{\otimes} \mathbb{Q}_l$$

is a $2d$-dimensional vector space over $\mathbb{Q}_l$ on which $\mathrm{Gal}(\overline{k}/k)$ acts. Thus, fixing a basis of $V_l(A)$, we obtain an l-adic Galois representation

$$\rho_{A,l} : \mathrm{Gal}(\overline{k}/k) \to GL_{2d}(\mathbb{Q}_l).$$

The following theorem of Faltings is important.

Theorem (0.1).
(i) (*Tate conjecture.*)

$$\mathrm{Hom}_k(A, A') \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_l \simeq \mathrm{Hom}_{\mathbb{Q}_l[\mathrm{Gal}(\overline{k}/k)]}(V_l(A), V_l(A')).$$

(ii) (*Semisimplicity conjecture.*) *$V_l(A)$ is a semisimple $\mathbb{Q}_l[\mathrm{Gal}(\overline{k}/k)]$-module.*

These conjectures can be also formulated for the l-adic Galois representations attached to more general motives, but they are still widely open.

Another problem is: What l-adic Galois representations come from abelian varieties (or motives)? We might hope for characterization of such representations in terms of p-adic theory at the places of k above $p = l$. In the case of abelian varieties, the following partial results are known (Serre, Tate, Raynaud, Deligne,...).

Theorem (0.2).
(i) *For each place v of k above l, $\rho_{A,l}|_{\mathrm{Gal}(\overline{k}_v/k_v)}$ is a Hodge-Tate representation, i. e. has a Hodge-Tate decomposition. (In fact, it seems to be known, moreover, to be a potentially semistable representation.)*
(ii) *Let ρ be an l-adic representation of $\mathrm{Gal}(\overline{k}/k)$ which is potentially abelian. (Namely, the image of $\mathrm{Gal}(\overline{k}/k)$ by ρ admits an abelian open subgroup.) If $\rho|_{\mathrm{Gal}(\overline{k}_v/k_v)}$ is a Hodge-Tate representation for all place v of k above l, then ρ is 'generated' by*

*This lecture was given in Japanese.

the representations attached to potentially CM abelian varieties and Artin representations.

In the present article, we consider t-adic Galois representations instead of l-adic Galois representations. A t-adic Galois representation is, by definition, a continuous group homomorphism $\mathrm{Gal}(k^{\mathrm{sep}}/k) \to GL_n(\mathbb{F}_q((t)))$, where k is a field of characteristic equal to $\mathrm{char}(\mathbb{F}_q)$. (For the definition, we do not have to restrict the characteristic of the field k, but we do not have any interesting theory so far in the case $\mathrm{char}(k) \neq \mathrm{char}(\mathbb{F}_q)$.) Here, the analogues of abelian varieties and motives are Drinfeld modules, Anderson's abelian t-modules, or more general objects, which yield t-adic Galois representations by taking their t-adic Tate modules.

In this new setting, the Tate conjecture has been proved independently by Taguchi [1][2] and the author [3]. See also [4]. In the present article, we consider mainly the semisimplicity conjecture and the problem of characterizing 'geometric' (or 'motivic') t-adic representations.

§1. Pink's restricted modules.

Pink introduced the concept of restricted modules (in 1994) in order to approach the semisimplicity conjecture for t-modules. (In fact, he also gave a proof of the conjecture, different from ours.) Roughly speaking, the category of restricted $k(t)\{\tau\}$-modules is that of t-motives modulo isogeny. Here, k is a field containing $\mathbb{F}_q$, t is an indeterminate, and the (generally noncommutative) ring $k(t)\{\tau\}$ is defined to be the ring whose underlying abelian group is a (left) $k(t)$-vector space with basis $\{\tau^i\}_{i=0,1,\dots}$ and whose multiplication rule is:

$$\left(\sum_i f_i\tau^i\right)\left(\sum_j g_j\tau^j\right) = \sum_i\sum_j f_i\sigma^i(g_j)\tau^{i+j},$$

where σ is defined by:

$$\sigma(\sum_i c_it^i) = \sum_i c_i^q t^i.$$

Definition (1.1). Let M be a left $k(t)\{\tau\}$-module.
(i) We say that M is *restricted*, if $\dim_{k(t)} M < \infty$ and

$$\tau_{\mathrm{linear}} : M^{(q)} \stackrel{\mathrm{def}}{=} k(t) \underset{\sigma,k(t)}{\otimes} M \to M,\ f\otimes x \mapsto f\tau x$$

is an isomorphism.
(ii) Assume M to be restricted. Then we say that M is *étale* (at $t=0$), if there exists an $O_{k(t)}\{\tau\}$-submodule $\mathcal{M}$ of M which is finitely generated as an $O_{k(t)}$-module such that τ_{linear} induces an isomorphism from $\mathcal{M}^{(q)} \stackrel{\mathrm{def}}{=} O_{k(t)} \underset{\sigma,O_{k(t)}}{\otimes} \mathcal{M}$ to $\mathcal{M}$. Here $O_{k(t)} \stackrel{\mathrm{def}}{=} k(t)\cap k[[t]] = k[t]_{(t)}$.

Remark (1.2).
(i) Similarly, we define the concept of restricted and étale restricted $F\{\tau\}$ -modules for each subfield F of $k((t))$ containing $k(t)$ with $\sigma(F)\subset F$. Examples of such F

are: $k((t))$, $Q \stackrel{\text{def}}{=} \mathrm{Frac}(k \underset{\mathbb{F}_q}{\otimes} \mathbb{F}_q((t)))$, $Q^h \stackrel{\text{def}}{=} \mathrm{Frac}(k \underset{\mathbb{F}_q}{\otimes} \mathbb{F}_q(t)^h)$, etc., where $\mathbb{F}_q(t)^h$ is the algebraic closure of $\mathbb{F}_q(t)$ in $\mathbb{F}_q((t))$.
(ii) In the definition above, the analogue of $(\mathbb{Q}, l, \mathbb{Q}_l)$ is $(\mathbb{F}_q(t), t, \mathbb{F}_q((t)))$. This is only for simplicity, and we can develop our theory for more general setting like [3].

Example (1.3). Let (G, ϕ) be a Drinfeld $\mathbb{F}_q[t]$-module or an abelian t-module of Anderson's. Then

$$M \stackrel{\text{def}}{=} k(t) \underset{k[t]}{\otimes} \mathrm{Hom}_{(\mathbb{F}_q\text{-module schemes}/k)}(G, \mathbb{G}_a)$$

becomes a restricted $k(t)\{\tau\}$-module. It is étale, unless the ideal (t) is the 'characteristic' of ϕ.

The following proposition gives a relation between restricted modules and t-adic Galois representations.

Proposition (1.4).
We have the following category equivalence:

$$\begin{array}{ccc} (\textit{étale restricted } k((t))\{\tau\}\textit{-modules}) & \simeq & (t\textit{-adic representations of } \mathrm{Gal}(k^{\mathrm{sep}}/k)) \\ M & \mapsto & \widehat{V}(M) \stackrel{\text{def}}{=} (k^{\mathrm{sep}}((t)) \underset{k((t))}{\otimes} M)^{\tau} \\ \widehat{D}(V) \stackrel{\text{def}}{=} (k^{\mathrm{sep}}((t)) \underset{\mathbb{F}_q((t))}{\otimes} V)^{\mathrm{Gal}(k^{\mathrm{sep}}/k)} & \leftarrow\!\shortmid & V. \end{array}$$

Here τ (resp. $\mathrm{Gal}(k^{\mathrm{sep}}/k)$) acts diagonally on $k^{\mathrm{sep}}((t)) \underset{k((t))}{\otimes} M$ (resp. $k^{\mathrm{sep}}((t)) \underset{\mathbb{F}_q((t))}{\otimes} V$), and $(\cdot)^{\tau}$ (resp. $(\cdot)^{\mathrm{Gal}(k^{\mathrm{sep}}/k)}$) means the τ-invariant (resp. $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-invariant) part. The action of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ on $\widehat{V}(M)$ (resp. τ on $\widehat{D}(V)$) is induced by its action on $k^{\mathrm{sep}}((t))$.

Definition (1.5). For an étale restricted $k(t)\{\tau\}$-module M, we write $\widehat{V}(M)$ instead of $\widehat{V}(k((t)) \underset{k(t)}{\otimes} M)$, and call it the ($t$-*adic*) *Tate module* of M. Similar notation is employed for an étale restricted $F\{\tau\}$-module. (cf. Remark (1.2)(i).)

The following example explains why we call $\widehat{V}(M)$ Tate module.

Example (1.6). In the case of Example (1.3), we have

$$\widehat{V}(M) \simeq V_t(G)^* = \mathrm{Hom}_{\mathbb{F}_q((t))}(V_t(G), \mathbb{F}_q((t))),$$

where

$$V_t(G) \stackrel{\text{def}}{=} (\varprojlim \mathrm{Ker}(\phi_{t^n} : G(\overline{k}) \to G(\overline{k}))) \underset{\mathbb{F}_q[[t]]}{\otimes} \mathbb{F}_q((t)).$$

§2. Tate conjecture and semisimplicity conjecture.

From now on, we assume that k is a *finitely generated* field over $\mathbb{F}_q$.

Theorem (2.1). (*Tate conjecture.*)
Let M and M' be étale restricted $k(t)\{\tau\}$-modules. Then,

$$\mathrm{Hom}_{k(t)\{\tau\}}(M,M') \underset{\mathbb{F}_q(t)}{\otimes} \mathbb{F}_q((t)) \simeq \mathrm{Hom}_{\mathbb{F}_q((t))[\mathrm{Gal}(k^{\mathrm{sep}}/k)]}(\widehat{V}(M),\widehat{V}(M')).$$

Theorem (2.2). (*Semisimplicity conjecture.*)
Let M be an étale restricted $k(t)\{\tau\}$-module, and assume that M is semisimple as a $k(t)\{\tau\}$-module. Then $\widehat{V}(M)$ is a semisimple $\mathbb{F}_q((t))[\mathrm{Gal}(k^{\mathrm{sep}}/k)]$-module.

Remark (2.3). In the semisimplicity conjecture, the assumption of semisimplicity of the $k(t)\{\tau\}$-module M excludes objects like semi-abelian varieties.

The outline of the proof of these theorems is given in the next section.

§3. 'Geometric' t-adic Galois representations.

The t-adic representations (of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$) attached to étale restricted $k(t)\{\tau\}$-modules or, more generally, those attached to étale restricted $Q^h\{\tau\}$-modules are worth calling *geometric* representations. (See Remark (1.2)(i) for the definition of Q^h and Q.)

Definition (3.1). We say that a t-adic representation of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ is *quasi-geometric*, if it is isomorphic to the t-adic representation attached to an étale restricted $Q\{\tau\}$-module.

Although we have not yet established any good theory of geometric t-adic representations, we have a good theory of quasi-geometric t-adic representations, as follows.

Remark (3.2). If k is finite, all t-adic representations are quasi-geometric, since Q then coincides with $k((t))$.

Now we have the following diagrams of categories and functors:

$$\begin{array}{ccc}
(\text{étale restricted } k(t)\{\tau\}\text{-modules}) & & \\
Q \underset{k(t)}{\otimes} \cdot \downarrow & & \\
(\text{étale restricted } Q\{\tau\}\text{-modules}) & \longrightarrow & (\text{quasi-geometric } t\text{-adic representations}) \\
k((t)) \underset{Q}{\otimes} \cdot \downarrow & & \cap \\
(\text{étale restricted } k((t))\{\tau\}\text{-modules}) & \simeq & (t\text{-adic representations}).
\end{array}$$

Lemma (3.3).
(i) *Let M and M' be étale restricted $k(t)\{\tau\}$-modules. Then,*

$$\mathrm{Hom}_{k(t)\{\tau\}}(M,M') \underset{\mathbb{F}_q(t)}{\otimes} \mathbb{F}_q((t)) \simeq \mathrm{Hom}_{Q\{\tau\}}(Q \underset{k(t)}{\otimes} M, Q \underset{k(t)}{\otimes} M').$$

(ii) *Let M be an étale restricted $k(t)\{\tau\}$-module, and assume that M is semisimple as a $k(t)\{\tau\}$-module. Then $Q \underset{k(t)}{\otimes} M$ is a semisimple $Q\{\tau\}$-module.*

This lemma, which is rather easy to prove, reduces the Tate conjecture (2.1) and the semisimplicity conjecture (2.2) to the following:

Theorem (3.4).
(i) *The functor*

$$(\textit{étale restricted } Q\{\tau\}\textit{-modules}) \xrightarrow{k((t))\underset{Q}{\otimes}\cdot} (\textit{étale restricted } k((t))\{\tau\}\textit{-modules})$$

is fully faithful.
(ii) *The subcategory (quasi-geometric t-adic representations) is stable under taking subquotients in the category (t-adic representations).*

Our proof of this theorem borrows a technique in p-adic Hodge theory. The main point is to construct a commutative ring B, which is a subring of $k^{\mathrm{sep}}((t))$ stable under the actions of τ and $\mathrm{Gal}(k^{\mathrm{sep}}/k)$, satisfying the following properties:
(i) $B^{\tau} = \mathbb{F}_q((t))$.
(ii) $B^{\mathrm{Gal}(k^{\mathrm{sep}}/k)} = Q$.
(iii) For each étale restricted $Q\{\tau\}$-module M, the canonical isomorphism

$$k^{\mathrm{sep}}((t)) \underset{\mathbb{F}_q((t))}{\otimes} \widehat{V}(M) \simeq k^{\mathrm{sep}}((t)) \underset{Q}{\otimes} M$$

comes from a (unique) isomorphism

$$B \underset{\mathbb{F}_q((t))}{\otimes} \widehat{V}(M) \simeq B \underset{Q}{\otimes} M.$$

Remark (3.5). Roughly speaking, the condition (iii) says that B contains the entries of a 'period matrix' of M.

Theorem (3.4)(i) follows directly from the properties of B. In fact, the inverse map of

$$\begin{aligned}\mathrm{Hom}_{Q\{\tau\}}(M, M') &\to \mathrm{Hom}_{k((t))\{\tau\}}(k((t)) \underset{Q}{\otimes} M, k((t)) \underset{Q}{\otimes} M') \\ &= \mathrm{Hom}_{\mathbb{F}_q((t))[\mathrm{Gal}(k^{\mathrm{sep}}/k)]}(\widehat{V}(M), \widehat{V}(M'))\end{aligned}$$

is defined to map $f \in \mathrm{Hom}_{\mathbb{F}_q((t))[\mathrm{Gal}(k^{\mathrm{sep}}/k)]}(\widehat{V}(M), \widehat{V}(M'))$ to the restriction of $\mathrm{id}_B \otimes f : B \underset{\mathbb{F}_q((t))}{\otimes} \widehat{V}(M) \to B \underset{\mathbb{F}_q((t))}{\otimes} \widehat{V}(M')$ to the $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-invariant parts.

Definition (3.6). For each t-adic representation V of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$, we define

$$D(V) = (B \underset{\mathbb{F}_q((t))}{\otimes} V)^{\mathrm{Gal}(k^{\mathrm{sep}}/k)}.$$

From the properties of B, we can easily deduce the following theorem, which completes the proof of Theorem (3.4)(ii).

Theorem (3.7).
Let V be a t-adic representation of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$. *Then the following are equivalent:*

(i) *V is quasi-geometric;*
(ii) $\dim_Q D(V) = \dim_{\mathbb{F}_q((t))} V$;
(iii) $k((t)) \underset{Q}{\otimes} D(V) \simeq \widehat{D}(V)$.

In particular, any subquotients of a quasi-geometric representation are again quasi-geometric.

Finally, we mention the construction of the ring B. Fix a proper normal model X of k over $\mathbb{F}_q$, and define Σ to be the set of the points of codimension 1 in X. Let X^{sep} be the normalization of X in k^{sep}, and define Σ^{sep} to be the set of the points of codimension 1 in X^{sep}. Denote by $w_{\bar{x}}$ the additive valuation of k^{sep} associated to $\bar{x} \in \Sigma^{\mathrm{sep}}$ (normalized as $w_{\bar{x}}(k^{\times}) = \mathbb{Z}$). Define the subring B^+ of $k^{\mathrm{sep}}((t))$ by: $f = \sum a_i t^i \in B^+ \iff$ for all $\bar{x} \in \Sigma^{\mathrm{sep}}$, $\{w_{\bar{x}}(a_i)\}_i$ is bounded below and, for almost all $\bar{x} \in \Sigma^{\mathrm{sep}}$, $w_{\bar{x}}(a_i) \geq 0$ for all i. Here 'for almost all $\bar{x} \in \Sigma^{\mathrm{sep}}$, ...' means 'there exists a finite subset Σ_0 of Σ and, for all $\bar{x} \in \Sigma^{\mathrm{sep}}$ not above Σ_0, ...'. Next define the subset S of $k^{\mathrm{sep}}((t))$ by

$$S = \{f \in k^{\mathrm{sep}}((t))^{\times} \mid \sigma(f)f^{-1} \in k \underset{\mathbb{F}_q}{\otimes} \mathbb{F}_q((t))\},$$

which turns out to be a multiplicative subset of B^+. Now the ring B is defined by

$$B = S^{-1}B^+.$$

References

[1] Y. Taguchi, *The Tate conjecture for t-motives*, preprint.
[2] ———, *On φ-modules*, preprint.
[3] A. Tamagawa, *The Tate conjecture for A-premotives*, preprint.
[4] ———, *Generalization of Anderson's t-motives and Tate conjecture*, 京都大学数理解析研究所講究録 **884** (1994), 154–159.

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