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A note on geometric changes of complete solutions of first order differential equations

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0. Introduction

Our purpose is to give a framework for understanding geometric changes of singularities appearing in solutions of completely integrable first order differential equations and then to study the special case of ordinary differential equations to get a well-known result.

This paper is closely related to the study [3] which uses Arnold's result([1], [2]). We consider first order differential equations in the context of contact geometry([5]). And we employ a method when classifying functions up to diffeomorphisms preserving discriminant sets, which uses explicit coordinate changes arising from vector fields preserving discriminant sets ([4]). We hope that the same method suffices (with suitable modifications) to describe generic changes of singularities of solutions for first order partial differential equations with complete integral.

We would like to thank Professor J. W. Bruce for introducing me to this method.

1. Complete solutions and discriminant sets

First we shall describe the geometric structure connected with first order differential equations following S. Izumiya's formulation([5]). Let $J^1(\mathbf{R}^n,\mathbf{R})$ be the 1-jet bundle of n-variables functions which may be considered as $\mathbf{R}^{2n+1}$ with natural coordinates given by $(x_1,\cdots,x_n,y,p_1,\cdots,p_n)$, where $(x_1,\cdots,x_n)$ is a coordinate system of $\mathbf{R}^n$. We have the natural projection $\pi\ :\ J^1(\mathbf{R}^n,\mathbf{R})\to\mathbf{R}^n\times\mathbf{R}\ ;\ \pi(x,y,p)=(x,y)$.

A system of first order differential equations (or, briefly, *an equation*) is defined to be an immersion germ $l:(\mathbf{R}^r,0)\to J^1(\mathbf{R}^n,\mathbf{R})$, where $n+1\leq r\leq 2n$. Let θ be the canonical contact form on $J^1(\mathbf{R}^n,\mathbf{R})$ which is given by $\theta=dy-\sum_{i=1}^n p_i dx_i$. By the philosophy of Lie, we may define the notion of solutions as follows. *An (abstract) solution of l* is a Legendrian immersion $i : L \to J^1(\mathbf{R}^n, \mathbf{R})$ such that $i(L) \subset l(\mathbf{R}^r)$, where L is a n-dimensional manifold and *the Legendrian immersion* is an immersion $i : L \to J^1(\mathbf{R}^n, \mathbf{R})$ such that $i^*\theta = 0$.

Let $f : \mathbf{R}^n \to \mathbf{R}$ be a smooth function. Then $j^1 f : \mathbf{R}^n \to J^1(\mathbf{R}^n, \mathbf{R})$ is a Legendrian embedding. Hence, in our terminology, the (classical) solution of l is a smooth function f such that $j^1 f(\mathbf{R}^n) \subset l(\mathbf{R}^r)$. On the other hand, we can show that an (abstract) solution $i : L \to J^1(\mathbf{R}^n, \mathbf{R})$ is given by (at least locally) a jet extension $j^1 f$ of a smooth function f if and only if $\pi \circ i$ is a non-singular map. Thus the graph of the (abstract) solution $\pi \circ i(L)$ in $\mathbf{R}^n \times \mathbf{R}$ may have singularities.

We say that l is *completely integrable* (or *l has an (abstract) complete solution*) if there exists a submersion germ $\mu = (\mu_1, \cdots, \mu_{r-n}) : (\mathbf{R}^r, 0) \to \mathbf{R}^{r-n}$ such that $l_t = l \mid \mu^{-1}(t) : \mu^{-1}(t) \to J^1(\mathbf{R}^n, \mathbf{R})$ is an abstract solution of l for any $t \in \mathbf{R}^{r-n}$. Then μ is called *a complete integral* of l and the pair $(\mu, l) : (\mathbf{R}^r, 0) \to \mathbf{R}^{r-n} \times J^1(\mathbf{R}^n, \mathbf{R})$ is called *an equation germ with complete integral.*

In order to study generic types of singularities appearing in solutions of completely integrable equations, we now introduce a natural equivalence relation among equations with complete integral([5]). Let $(\mu, l) : (\mathbf{R}^r, 0) \to (\mathbf{R}^{r-n} \times J^1(\mathbf{R}^n, \mathbf{R}), (t_0, (x_0, y_0, p_0)))$ and $(\mu', l') : (\mathbf{R}^r, 0) \to (\mathbf{R}^{r-n} \times J^1(\mathbf{R}^n, \mathbf{R}), (t_1, (x_1, y_1, p_1)))$ be equation germs with complete integral. We say that (μ, l) and (μ', l') are *equivalent as equations with complete integral* if there exist diffeomorphism germs $\phi : (\mathbf{R}^{r-n}, t_0) \to (\mathbf{R}^{r-n}, t_1)$, $\Phi : (\mathbf{R}^r, 0) \to (\mathbf{R}^r, 0)$, $\kappa : (\mathbf{R}^n \times \mathbf{R}, (x_0, y_0)) \to (\mathbf{R}^n \times \mathbf{R}, (x_1, y_1))$ and a contact diffeomorphism germ $K : (J^1(\mathbf{R}^n, \mathbf{R}), (x_0, y_0, p_0)) \to (J^1(\mathbf{R}^n, \mathbf{R}), (x_1, y_1, p_1))$ such that the following diagram is commute:

$$
\begin{array}{ccccccc}
(\mathbf{R}^{r-n}, t_0) & \xleftarrow{\mu} & (\mathbf{R}^r, 0) & \xrightarrow{l} & (J^1(\mathbf{R}^n, \mathbf{R}), (x_0, y_0, p_0)) & \xrightarrow{\pi} & \mathbf{R}^n \times \mathbf{R} \\
\downarrow \phi & & \downarrow \Phi & & \downarrow K & & \downarrow \kappa \\
(\mathbf{R}^{r-n}, t_1) & \xleftarrow{\mu'} & (\mathbf{R}^r, 0) & \xrightarrow{l'} & (J^1(\mathbf{R}^n, \mathbf{R}), (x_1, y_1, p_1)) & \xrightarrow{\pi} & \mathbf{R}^n \times \mathbf{R}
\end{array}
$$

Let $f : (\mathbf{R}^{r-n} \times \mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ be a function germ such that $rank(\partial f / \partial t_i, \partial^2 f / \partial t_i \partial q_j) = r - n$. We call such a function germ *a complete family of function germs.* We now define a map germ $L_f : (\mathbf{R}^{r-n} \times \mathbf{R}^n, 0) \to J^1(\mathbf{R}^n, \mathbf{R})$ by $L_f(t, q) = (\partial f / \partial q(t, q), \sum_{i=1}^n \partial f / \partial q_i(t, q) \cdot q_i - f(t, q), q)$.

Then L_f is an immersion germ if and only if f is a complete family of function germs. Hence (π_1, L_f) is an equation germ with complete integral, where $\pi_1(\mathbf{R}^{r-n} \times \mathbf{R}^n, 0) \to (\mathbf{R}^{r-n}, 0)$ is the canonical projection. Then we have the following proposition.

Proposition 1.1. ([5]). *Let $(\mu, l) : (\mathbf{R}^r, 0) \to (\mathbf{R}^{r-n} \times J^1(\mathbf{R}^n, \mathbf{R}), (t_0, (x_0, y_0, p_0)))$ be an equation germ with complete integral. Then there exists a complete family of function germs $f : (\mathbf{R}^{r-n} \times \mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ such that (μ, l) and (π_1, L_f) are equivalent as equations with complete integral.*

This proposition guaranties that it is enough to study L_f for studying singularities of solutions of equations with complete integral.

Now we show how the graphs of abstract complete solutions of equations relate to discriminant sets of an unfolding of some function (a family of height functions).

Let $f : (\mathbf{R}^{r-n} \times \mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ be a complete family of function germs.

We consider the following set:

$\Sigma^f = \{(t, \partial f/\partial q(t,q), \partial f/\partial q(t,q)\cdot q - f(t,q)) \mid t \in \mathbf{R}^{r-n}, q \in \mathbf{R}^n\} \subset \mathbf{R}^{r-n} \times \mathbf{R}^n \times \mathbf{R}$.

For a fixed $t \in \mathbf{R}^{r-n}$, $\Sigma^f_t = \Sigma^f \cap \{t\} \times \mathbf{R}^n \times \mathbf{R}$ is the graph of an abstract solution of the completely integrable equation L_f and is clearly the affine dual of the graph $\Gamma^f_t = \{(q, f(t,q)) \mid q \in \mathbf{R}^n\}$ of f. So we refer to the assembled family of duals Σ^f as *the big dual.*

The big dual can be studied by considering the following $(r-n)$ parameter family of height functions([2],[3]),

$H_f : \mathbf{R}^n \times (\mathbf{R}^{r-n} \times S^n \times \mathbf{R}) \to \mathbf{R}$, where $H_f(q,t,u,z) = (q, f(t,q)).u - z$, S^n is the unit vectors in $\mathbf{R}^{n+1}$ and . denotes the usual inner product in $\mathbf{R}^{n+1}$.

Since we are only interested in these graphs near $(0, f(0,0))$, we consider the germ

$$F : \mathbf{R}^n \times (\mathbf{R}^{r-n} \times \mathbf{R}^n \times \mathbf{R}), (0,0,0,0) \to \mathbf{R}, 0 \quad \text{defined by}$$

$$F(q,t,\lambda,z) = (q, f(t,q)).(\lambda_1 - \partial f/\partial q_1(0,0), \lambda_2 - \partial f/\partial q_2(0,0), \cdots, \lambda_n - \partial f/\partial q_n(0,0), 1) - z.$$

We can naturally regard F as a $(r+1)$-parameter unfolding of $F_0(q) = F(q,0,0,0) : (\mathbf{R}^n, 0) \to \mathbf{R}, 0$. Then *the discriminant set* of F is (by definition) the set germ

$$D_F, 0 = \{(t,\lambda,z) \in \mathbf{R}^{r-n} \times \mathbf{R}^n \times \mathbf{R} \mid F(q,t,\lambda,z) = \partial F/\partial q(q,t,\lambda,z) = 0 \textit{ for some } q\}, 0.$$

Geometrically the discriminant set can be thought of as the big dual, that is, the sections $t=$ constant of D_F are locally diffeomorphic to the duals Σ^f_t of the graphs Γ^f_t.

Therefore in order to see geometrically how the graphs of abstract complete solution of equations change, we need to consider the natural projection germ of the discriminant set D_F to the t-parameter, i.e. $p_1 : (\mathbf{R}^{r+1}, D_F), 0 \to \mathbf{R}^{r-n}, 0 : p_1(t,\lambda,z) = t$.

2. Functions on discriminant sets

In this section we study the special case $n = 1$ and $r = 2$, i.e. the case of ordinary differential equations. Then we need to consider the following 3-parameter unfolding.

$F : \mathbf{R} \times \mathbf{R}^3, (0,0) \to \mathbf{R}, 0$ given by $F(q,t,\lambda,z) = (q, f(t,q)).(\lambda - \partial f/\partial q(0,0), 1) - z$, where $f : (\mathbf{R} \times \mathbf{R}, 0) \to (\mathbf{R}, 0)$ is a complete family of function germs, i.e. $rank(\partial f/\partial t, \partial^2 f/\partial t\partial q) \mid_0 = 1$.

First we consider the discriminant set D_F of F for "generic" complete family of function germs f. We shall define the genericity of complete family of function germs ([5]). Let $U \times V$ be an open subset of $\mathbf{R} \times \mathbf{R}$ and $CF(U \times V, \mathbf{R}) = \{f \in C^\infty(U \times V, \mathbf{R}) \mid rank(f_t, f_{tq}) = 1$ at any $(t,q) \in U \times V\}$. By Proposition 1.1 we may consider that $CF(U \times V, \mathbf{R})$ is the space of equations with complete integral. A subset of $CF(U \times V, \mathbf{R})$ is called generic if it is open and dense in $CF(U \times V, \mathbf{R})$.

Let P be a property of complete family of function germs $f : \mathbf{R} \times \mathbf{R}, 0 \to \mathbf{R}$. The property P is said to be *generic* if for some neighbourhood $U \times V$ of 0 in $\mathbf{R} \times \mathbf{R}$ the set $P(U \times V) = \{f \in CF(U \times V, \mathbf{R}) \mid$ the germ $f : (U \times V, (t, q)) \to \mathbf{R}$ has the property P for any point $(t, q) \in U \times V\}$ is generic in $CF(U \times V, \mathbf{R})$.

Now we obtain the following:

Proposition 2.1. *For a generic complete family of function germs f, F contains only A_1, A_2 and A_3 singularities and all these singularities are versally unfolded by F.*

Proof. $F_0 = F(-, 0)$ has an A_k $(k \geq 1)$ singularity at $q = 0$ if $f^{(2)}(0,0) = \cdots = f^{(k)}(0,0) = 0$ and $f^{(k+1)}(0,0) \neq 0$.

For $\partial F/\partial z(q,0) = -1$, $\partial F/\partial \lambda(q,0) = q$ and $\partial F/\partial t(q,0) = \partial f/\partial t(0,q)$, so A_1 and A_2 singularities are always versally unfolded and an A_3 is versally unfolded if and only if $\partial^3 f/\partial q^2 \partial t(0,0) \neq 0$.

We now define subsets of $J^4(\mathbf{R} \times \mathbf{R}, \mathbf{R})$ as follows: $S_1 = \{\partial^2 f/\partial q^2(t,q) = \partial^3 f/\partial q^3(t,q) = \partial^4 f/\partial q^4(t,q) = 0\}$, $S_2 = \{\partial^2 f/\partial q^2(t,q) = \partial^3 f/\partial q^3(t,q) = \partial^3 f/\partial q^2 \partial t(t,q) = 0\}$. Then consider $j^4 f : \mathbf{R} \times \mathbf{R}, 0 \to J^4(\mathbf{R} \times \mathbf{R}, \mathbf{R})$. By the transversality theorem we get the result.

Then for generic complete family of function germs f, the discriminant set germ D_F at 0 of F is diffeomorphic to a plane, cuspidal edge or swallowtail in 3-space. To see how the duals change, we need to consider the natural projection of these discriminant sets D_F to the t parameter, i.e. $p_1 : (\mathbf{R}^3, D_F), 0 \to \mathbf{R}, 0$, where $p_1(t, \lambda, z) = t$.

Let G be the standard versal unfolding $G(q, b) = \pm q^{k+1} + b_1 q^{k-1} + \cdots + b_{k-1} q + b_k$, where $b \in \mathbf{R}^3$ and $1 \leq k \leq 3$. Using the fact that F is a versal unfolding of an A_k-singularity$(k \leq 3)$ we can find smooth germs $\phi : \mathbf{R} \times \mathbf{R}^3, 0 \to \mathbf{R}, 0$, $\psi : \mathbf{R}^3 \to \mathbf{R}^3, 0$ with $\phi(-,0) : \mathbf{R}, 0 \to \mathbf{R}, 0$ a diffeomorphism, ψ a diffeomorphism and $F(\phi(q,b), \psi(b)) = G(q,b)$. The discriminant set of G is mapped by ψ to the discriminant set of F (ψ being *a discriminant preserving diffeomorphism*), and ψ_1, the first component of ψ, is the function on D_G corresponding to the natural projection p_1 of D_F to the t-axis, as in the following commutative diagram:

$$\begin{array}{ccccccc}
\mathbf{R} \times \mathbf{R}^3 & \xrightarrow{(F,id)} & \mathbf{R} \times \mathbf{R}^3 & \xrightarrow{p} & \mathbf{R}^3 & \xrightarrow{p_1} & \mathbf{R} \\
\uparrow (\phi, \psi) & & \uparrow id \times \psi & & \uparrow \psi & \nearrow \psi_1 & \\
\mathbf{R} \times \mathbf{R}^3 & \xrightarrow{(G,id)} & \mathbf{R} \times \mathbf{R}^3 & \xrightarrow{p} & \mathbf{R}^3 & & ,
\end{array}$$

where $p : \mathbf{R} \times \mathbf{R}^3 \to \mathbf{R}^3$ is the natural projection.

We use the discriminant preserving diffeomorphism ψ to study the function ψ_1 instead of the natural projection p_1 (see [2], the proof of Theorem 1.2).

Proposition 2.2. *Let $F(q,t,\lambda,z)$ and $\psi_1(b)$ be as above.*

(1)If F is a versal unfolding of A_1-singularity and $G(q,b) = q^2 + b_1$, then we have $\partial\psi_1/\partial b_2(0) \neq 0$ or $\partial\psi_1/\partial b_3(0) \neq 0$.

(2)If F is a versal unfolding of A_2-singularity and $G(q,b) = q^3 + b_1 q + b_2$, then we have $\partial\psi_1/\partial b_3(0) \neq 0$.

(3)If F is a versal unfolding of A_3-singularity and $G(q,b) = q^4 + b_1 q^2 + b_2 q + b_3$, then we have $\partial\psi_1/\partial b_1(0) \neq 0$.

Proof. (1) From the chain rule we find that

$\frac{\partial F}{\partial q}(\phi(q,0),0)\frac{\partial\phi}{\partial b_i}(q,0)+\frac{\partial F}{\partial t}(\phi(q,0),0)\frac{\partial\psi_1}{\partial b_i}(0)+\frac{\partial F}{\partial\lambda}(\phi(q,0),0)\frac{\partial\psi_2}{\partial b_i}(0)+\frac{\partial F}{\partial z}(\phi(q,0),0)\frac{\partial\psi_3}{\partial b_i}(0) = \delta_{1i}$, where δ is the usual Kronecker symbol and $i = 1 \sim 3$.

Since F is an unfolding of an A_1-singularity, $\partial F/\partial q(\phi(q,0),0)$ has a Taylor series starting with terms of degree at least 1. So we get

$\frac{\partial F}{\partial t}(\phi(q,0),0)\frac{\partial\psi_1}{\partial b_i}(0) + \frac{\partial F}{\partial\lambda}(\phi(q,0),0)\frac{\partial\psi_2}{\partial b_i}(0) + \frac{\partial F}{\partial z}(\phi(q,0),0)\frac{\partial\psi_3}{\partial b_i}(0) \equiv \delta_{1i} \ mod\langle q\rangle$.

If $\partial F/\partial t(\phi(q,0),0) \equiv a_1, \partial F/\partial\lambda(\phi(q,0),0) \equiv a_2$ *and* $\partial F/\partial z(\phi(q,0),0) \equiv a_3 \ mod\langle q\rangle$, then $(a_1,a_2,a_3)(\partial\psi_i/\partial b_j(0))_{i,j=1,2,3} = (1,0,0)$, *where* $a_3 \neq 0$. If $\partial\psi_1/\partial b_2(0) = \partial\psi_1/\partial b_3(0) = 0$, then $(\partial\psi_i/\partial b_j(0))_{i,j=2,3}$ is regular and $(a_2,a_3)(\partial\psi_i/\partial b_j(0))_{i,j=2,3} = (0,0)$. Therefore $a_2 = a_3 = 0$, which is a contradiction. Hence we have $\partial\psi_1/\partial b_2(0) \neq 0$ or $\partial\psi_1/\partial b_3(0) \neq 0$.

(2) In the same way as in (1) we get

$\frac{\partial F}{\partial t}(\phi(q,0),0)\frac{\partial\psi_1}{\partial b_i}(0) + \frac{\partial F}{\partial\lambda}(\phi(q,0),0)\frac{\partial\psi_2}{\partial b_i}(0) + \frac{\partial F}{\partial z}(\phi(q,0),0)\frac{\partial\psi_3}{\partial b_i}(0) \equiv \begin{cases} q^{2-i} & (i=1,2) \\ 0 & (i=3) \end{cases} mod\langle q^2\rangle$,

If $\partial F/\partial t(\phi(q,0),0) \equiv a_{11}q + a_{12}, \partial F/\partial\lambda(\phi(q,0),0) \equiv a_{21}q + a_{22}$ *and* $\partial F/\partial z(\phi(q,0),0) \equiv a_{31}q + a_{32} \ mod\langle q^2\rangle$, then

$$\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{pmatrix} (\partial\psi_i/\partial b_j(0))_{i,j=1,2,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

where $\begin{pmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{pmatrix}$ is regular. If $\partial\psi_1/\partial b_3(0) = 0$, then $\partial\psi_2/\partial b_3(0) \neq 0$ (or $\partial\psi_3/\partial b_3(0) \neq 0$) and $\begin{pmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{pmatrix} \begin{pmatrix} \partial\psi_2/\partial b_3(0) \\ \partial\psi_3/\partial b_3(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Therefore $\partial\psi_2/\partial b_3(0) = \partial\psi_3/\partial b_3(0) = 0$, which is a contradiction. Hence we have $\partial\psi_1/\partial b_3(0) \neq 0$.

(3) In the same way as in (1) we get

$\frac{\partial F}{\partial t}(\phi(q,0),0)\frac{\partial\psi_1}{\partial b_i}(0)+\frac{\partial F}{\partial\lambda}(\phi(q,0),0)\frac{\partial\psi_2}{\partial b_i}(0)+\frac{\partial F}{\partial z}(\phi(q,0),0)\frac{\partial\psi_3}{\partial b_i}(0) \equiv q^{3-i} \ mod\langle q^3\rangle \ (i = 1 \sim 3)$.

If $\partial F/\partial t(\phi(q,0),0) \equiv a_{11}q^2 + a_{12}q + a_{13}$, $\partial F/\partial\lambda(\phi(q,0),0) \equiv a_{21}q^2 + a_{22}q + a_{23}$ and $\partial F/\partial z(\phi(q,0),0) \equiv a_{31}q^2 + a_{32}q + a_{33} \ mod\langle q^3\rangle$, then

$$\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} (\partial\psi_i/\partial b_j(0))_{i,j=1,2,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $(a_{ij})_{i,j=1,2,3}$ and $\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$ are regular. If $\partial\psi_1/\partial b_1(0) = 0$, then $\partial\psi_2/\partial b_1(0) \neq$

0 (or $\partial\psi_3/\partial b_1(0) \neq 0$) and $\begin{pmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} \partial\psi_2/\partial b_1(0) \\ \partial\psi_3/\partial b_1(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Therefore $\partial\psi_2/\partial b_1(0) = \partial\psi_3/\partial b_1(0) = 0$, which is a contradiction. Hence we have $\partial\psi_1/\partial b_1(0) \neq 0$. This completes the proof.

Using the conditions on ψ_1 of Proposition 2.2 we classify function germs $\psi_1 : (\mathbf{R}^3, 0) \to (\mathbf{R}, 0)$ up to local diffeomorphisms of $\mathbf{R}^3$ preserving the standard discriminant set D_G of G. Then we get the following.

Proposition 2.3. *Let $\psi_1(b)$ be as above.*

(1)If $G(q,b) = q^2 + b_1$ and $\partial\psi_1/\partial b_2(0) \neq 0$ (or $\partial\psi_1/\partial b_3(0) \neq 0$), then ψ_1 is equivalent, via a discriminant preserving diffeomorphism, to the trivial projection onto b_2-coordinate (or b_3-coordinate) of a product discriminant set (i.e. a plane).

(2)If $G(q,b) = q^3 + b_1 q + b_2$ and $\partial\psi_1/\partial b_3(0) \neq 0$, then ψ_1 is equivalent, via a discriminant preserving diffeomorphism, to the trivial projection onto b_3-coordinate of a product discriminant set (i.e. a cuspidal edge).

(3)If $G(q,b) = q^4 + b_1 q^2 + b_2 q + b_3$ and $\partial\psi_1/\partial b_1(0) \neq 0$, then ψ_1 is equivalent, via a discriminant preserving diffeomorphism, to the projection of the standard discriminant set (i.e. the swallowtail) onto b_1-coordinate. We call it the standard swallowtail projection.

Proof. The standard method of obtaining diffeomorphisms is to integrate smooth vector fields. If the diffeomorphism is to preserve the discriminant, then the vector fields must be tangent to the discriminant (in the sense of being tangent to the smooth strata in the natural stratification of the discriminant).

(1) The discriminant of $G(q,b)$ is the set $D_G = \{(0, b_2, b_3)\}$. Then we can obtain a free basis for the $\mathcal{E}(3,1)$-module of vector fields tangent to the set D_G as follows([4]):

$$\Omega = \mathcal{E}(3,1)\{b_1\partial/\partial b_1, \partial/\partial b_2, \partial/\partial b_3\},$$

where $\mathcal{E}(3,1)$ is the ring of C^∞ map-germs $f : (\mathbf{R}^3, 0) \to \mathbf{R}^1$. Then the integration yields diffeomorphisms $\phi : (\mathbf{R}^3, 0) \to (\mathbf{R}^3, 0)$ preserving D_G, whose 1-jet $j^1\phi(0)$ are

$$\begin{cases} (b_1, b_2, b_3) \to (b_1, k_1 b_1 + l_{11} b_2 + l_{12} b_3, k_2 b_1 + l_{21} b_2 + l_{22} b_3), & \textit{where } det(l_{ij})_{i,j=1,2} \neq 0, \\ (b_1, b_2, b_3) \to (k b_1, b_2, b_3), & \textit{where } k \neq 0. \end{cases}$$

Let $j^1\psi_1(0) = c_1 b_1 + c_2 b_2 + c_3 b_3$, where $c_2 \neq 0$ or $c_3 \neq 0$. Hence changing coordinates $(b_1, b_2, b_3) \to (b_1, c_1 b_1 + c_2 b_2 + c_3 b_3, b_3)$ or $(b_1, b_2, b_3) \to (b_1, c_1 b_1 + c_2 b_2 + c_3 b_3, b_2)$ turns $j^1\psi_1(0)$ into $f : (b_1, b_2, b_3) \to b_2$, which satisfy the following determinacy condition $\Omega_0.f \supset \mathcal{M}_3$, where $\mathcal{M}_3$ is the maximal ideal of $\mathcal{E}(3,1)$ and $\Omega_0 = \{\xi \in \Omega : \xi \mid_0 = 0\}$. (We can get the following as in the similar way to the ordinary determinacy theorem. That is, if $\Omega_0.f \supset \mathcal{M}_3{}^k$, then f is k-determined with respect to D_G-preserving diffeomorphisms.) Therefore f is 1-determined with respect to the D_G-preserving diffeomorphisms and hence ψ_1 and f are equivalent, via a D_G-preserving diffeomorphism.

(2) The discriminant of $G(q,b)$ is the set $D_G = \{(b_1,b_2,b_3) \mid 4b_1^3 + 27b_2^2 = 0\}$. We can obtain a free basis for the $\mathcal{E}(3,1)$-module of vector fields tangent to the set D_G as follows([4]):

$$\Omega = \mathcal{E}(3,1)\{9b_2\partial/\partial b_1 - 2b_1^2\partial/\partial b_2, 2b_1\partial/\partial b_1 + 3b_2\partial/\partial b_2, \partial/\partial b_3\}.$$

Then the integration yields diffeomorphisms $\phi : (\mathbf{R}^3,0) \to (\mathbf{R}^3,0)$ preserving D_G, whose 1-jet $j^1\phi(0)$ are

$$\begin{cases} (b_1,b_2,b_3) \to (b_1,b_2,lb_1+mb_2+nb_3),\ \mathit{where}\ n \neq 0, \\ (b_1,b_2,b_3) \to (b_1+kb_2,b_2,b_3), \\ (b_1,b_2,b_3) \to (kb_1,lb_2,b_3),\ \mathit{where}\ k^3 = l^2\ (k,l>0). \end{cases}$$

Let $j^1\psi_1(0) = c_1b_1 + c_2b_2 + c_3b_3$, where $c_3 \neq 0$. Hence changing coordinates $(b_1,b_2,b_3) \to (b_1,b_2,c_1b_1+c_2b_2+c_3b_3)$ turns $j^1\psi_1(0)$ into $f : (b_1,b_2,b_3) \to b_3$, which satisfy the following determinacy condition $\Omega_0.f \supset \mathcal{M}_3$. Therefore f is 1-determined with respect to the D_G-preserving diffeomorphisms and hence ψ_1 and f are equivalent, via a D_G-preserving diffeomorphism.

(3) The discriminant of $G(q,b)$ is the standard swallowtail set. We can obtain a free basis for the $\mathcal{E}(3,1)$-module of vector fields tangent to the set D_G as follows([4]):

$$\Omega = \mathcal{E}(3,1)\{2b_1\partial/\partial b_1 + 3b_2\partial/\partial b_2 + 4b_3\partial/\partial b_3, 6b_2\partial/\partial b_1 + (8b_3 - 2b_1^2)\partial/\partial b_2 - b_1b_2\partial/\partial b_3,$$
$$(16b_3 - 4b_1^2)\partial/\partial b_1 - 8b_1b_2\partial/\partial b_2 - 3b_2^2\partial/\partial b_3\}.$$

Then the integration yields diffeomorphisms $\phi : (\mathbf{R}^3,0) \to (\mathbf{R}^3,0)$ preserving D_G, whose 1-jet $j^1\phi(0)$ are

$$\begin{cases} (b_1,b_2,b_3) \to (b_1+3kb_2+6k^2b_3, b_2+4kb_3, b_3) & (i) \\ (b_1,b_2,b_3) \to (b_1+tb_3,b_2,b_3) & (ii) \\ (b_1,b_2,b_3) \to (kb_1,lb_2,mb_3),\ \mathit{where}\ k^3 = l^2,\ k^2 = m,\ l^4 = m^3\ (k,l,m \neq 0) & (iii) \end{cases}$$

Let $j^1\psi_1(0) = c_1b_1 + c_2b_2 + c_3b_3$, where $c_1 \neq 0$. By (iii) $k = |c_1|$, $j^1\psi_1(0)$ is equivalent to $\pm b_1 + c_2'b_2 + c_3'b_3$. By (ii) $t = \pm c_3'$ we get $\pm b_1 + c_2'b_2$. Then we get $\pm(b_1 - \frac{2}{3}c_2'^2b_3)$ by (i) $k = \pm\frac{1}{3}c_2'$. Finally by (ii) $t = -\frac{2}{3}c_2'^2$ we get $f : (b_1,b_2,b_3) \to \pm b_1$, which satisfy the following determinacy condition $\Omega_0.f \supset \mathcal{M}_3$. Therefore f is 1-determined with respect to the D_G-preserving diffeomorphisms and hence ψ_1 and f are equivalent, via a D_G-preserving diffeomorphism. This completes the proof.

From Proposition 1.1 $\sim$ 2.3, for almost all first order ordinary differential equations with complete integral the local models for the changes in the graphs of solutions are the followings.

(1) the graphs of solutions near q_0 are all diffeomorphic to lines.

(2) the graphs of solutions near q_0 are all diffeomorphic to cusps.

(3) the family of graphs of solutions near q_0 are obtained as sections of the standard swallowtail projection.

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