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Algebraic Kripke sheaf semantics for super-intuitionistic predicate logics

NOBU-YUKI SUZUKI
鈴木信行

Department of Computer and Information Sciences
Faculty of Engineering
Ibaraki University
Nakanarusawa, Hitachi 316
JAPAN

Abstract

In so-called Kripke-type models, each sentence is assigned either to *true* or to *false* at each possible world. In this setting, every possible world has the two-valued Boolean algebra as the set of the truth values. Instead, we take a collection of algebras each of which is attached to a world as the set of the truth values at the world, and obtain an extended semantics based on the traditional Kripke-type semantics, which we call here the *algebraic Kripke* semantics.

We introduce algebraic Kripke sheaf semantics for super-intuitionistic predicate logics, and discuss some basic properties. As application, we show a new result on super-intuitionistic predicate logics. We prove that there exists a continuum of super-intuitionistic predicate logics each of which has both of the disjunction and existence properties and moreover the same propositional fragment as the intuitionistic logic.

Introduction

In so-called Kripke-type models, we assign *true* or *false* to each sentence at each possible world. In this setting, each possible world w is supposed to have the two-valued Boolean algebra $\mathbf{2} = \{0, 1\}$ as the set of the truth values. Instead of $\mathbf{2}$, we take an algebra $P(w)$ for each possible world w as the set of the truth values at w. In the case that a possible world w is accessible from another world v, it is desirable that computational process in $P(w)$ can be traced in $P(v)$. For this request, we arrange an apparatus in category theory, and obtain an extended semantics based on the traditional Kripke-type semantics, which we call here the *algebraic Kripke* semantics. This kind of generalization yields non-Kripke-type semantics in the sense of Skvortsov-Shehtman [13].

In this article, we introduce *algebraic Kripke sheaf semantics* for super-intuitionistic predicate logics based on the Kripke sheaf semantics, which is introduced by Shehtman-Skvortsov [11] as an extension of the Kripke frame semantics. In this case, each $P(w)$ is a Heyting algebra, and the collection of $P(w)$'s shall form a functor whose codomain is an appropriate category with objects being Heyting algebras.

After this idea came to the author, Prof. Ono informed the author of that Nagai [8] introduced in 1973 an algebraic Kripke-type semantics based on the Kripke frame semantics such that all $P(w)$'s are identical to each other. Our algebraic Kripke sheaf semantics contains his semantics as a special case, and is proved to be properly more powerful than the Nagai semantics.

We discuss some basic properties and give a new result as application of this semantics. This result cannot be proved, at present, by using the Kripke sheaf semantics. We present a new method of constructing a continuum of super-intuitionistic predicate logics. This method is obtained from the modified Jankov method in [15] by re-modification. We show that there exists a continuum of super-intuitionistic predicate logics each of which has both of the disjunction and existence properties and moreover the same propositional fragment as the intuitionistic logic. For this aim, we must find a criterion for a logic to have both of the disjunction and existence properties. Our idea comes from an observation of the delta operation in super-intuitionistic predicate logics.

In section 1, we repeat some basic definitions and properties of Kripke sheaf semantics for super-intuitionistic predicate logics. We introduce algebraic Kripke sheaf semantics in section 2. Some concepts and results in the Kripke sheaf semantics can be repeated for this semantics. An advantage of this semantics is that the set of formulas valid in an algebraic Kripke sheaf is always closed under the rule of substitution. This property fails to be possessed by extended Kripke-type semantics presented in Ghilardi [2, 3], Shehtman-Skvortsov [11], and Skvortsov-Shehtman [13]. We give in section 3 concrete application of the algebraic Kripke sheaf semantics stated above.

1 Starting from the Kripke sheaf semantics

Basic definitions and properties of the Kripke sheaf semantics are stated here to make this article self-contained. We refer readers to [14, 15] for details.

We fix a first-order language $\mathcal{L}$, which consists of logical connectives $\vee$ (disjunction), $\wedge$ (conjunction), $\supset$ (implication), $\neg$ (negation), and quantifiers $\exists$ (existential quantifier) and $\forall$ (universal quantifier), a denumerable list of individual variables and a denumerable list of m-ary predicate variables for each $m < \omega$. As usual, 0-ary predicate variables are identified with propositional variables. Note that $\mathcal{L}$ contains neither individual constants nor function symbols. For each non-empty set U, we denote by $\mathcal{L}[U]$ the language obtained from $\mathcal{L}$ by adding the name $\overline{u}$ of each $u \in U$. In what follows, we will sometimes use the same letter u for the name of u. We sometimes identify

$\mathcal{L}[U]$ with the set of all sentences of $\mathcal{L}[U]$. For each super-intuitionistic propositional logic $\mathbf{J}$, we denote, by $\mathbf{J}_*$, the smallest predicate extension of $\mathbf{J}$, i.e., $\mathbf{J}_* = \mathbf{H}_* + \mathbf{J}$, where $\mathbf{H}_*$ is the intuitionistic predicate logic.

Definition 1.1 A partially ordered set $\mathbf{M} = \langle M, \leq \rangle$ with the least element $0_{\mathbf{M}}$ is said to be a *Kripke base.* We can regard a Kripke base $\mathbf{M}$ as a category in the usual way. Let $\mathcal{S}$ denote the category of all non-empty sets. A covariant functor D from a Kripke base $\mathbf{M}$ to $\mathcal{S}$ is called a *domain-sheaf* over $\mathbf{M}$. That is,

DS1) $D(a)$ is a non-empty set for every $a \in M$,
DS2) for every $a, b \in M$ with $a \leq b$, there exists a mapping $D_{ab} : D(a) \to D(b)$,
DS3) D_{aa} is the identity mapping of $D(a)$ for every $a \in M$,
DS4) $D_{ac} = D_{bc} \circ D_{ab}$ for every $a, b, c \in M$ with $a \leq b \leq c$.

A pair $\mathcal{K} = \langle \mathbf{M}, D \rangle$ of a Kripke base $\mathbf{M}$ and a domain-sheaf D over $\mathbf{M}$ is called a *Kripke sheaf.*

If every D_{ab} $(a \leq b)$ is the set-theoretic inclusion, $\langle \mathbf{M}, D \rangle$ is said to be a *Kripke frame* (for predicate logics). Intuitively, each $D(a)$ is the individual domain of the world $a \in M$. For each $d \in D(a)$ and each $b \in M$ with $a \leq b$, $D_{ab}(d)$ is said to be the *inheritor* of d at b. For each formula A of $\mathcal{L}[D(a)]$ and each $b \in M$ with $a \leq b$, the *inheritor* $A_{a,b}$ of A at b is a formula of $\mathcal{L}[D(b)]$ obtained from A by replacing occurrences of $\overline{u}$ $(u \in D(a))$ by the name $\overline{v}$ of the inheritor v of u at b.

A binary relation $\models$ between each $a \in M$ and each atomic sentence of $\mathcal{L}[D(a)]$ is said to be a *valuation* on $\langle \mathbf{M}, D \rangle$ if for every $a, b \in M$ and every atomic sentence A of $\mathcal{L}[D(a)]$, $a \models A$ and $a \leq b$ imply $b \models A_{a,b}$. We extend $\models$ to a relation between each $a \in M$ and each sentence of $\mathcal{L}[D(a)]$ inductively as follows:

- $a \models A \wedge B$ if and only if $a \models A$ and $a \models B$,
- $a \models A \vee B$ if and only if $a \models A$ or $a \models B$,
- $a \models A \supset B$ if and only if for every $b \in M$ with $a \leq b$, either $b \not\models A_{a,b}$ or $b \models B_{a,b}$,
- $a \models \neg A$ if and only if for every $b \in M$ with $a \leq b$, $b \not\models A_{a,b}$,
- $a \models \forall x A(x)$ if and only if for every $b \in M$ with $a \leq b$ and every $u \in D(b)$, $b \models A_{a,b}(\overline{u})$,
- $a \models \exists x A(x)$ if and only if there exists $u \in D(a)$ such that $a \models A(\overline{u})$.

A pair $(\mathcal{K}, \models)$ of a Kripke sheaf $\mathcal{K}$ and a valuation $\models$ on it is said to be a *Kripke-sheaf model.* A formula A of $\mathcal{L}$ is said to be *true* in a Kripke-sheaf model $(\mathcal{K}, \models)$ if $0_{\mathbf{M}} \models \overline{A}$, where $\overline{A}$ is the universal closure of A. A formula A of $\mathcal{L}$ is said to be *valid* in a Kripke sheaf $\mathcal{K}$ if for every valuation $\models$ on $\mathcal{K}$, A is true in $(\mathcal{K}, \models)$. The set of formulas of $\mathcal{L}$ valid in $\mathcal{K} = \langle \mathbf{M}, D \rangle$ is denoted by $L(\mathcal{K})$ or $L\langle \mathbf{M}, D \rangle$. The following propositions are fundamental properties of Kripke-sheaf semantics.

Proposition 1.2 *For every Kripke-sheaf model* $(\langle \mathbf{M}, D\rangle, \models)$, *every* $a, b \in M$, *and every sentence* $A \in \mathcal{L}[D(a)]$, *if* $a \models A$ *and* $a \leq b$, *then* $b \models A_{a,b}$.

Proposition 1.3 *For each Kripke-sheaf* $\mathcal{K}$, *the set* $L(\mathcal{K})$ *contains all formulas provable in* $\mathbf{H}_*$, *and is closed under the modus ponens, the rule of generalization and the rule of substitution. Namely,* $L(\mathcal{K})$ *is a super-intuitionistic predicate logic.*

The Kripke sheaf semantics is properly more powerful than the Kripke frame semantics, which is stated in the following.

Theorem 1.4 **([14])** *Let* Z *be a formula* $\exists x p(x) \supset \forall x p(x)$, *where* p *is a unary predicate variable. The logic* $\mathbf{H}_* + \neg\neg Z$ *is Kripke-sheaf complete, but not Kripke-frame complete.*

We can introduce p-morphisms between Kripke sheaves.

Definition 1.5 Let $\mathbf{M} = \langle M, \leq_{\mathbf{M}}\rangle$ and $\mathbf{N} = \langle N, \leq_{\mathbf{N}}\rangle$ be Kripke bases. A mapping f of M to N is said to be a *p-morphism* of $\mathbf{M}$ to $\mathbf{N}$, if for every $a, b \in M$, and every $c \in N$,

(1) $a \leq_{\mathbf{M}} b$ implies $f(a) \leq_{\mathbf{N}} f(b)$,
(2) if $f(a) \leq_{\mathbf{N}} c'$, that there exists a $c \in M$ such that $a \leq_{\mathbf{M}} c$ and $f(c) = c'$.

Lemma 1.6 *Let* $\langle \mathbf{M}, D\rangle$ *and* $\langle \mathbf{N}, E\rangle$ *be Kripke sheaves. Suppose that* f *is a p-morphism of* $\mathbf{M}$ *to* $\mathbf{N}$. *Then* $E \circ f$ *defined below is a covariant functor from* $\mathbf{M}$ *to* $\mathcal{S}$, *i.e., a domain-sheaf over* $\mathbf{M}$. *Hence* $\langle \mathbf{M}, E \circ f\rangle$ *is a Kripke sheaf:*

$$\begin{array}{rcll} (E \circ f)(a) & = & E(f(a)) & \text{for every } a \in M, \\ (E \circ f)_{ab} & = & E_{f(a)\,f(b)} & \text{for every } a, b \in M \text{ with } a \leq_{\mathbf{M}} b. \end{array}$$

Definition 1.7 Let $\langle \mathbf{M}, D\rangle$ and $\langle \mathbf{N}, E\rangle$ be Kripke sheaves. A pair (f, τ) of a p-morphism f of $\mathbf{M}$ to $\mathbf{N}$ and a natural transformation τ of D to $E \circ f$ is said to be a *p-morphism* of $\langle \mathbf{M}, D\rangle$ to $\langle \mathbf{N}, E\rangle$ if each $\tau_a : D(a) \to (E \circ f)(a)$ $(a \in M)$ is surjective. That is, τ assigns to each $a \in M$ a surjective mapping $\tau_a : D(a) \to (E \circ f)(a)$ in such a way that every $a, b \in M$ with $a \leq_{\mathbf{M}} b$ yield the following commutative diagram:

$$\begin{array}{ccc} D(a) & \xrightarrow{\tau_a} & (E \circ f)(a) \\ D_{ab} \downarrow & & \downarrow (E \circ f)_{ab} \\ D(b) & \xrightarrow[\tau_b]{} & (E \circ f)(b) \end{array}.$$

Theorem 1.8 **(the p-morphism theorem for Kripke sheaves)** *Let* $\mathcal{K}_1$ *and* $\mathcal{K}_2$ *be Kripke sheaves. If there exists a p-morphism of* $\mathcal{K}_1$ *to* $\mathcal{K}_2$, *then* $L(\mathcal{K}_1) \subseteq L(\mathcal{K}_2)$.

Let CD be the sentence $\forall x(p(x) \vee q) \supset (\forall x p(x) \vee q)$ where p is a unary predicate variable and q is a propositional variable. A Kripke frame $\langle \mathbf{M}, D\rangle$ is said to be *with constant domain*, if $D(a) = D(0_{\mathbf{M}})$ for every $a \in \mathbf{M}$. It is well-known that CD is valid in a Kripke frame if and only if it is with constant domain.

Definition 1.9 A Kripke sheaf $\langle \mathbf{M}, D\rangle$ is said to be *of type CD*, if D_{ab} is surjective for every $a, b \in M$ with $a \leq_{\mathbf{M}} b$.

Lemma 1.10 *The sentence CD is valid in a Kripke sheaf if and only if it is of type CD.*

Theorem 1.11 *(1) For every Kripke sheaf $\mathcal{K} = \langle \mathbf{M}, D\rangle$, there exist a Kripke frame $\mathcal{K}^* = \langle \mathbf{M}, D^*\rangle$ and a p-morphism of $\mathcal{K}^*$ to $\mathcal{K}$.*

(2) For every Kripke sheaf $\mathcal{K} = \langle \mathbf{M}, D\rangle$ of type CD, there exist a Kripke frame $\mathcal{K}^* = \langle \mathbf{M}, D^*\rangle$ with constant domain and a p-morphism of $\mathcal{K}^*$ to $\mathcal{K}$.

Corollary 1.12 *(1) For every intermediate propositional logic $\mathbf{J}$, $\mathbf{J}_*$ is Kripke-sheaf complete if and only if $\mathbf{J}_*$ is Kripke-frame complete.*

(2) For every intermediate propositional logic $\mathbf{J}$, $\mathbf{J}_* + CD$ is Kripke-sheaf complete if and only if $\mathbf{J}_* + CD$ is Kripke-frame complete.

2 Algebraic Kripke sheaf semantics

In the definition of Kripke sheaf model $(\langle \mathbf{M}, D\rangle, \models)$, we can replace a relation $\models$ by a function V whose codomain is the two-valued Boolean algebra $\mathbf{2} = \{0, 1\}$. Namely,

$$V(a, A) = \begin{cases} 1 & \text{if } a \models A \\ 0 & \text{if } a \not\models A. \end{cases}$$

for every $a \in \mathbf{M}$ and every sentence $A \in \mathcal{L}[D(a)]$. In this setting, each a $(\in \mathbf{M})$ has $\mathbf{2}$ as the set of the truth values. Instead of $\mathbf{2}$, we take an algebra $P(a)$ for each a $(\in \mathbf{M})$ as the set of the truth values at a. It is natural that $P(a)$ coincides with an algebraic semantics, whenever $\mathbf{M}$ is a singleton $\{a\}$. Since we intend to provide semantics for super-intuitionistic predicate logics, $P(a)$'s must be Heyting algebras.

If a is accessible from b ($b \leq_{\mathbf{M}} a$), computation in $P(a)$ is to be traced in $P(b)$. For this request, we take a monomorphism[1] of $P(b)$ into $P(a)$. We design that the collection $\{P(a)\ ;\ a \in \mathbf{M}\}$ forms a functor with domain $\mathbf{M}$, whose codomain shall be an appropriate category with objects being Heyting algebras.

[1]One might just as well try to take a homomorphism instead of a monomorphism. In the Kripke frame semantics for super-intuitionistic predicate logics, we have a fundamental request that if A is true at b and $b \leq_{\mathbf{M}} a$, then A is true at a. In our semantics, this is translated into: if $V(b, A) = 1$ and $b \leq_{\mathbf{M}} a$, then $V(a, A) = 1$. This request corresponds to the condition that each homomorphism $h : P(a) \rightarrow P(b)$ satisfies that $h^{-1}(1) = \{1\}$. This is equivalent to that h is a monomorphism.

Definition 2.1 Let $\mathcal{H}$ denote the category of all non-degenerate complete Heyting algebras with arrows being complete monomorphisms between complete Heyting algebras. A contravariant functor P from a Kripke base $\mathbf{M} = \langle M, \leq \rangle$ to $\mathcal{H}$ is called a *truth-value-sheaf* over $\mathbf{M}$. That is,

TVS1) $P(a)$ is a non-degenerate complete Heyting algebra for every $a \in M$,
TVS2) for every $a, b \in M$ with $a \leq b$, there exists a complete monomorphism $P_{ab} : P(b) \to P(a)$,
TVS3) P_{aa} is the identity mapping of $P(a)$ for every $a \in M$,
TVS4) $P_{ac} = P_{ab} \circ P_{bc}$ for every $a, b, c \in M$ with $a \leq b \leq c$.

A triple $\mathcal{K} = \langle \mathbf{M}, D, P \rangle$ of a Kripke base $\mathbf{M}$, a domain-sheaf D over $\mathbf{M}$, and a truth-value-sheaf P is called an *algebraic Kripke sheaf.*

Intuitively, each $P(a)$ $(a \in M)$ is the set of truth values of the world a. Since we assume the existence of the least element $0_{\mathbf{M}}$ of $\mathbf{M}$, every $P(a)$ can be identified with a complete subalgebra of $P(0_{\mathbf{M}})$. We write $P(a) = \langle P(a), \cap_P, \cup_P, \rightarrow_P, 0_P, 1_P \rangle$ or $P(a) = \langle P(a), \cap, \cup, \rightarrow, 0, 1 \rangle$ for every $a \in \mathbf{M}$, if there occurs no confusion. The canonical order in $P(0_{\mathbf{M}})$ (and also in $P(a)$ for $a \in \mathbf{M}$) is denoted by $\leq_P$ or simply by $\leq$.

A mapping V which assigns to each pair (a, A) of $a \in M$ and each atomic sentence A of $\mathcal{L}[D(a)]$ an element of $P(a)$ is said to be a *valuation* on $\langle \mathbf{M}, D, P \rangle$ if for every $a, b \in M$ and every atomic sentence A of $\mathcal{L}[D(a)]$, $a \leq_{\mathbf{M}} b$ implies $V(a, A) \leq_P V(b, A_{a,b})$. We extend V to a mapping which assigns to each pair (a, A) of $a \in M$ and each sentence A of $\mathcal{L}[D(a)]$ an element of $P(a)$ inductively as follows:

- $V(a, A \wedge B) = V(a, A) \cap V(a, B)$,
- $V(a, A \vee B) = V(a, A) \cup V(a, B)$,
- $V(a, A \supset B) = \bigcap_{b; a \leq b} (V(b, A_{a,b}) \to V(b, B_{a,b}))$,
- $V(a, \neg A) = \bigcap_{b; a \leq b} (V(b, A_{a,b}) \to 0)$,
- $V(a, \forall x A(x)) = \bigcap_{b; a \leq b} \bigcap_{u \in D(b)} V(b, A_{a,b}(\overline{u}))$,
- $V(a, \exists x A(x)) = \bigcup_{d \in D(a)} V(a, A(\overline{u}))$.

A pair $(\mathcal{K}, V)$ of an algebraic Kripke sheaf $\mathcal{K}$ and a valuation V on it is said to be an *algebraic Kripke-sheaf model.* A formula A of $\mathcal{L}$ is said to be *true* in an algebraic Kripke-sheaf model $(\mathcal{K}, V)$ if $V(0_{\mathbf{M}}, \overline{A}) = 1$. A formula A of $\mathcal{L}$ is said to be *valid* in an algebraic Kripke sheaf $\mathcal{K}$ if for every valuation V on $\mathcal{K}$, A is true in $(\mathcal{K}, V)$. The set of formulas of $\mathcal{L}$ valid in $\mathcal{K} = \langle \mathbf{M}, D, P \rangle$ is denoted by $L(\mathcal{K})$ or $L\langle \mathbf{M}, D, P \rangle$. The following propositions are fundamental properties of algebraic Kripke sheaf semantics.

Proposition 2.2 *For every algebraic Kripke-sheaf model* $(\langle \mathbf{M}, D, P \rangle, V)$, *every* $a, b \in M$, *and every sentence* $A \in \mathcal{L}[D(a)]$, *if* $a \leq b$, *then* $V(a, A) \leq_P V(b, A_{a,b})$.

Proposition 2.3 *For each algebraic Kripke sheaf $\mathcal{K}$, the set $L(\mathcal{K})$ contains all formulas provable in $\mathbf{H}_*$, and is closed under the modus ponens, the rule of generalization and the rule of substitution. Namely, $L(\mathcal{K})$ is a super-intuitionistic predicate logic.*

Ghilardi [2, 3], Shehtman-Skvortsov [11], and Skvortsov-Shehtman [13] presented some extended Kripke-type semantics. The above proposition fails to hold for their semantics. For their semantical object, the set of formulas valid in it is not always closed under the rule of substitution.

In 1973, Nagai [8] introduced an extended Kripke frame semantics for which the above proposition holds. His semantics is a special case of our algebraic Kripke sheaf semantics. An algebraic Kripke sheaf $\langle \mathrm{M}, D, P\rangle$ is said to be a *Nagai sheaf*, if all $P(a)$ $(a \in M)$ are identical to the same Heyting algebra. A Nagai sheaf $\langle \mathrm{M}, D, P\rangle$ is said to be a *Nagai frame*, if $\langle \mathrm{M}, D\rangle$ is a Kripke frame. (cf. Nagai [8]). We can also introduce *algebraic Kripke frames*. An algebraic Kripke sheaf $\langle \mathrm{M}, D, P\rangle$ is said to be an algebraic Kripke frame, if $\langle \mathrm{M}, D\rangle$ is a Kripke frame.

The algebraic Kripke sheaf semantics is properly more powerful than the Nagai semantics. Let G and Lin be the following sentences;

$$\begin{aligned} G &\equiv \exists x \forall y (p(y) \supset p(x)), \\ Lin &\equiv (q \supset r) \vee (r \supset q), \end{aligned}$$

where p is a unary predicate variable, and q and r are propositional variables. The we have the

Theorem 2.4 *(1) The logic $\mathbf{H}_* + \neg\neg Z$ is Kripke-sheaf complete, but not algebraic-Kripke-frame complete.*

(2) For every Kripke sheaf $\mathcal{K}$, if $G \in L(\mathcal{K})$ then $CD \in L(\mathcal{K})$. On the other hand, there exists a Nagai frame $\mathcal{N}$ such that $G \in L(\mathcal{N})$ and $CD \notin L(\mathcal{N})$.

(3) For every Nagai sheaf $\mathcal{N}$, if $\mathbf{H}_* + Lin + G \subseteq L(\mathcal{N})$ then $CD \in L(\mathcal{N})$. On the other hand, there exists an algebraic Kripke sheaf $\mathcal{K}$ such that $\mathbf{H}_* + Lin + G \subseteq L(\mathcal{K})$ and $CD \notin L(\mathcal{K})$.

Proof. (1) : We can show that for every algebraic Kripke frame $\mathcal{F}$, if $\neg\neg Z \in L(\mathcal{F})$ then $Z \in L(\mathcal{F})$.

(2) : In [17], Umezawa proved that there exists an algebraic frame $(\mathbf{A}, U)$ which validates G but does not CD (Model 9 in [17]). Since each algebraic frame is a special Nagai frame, we have proved the latter half of (2).

(3) : Let $\mathbf{W} = \{0, 1\}$ be a Kripke base with $0 \leq 0$, $0 \leq 1$ and $1 \leq 1$. Define a domain-sheaf D and a truth-value-sheaf P by

$$D(0) = U,\ D(1) = \{0\},$$

$$D_{01}(u) = 0 \text{ for every } u \in U,$$

$$P(0) = \mathbf{A},\ P(1) = \mathbf{2},$$
$$P_{01}(0) = 0,\ P_{01}(1) = 1.$$

Then $\mathbf{H}_* + Lin + G \subseteq L\langle \mathbf{W}, D, P\rangle$ and $CD \notin L\langle \mathbf{W}, D, P\rangle$. □

We illustrate the situation in Figure 1. Until now, we have not found an appropriate example to separate Nagai frames and algebraic Kripke frames (i.e., dotted line $\cdots$ in Figure 1).

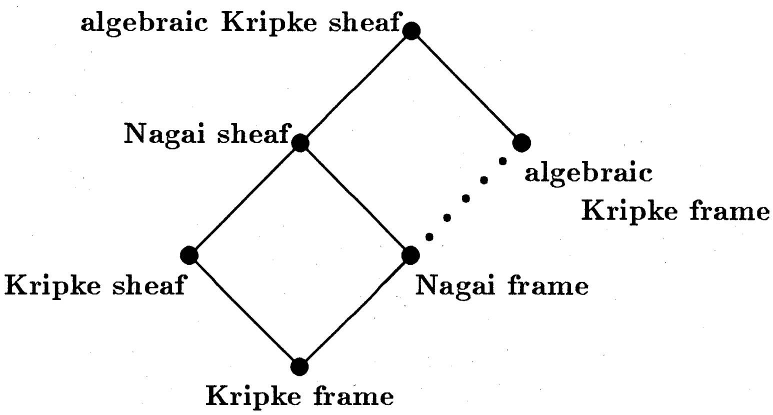

Figure 1

In the rest of this section, we state some basic results on the algebraic Kripke sheaf semantics.

Definition 2.5 Let $\mathrm{M} = \langle M, \leq\rangle$ and P be a Kripke base and a truth-value-sheaf over M, respectively. A mapping $f : \mathrm{M} \to P(0_{\mathrm{M}})$ is said to be *order-preserving*, if for every $a, b \in M$, $a \leq b$ implies $f(a) \leq_P f(b)$. An order-preserving mapping f is said to be *tame*, if for every $a \in M$, $f(a) \in P(a)$. The set of all tame and order-preserving mappings is denoted by $[P^{\mathrm{M}}]$.

Proposition 2.6 (Cf. Ono-Nagai [9]) *Let $\langle \mathrm{M}, D, P\rangle$ be an algebraic Kripke sheaf. Then, $[P^{\mathrm{M}}]$ is a Heyting algebra. Moreover, for every propositional formula A, A is valid in $\langle \mathrm{M}, D, P\rangle$ if and only if A is valid in $[P^{\mathrm{M}}]$.*

Definition 2.7 Let $\langle \mathrm{M}, D, P\rangle$ and $\langle \mathrm{N}, E, Q\rangle$ be algebraic Kripke sheaves. The truth-value-sheaf $Q \circ f$ over M is defined in a similar way that is used in the previous section. A *p-morphism* of $\langle \mathrm{M}, D, P\rangle$ to $\langle \mathrm{N}, E, Q\rangle$ is a triple (f, τ, σ) of a p-morphism (f, τ) of $\langle \mathrm{M}, D\rangle$ to $\langle \mathrm{N}, E\rangle$ (as Kripke sheaves) and a natural transformation σ of $Q \circ f$ to P. That is, σ assigns to each $a \in M$ a complete monomorphism (an arrow of $\mathcal{H}$)

$\sigma_a : (Q \circ f)(a) \to P(a)$ in such a way that every $a, b \in M$ with $a \leq_{\mathbf{M}} b$ yield the following commutative diagram:

$$\begin{array}{ccc} & \sigma_b & \\ (Q \circ f)(b) & \longrightarrow & P(b) \\ (Q \circ f)_{ab} \downarrow & & \downarrow P_{ab} \\ (Q \circ f)(a) & \longrightarrow & P(a) \\ & \sigma_a . & \end{array}$$

Theorem 2.8 (the p-morphism theorem for algebraic Kripke sheaves) *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be algebraic Kripke sheaves. If there exists a p-morphism of $\mathcal{K}_1$ to $\mathcal{K}_2$, then $L(\mathcal{K}_1) \subseteq L(\mathcal{K}_2)$.*

Definition 2.9 Let λ be an arbitrary non-zero cardinal. A Heyting algebra $\mathbf{P} = \langle P, \cap, \cup, \rightarrow, 0, 1\rangle$ is said to be *λ-distributive*, if for every element $a \in P$ and every subset $S \subseteq P$ with $\mathrm{Card}(S) \leq \lambda$, it holds that $\bigcap_{s \in S}(s \cup a) = (\bigcap_{s\in S} s) \cup a$, where $\mathrm{Card}(S)$ is the cardinality of S. An algebraic Kripke sheaf $\langle \mathbf{M}, D, P\rangle$ is said to be *of type CD*, if $\langle \mathbf{M}, D\rangle$ is a Kripke sheaf of type *CD* and for every $a \in M$, $P(a)$ is $\mathrm{Card}(D(a))$-distributive.

Lemma 2.10 *The sentence CD is valid in an algebraic Kripke sheaf if and only if it is of type CD.*

Theorem 2.11 *(1) For every algebraic Kripke sheaf $\mathcal{K} = \langle \mathbf{M}, D, P\rangle$, there exist a Nagai frame $\mathcal{K}^* = \langle \mathbf{M}, D^*, P^*\rangle$ and a p-morphism of $\mathcal{K}^*$ to $\mathcal{K}$.*

(2) For every algebraic Kripke sheaf $\mathcal{K} = \langle \mathbf{M}, D, P\rangle$ of type CD, there exist a Nagai frame $\mathcal{K}^* = \langle \mathbf{M}, D^*, P^*\rangle$ of type CD and a p-morphism of $\mathcal{K}^*$ to $\mathcal{K}$.

Along lines with the above discussion, we can develop algebraic Kripke semantics based on extended Kripke-type semantics presented in Ghilardi [2, 3], Shehtman-Skvortsov [11], and Skvortsov-Shehtman [13]. E.g., *algebraic Kripke bundles*, *algebraic* **C**-*sets* and *algebraic Kripke metaframes* can be, in principle, introduced.

3 Application

As application of our algebraic Kripke sheaf semantics, we show here new results in super-intuitionistic predicate logics. This result cannot be proved, at present, by using the Kripke sheaf semantics. We present a new method of constructing a continuum of super-intuitionistic predicate logics. This method is obtained from the modified Jankov method in [15] by re-modification with adopting a suitable subclass of algebraic Kripke sheaves. We show that there exists a continuum of super-intuitionistic predicate logics each of which has both of the disjunction and existence properties and moreover the same propositional fragment as the intuitionistic logic. For this aim, we must find a criterion for a logic to have both of the disjunction and existence properties. Our idea comes from an observation of the delta operation in super-intuitionistic predicate logics.

We explain how to modify the Jankov method used in [15] by making use of algebraic Kripke sheaf semantics. First, we give here a brief sketch of the original Jankov method ([5]). We recall some terminology and notation mainly from [15]. The Jankov method is a method constructing a *strongly independent sequence* of logics.

Definition 3.1 A denumerable sequence $\{\mathbf{L}_i\}_{i<\omega}$ of logics is said to be *strongly independent*, if $\mathbf{L}_i \not\subseteq \bigcup_{j\neq i}\mathbf{L}_j$ for each $i<\omega$, where $\bigcup_{j\neq i}\mathbf{L}_j$ is the smallest logic containing all $\mathbf{L}_j$ $(j\neq i)$.

If there exists a strongly independent sequence $\{\mathbf{L}_i\}_{i<\omega}$ of logics, we have that for every I and $J\subseteq\omega$, $I=J$ if and only if $\bigcup_{i\in I}\mathbf{L}_i=\bigcup_{j\in J}\mathbf{L}_j$. Hence the cardinality of $\{\bigcup_{i\in I}\mathbf{L}_i\ ;\ I\subseteq\omega\}$ equals that of a continuum.

A Heyting algebra $\mathbf{A}$ is said to be *strongly compact* if there exists the second greatest element in $\mathbf{A}$. The second greatest element of a strongly compact Heyting algebra $\mathbf{A}$ is denoted by $\star_{\mathbf{A}}$ or simply by $\star$. Let $\mathbf{A}=\langle A,\cap,\cup,\rightarrow,0,1\rangle$ be a finite and strongly compact Heyting algebra. Since its underlying set A is finite, for each element $a\in A$, we can attach a unique propositional variable P_a. The *diagram* $\delta(\mathbf{A})$ of $\mathbf{A}$ is defined by

$$\begin{aligned}\delta(\mathbf{A}) \ = \ & \{P_{a\cap b}\supset(P_a\wedge P_b),\ (P_a\wedge P_b)\supset P_{a\cap b}\ ;\ a,b\in A\}\\ & \bigcup\{P_{a\cup b}\supset(P_a\vee P_b),\ (P_a\vee P_b)\supset P_{a\cup b}\ ;\ a,b\in A\}\\ & \bigcup\{P_{a\rightarrow b}\supset(P_a\supset P_b),\ (P_a\supset P_b)\supset P_{a\rightarrow b}\ ;\ a,b\in A\}\\ & \bigcup\{P_{a\rightarrow 0}\supset\neg P_a,\ \neg P_a\supset P_{a\rightarrow 0}\ ;\ a\in A\}.\end{aligned}$$

The *Jankov formula* of $\mathbf{A}$ is the formula

$$J(\mathbf{A})=(\bigwedge\delta(\mathbf{A}))\supset P_\star,$$

where $\bigwedge\delta(\mathbf{A})$ is the conjunction of all formulas in $\delta(\mathbf{A})$. Then it is easy to see that $J(\mathbf{A})$ is not valid in $\mathbf{A}$. The following Lemmas is the Key Lemma for the original Jankov method.

Lemma 3.2 *Let* $\mathbf{A}$ *be a finite and strongly compact Heyting algebra. For each Heyting algebra* $\mathbf{B}$, *the following two conditions are equivalent:*
(1) $J(\mathbf{A})$ *is not valid in* $\mathbf{B}$,
(2) $\mathbf{A}$ *is embeddable into a quotient algebra of* $\mathbf{B}$.

By this Lemma, Jankov established a relation between validity of a certain propositional formula and an algebraic property. We extend this to an appropriate relation in predicate logics.

A subset A of a Kripke base $\mathbf{M}$ is said to be *open* if for every $u \in A$ and every $v \in M$, $u \leq v$ implies $v \in A$. The set $\mathcal{O}(\mathbf{M})$ of all open subsets is a strongly compact Heyting algebra with the second greatest element $M \setminus \{0_{\mathbf{M}}\}$, which we will denote by $\star_{\mathbf{M}}$. For each Heyting algebra $\mathbf{A}$, the set $\mathcal{P}(\mathbf{A})$ of all prime filters is a Kripke frame. If $\mathbf{A}$ is strongly compact, then $\mathcal{P}(\mathbf{A})$ is a Kripke base. The following facts and lemma are well-known.

Fact 3.3 *(1) For each finite Kripke base* $\mathbf{M}$, $\mathcal{P}(\mathcal{O}(\mathbf{M}))$ *is isomorphic to* $\mathbf{M}$.

(2) For each finite and strongly compact Heyting algebra $\mathbf{A}$, $\mathcal{O}(\mathcal{P}(\mathbf{A}))$ *is isomorphic to* $\mathbf{A}$.

Lemma 3.4 *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be Kripke bases. Suppose that* $\mathbf{N}$ *is finite. Then, there exists a p-morphism from* $\mathbf{M}$ *to* $\mathbf{N}$ *if and only if there exists an embedding of* $\mathcal{O}(\mathbf{N})$ *into* $\mathcal{O}(\mathbf{M})$.

Hence, we can identify *finite* Kripke bases with *finite* strongly compact Heyting algebras. In the following we implicitly use the above fact and lemma in order to make our exposition simple.

Definition 3.5 For a strongly compact Heyting algebra $\mathbf{A}$, there exists a Heyting algebra $\mathbf{A}^-$ such that $\mathbf{A}$ is isomorphic to $\mathbf{A}^- \oplus \mathbf{2}$, where $\oplus$ is the sum operation of Heyting algebras. (See e.g., Troelstra [16].) For a strongly compact Heyting algebra $\mathbf{A}$ and a Heyting algebra $\mathbf{B}$, we denote by $\mathbf{A} \lhd \mathbf{B}$ the Heyting algebra $\mathbf{A}^- \oplus \mathbf{B}$.

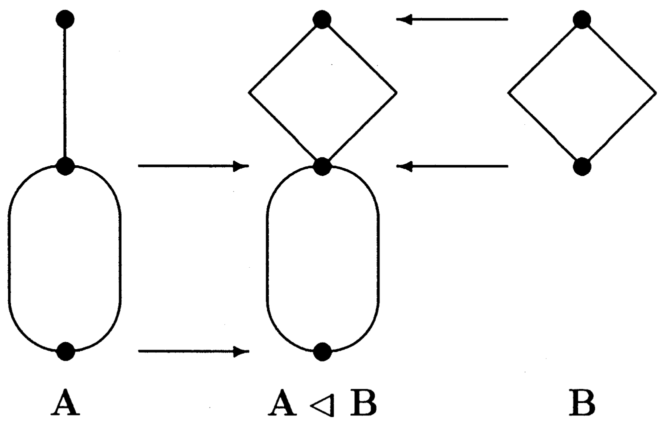

Figure 2

Lemma 3.6 *Every quotient algebra of* $\mathbf{A} \lhd \mathbf{B}$ *is isomorphic either to a quotient algebra of* $\mathbf{A}$ *or to* $\mathbf{A} \lhd \mathbf{B}'$ *for some quotient algebra* $\mathbf{B}'$ *of* $\mathbf{B}$.

Lemma 3.7 *Suppose that a strongly compact Heyting algebra* $\mathbf{A}$ *contains at least three elements. If a finite and strongly compact Heyting algebra* $\mathbf{C}$ *satisfies the following conditions:*

(#1) $4 \leq \mathrm{Card}(\mathbf{C})$,

(#2) there are exactly three elements in $\mathbf{C}$ *having no incomparable element (i.e.,* 0, 1 *and* $\star$*),*

then the following are equivalent:
(a) $\mathbf{C}$ *is embeddable into a quotient algebra of* $\mathbf{A} \lhd \mathbf{B}$,
(b) $\mathbf{C}$ *is embeddable into a quotient algebra of* $\mathbf{A}$.

Definition 3.8 An algebraic Kripke sheaf $\langle \mathrm{M}, D, P \rangle$ is said to be a *pendulum*, if for every $a \in M$, $a \neq 0_{\mathrm{M}}$ implies $P(a) = \mathbf{2}$. A pendulum $\langle \mathrm{M}, D, P \rangle$ is said to be an $\mathbf{A}$-*pendulum*, if $P(0_{\mathrm{M}}) = \mathbf{A}$. An algebraic Kripke sheaf model $(\langle \mathrm{M}, D, P \rangle, V)$ is said to be a *pendulum model* (an $\mathbf{A}$-*pendulum model*), if $\langle \mathrm{M}, D, P \rangle$ is a pendulum (an $\mathbf{A}$-pendulum, respectively).

Lemma 3.9 *Let* $\langle \mathrm{M}, D, P \rangle$ *be an* $\mathbf{A}$*-pendulum with* $\mathrm{Card}(\mathrm{M}) \geq 2$.
(1) $[P^{\mathrm{M}}]$ *is isomorphic to* $\mathcal{O}(\mathrm{M}) \lhd \mathbf{A}$.
(2) Suppose a finite and strongly compact Heyting algebra $\mathbf{C}$ *satisfies the conditions (#1) and (#2) in Lemma 3.7. Then, for every pendulum* $\langle \mathrm{M}, D, P \rangle$*, the following are equivalent:*
(a) $\mathbf{C}$ *is embeddable into a quotient algebra of* $[P^{\mathrm{M}}]$,
(b) $\mathbf{C}$ *is embeddable into a quotient algebra of* $\mathcal{O}(\mathrm{M})$.

Let M be a finite Kripke base. For each sentence S of $\mathcal{L}$, we denote by $J(\mathrm{M}\,;S)$ the sentence obtained from $J(\mathcal{O}(\mathrm{M}))$ by replacing all occurrences of $P_\star$ by S. We implicitly assume that S and $J(\mathcal{O}(\mathrm{M}))$ contain no propositional variables in common.

Lemma 3.10 *Let* M *be a finite Kripke base such that* $\mathcal{O}(\mathrm{M})$ *satisfies the conditions (#1) and (#2) in Lemma 3.7. For each pendulum* $\mathcal{K} = \langle \mathrm{N}, D, P \rangle$ *with* $\mathrm{Card}(\mathrm{N}) \geq 2$, *if* $J(\mathrm{M}\,;S) \notin L(\mathcal{K})$, *then* $\mathcal{O}(\mathrm{M})$ *is embeddable into a quotient algebra of* $\mathcal{O}(\mathrm{N})$.

We state our newly modified Jankov method as follows.

Lemma 3.11 (cf. Lemma 2.3 in [15]) *Suppose* $\{(\langle \mathrm{M}_i, D_i, P_i \rangle, V_i)\}_{i<\omega}$ *is a sequence of algebraic Kripke sheaf models satisfying:*
(0) each $\langle \mathrm{M}_i, D_i, P_i \rangle$ *is a pendulum,*
(1) each $\mathcal{O}(\mathrm{M}_i)$ *is finite and satisfies the conditions (#1) and (#2) in Lemma 3.7,*
(2) for every $i, j < \omega$, $i \neq j$ *implies that* $\mathcal{O}(\mathrm{M}_i)$ *cannot be embedded into any quotient algebra of* $\mathcal{O}(\mathrm{M}_j)$,
(3) $J(\mathrm{M}_i\,;S)$ *is not true in* $(\langle \mathrm{M}_i, D_i, P_i \rangle, V_i)$.
If a super-intuitionistic predicate logic $\mathbf{L}$ *is contained in every* $L\langle \mathrm{M}_i, D_i, P_i \rangle$ $(i < \omega)$, *then the sequence* $\{\mathbf{L}_i\}_{i<\omega}$ *of super-intuitionistic predicate logics defined by* $\mathbf{L}_i = \mathbf{L} + J(\mathrm{M}_i\,;S)$ $(i < \omega)$ *is strongly independent. In particular, the sequence* $\{\mathbf{K}_i\}_{i<\omega}$ *of super-intuitionistic predicate logics defined by* $\mathbf{K}_i = \mathbf{H}_* + J(\mathrm{M}_i\,;S)$ $(i < \omega)$ *is strongly independent.*

By the virtue of this Lemma, we can present a concrete example of a continuum of logics whenever we have appropriate sequence $\{(\langle \mathbf{M}_i, D_i, P_i\rangle, V_i)\}_{i<\omega}$ and sentence S. Since all of the additional axioms $J(\mathbf{M}_i\,;S)$ $(i<\omega)$ are provable in $\mathbf{H}_*+S$, all of $\mathbf{K}_i$'s constructed above are locate below $\mathbf{H}_*+S$.

In [18], Wroński constructed a sequence $\{\mathbf{M}_i\}_{i<\omega}$ of finite Kripke bases satisfying the conditions (1) and (2) in Lemma 3.11. We fix this sequence and use it to show our results. The following Lemma provides a criterion for a sentence S to have sequences $\{D_i\}_{i<\omega}$, $\{P_i\}_{i<\omega}$ and $\{V_i\}_{i<\omega}$ of domain-sheaves, truth-value-sheaves and valuations, respectively, such that $\{(\langle \mathbf{M}_i, D_i, P_i\rangle, V_i)\}_{i<\omega}$ and S satisfy the conditions in Lemma 3.11.

Lemma 3.12 *Let S be a sentence. Suppose that there exists a pendulum model $(\langle \mathbf{W}, D, P\rangle, V)$ such that $V(0,S)=0_P$ and $V(1,S)=1_P$, where $\mathbf{W}=\{0,1\}$ is a two-element Kripke base. Then, for every finite Kripke base $\mathbf{M}$ with $\mathrm{Card}(\mathbf{M})\geq 2$, there exists a pendulum model $(\langle \mathbf{M}, D^*, P^*\rangle, V^*)$ such that $J(\mathbf{M}\,;S))$ is not true in it.*

Let W^* be a sentence defined by:

$$W^* \equiv \forall x((p(x)\supset \forall y p(y))\supset \forall y p(y))\supset \forall x p(x).$$

where p is a unary predicate variable.

Theorem 3.13 *There exists a continuum of super-intuitionistic predicate logics between $\mathbf{H}_*$ and $\mathbf{H}_*+W^*$.*

Define a sequence $\{\widetilde{\mathsf{P}_n}\}_{n<\omega}$ of sentences inductively by:

$$\begin{aligned}\widetilde{\mathsf{P}_0} &\equiv q\wedge\neg q,\\ \widetilde{\mathsf{P}_{n+1}} &\equiv \forall x(p_{n+1}(x)\vee(p_{n+1}(x)\supset \widetilde{\mathsf{P}_n})),\end{aligned}$$

where q is a propositional variable and each p_n $(n<\omega)$ is a unary predicate variable.

The following Theorem can be proved similarly.

Theorem 3.14 *For each $n<\omega$, there exists a continuum of super-intuitionistic predicate logics between $\mathbf{H}_*$ and $\mathbf{H}_*+\widetilde{\mathsf{P}_n}$.*

We make a remark here that the sentence $\widetilde{\mathsf{P}_n}$ is valid in every Kripke frames with height not greater than n. By Lemma 1.11, $\widetilde{\mathsf{P}_n}$ is valid in every Kripke sheaves with Kripke bases having height not greater than n.

Now we deal with the disjunction and existence properties.

Definition 3.15 (1) A super-intuitionistic predicate logic $\mathbf{L}$ is said to have the *disjunction property*, if for every formula A and B, either A or B is provable in $\mathbf{L}$ whenever $A\vee B$ is provable in $\mathbf{L}$.

(2) A super-intuitionistic predicate logic $\mathbf{L}$ is said to have the *existence property*, if for every formula $\exists x A(x)$, there exists a such that $A(a)$ is provable in $\mathbf{L}$ whenever $\exists x A(x)$ is provable in $\mathbf{L}$.

We would like to find a sufficient condition for a logic to have both of the disjunction and existence properties. Our idea comes from an observation of the delta operation in super-intuitionistic predicate logics.

Definition 3.16 For each formula A, define

$$\Delta(A) \equiv p \vee (p \supset A),$$

where p is a propositional variable not occurring in A. Let $\mathbf{L}$ be a super-intuitionistic predicate logic. We define a super-intuitionistic predicate logic $\Delta(\mathbf{L})$ by

$$\Delta(\mathbf{L}) = \mathbf{H}_* + \{\Delta(A)\ ;\ A \in \mathbf{L}\}.$$

The Δ is originally defined on the set of super-intuitionistic propositional logics (see [4]). Some properties of Δ on super-intuitionistic propositional logics can be rather faithfully translated to those of the extended Δ on super-intuitionistic predicate logics. The following Fact can be shown in ways that are similar to those used for the case of super-intuitionistic propositional logics.

Fact 3.17 *(1) For every super-intuitionistic predicate logic* $\mathbf{L}$, $\Delta(\mathbf{L}) \subseteq \mathbf{L}$.

(2) For every super-intuitionistic predicate logics $\mathbf{L}_1$ *and* $\mathbf{L}_2$, $\mathbf{L}_1 \subseteq \mathbf{L}_2$ *implies* $\Delta(\mathbf{L}_1) \subseteq \Delta(\mathbf{L}_2)$.

(3) For every super-intuitionistic predicate logic $\mathbf{L}$ *and every formula* A, $A \in \mathbf{L}$ *if and only if* $\Delta(A) \in \Delta(\mathbf{L})$.

(4) Δ *is one-to-one as a mapping of the set of all super-intuitionistic predicate logics to itself.*

As is often the case with logicians who try to extend things in propositional logics to those in predicate logics, we can find contrast between propositional and predicate logics. One of the most interesting results is the following.

Fact 3.18 *(Propositional) For every propositional logic* $\mathbf{J}$, $\Delta(\mathbf{J}) = \mathbf{J}$ *if and only if* $\mathbf{J}$ *is the intuitionistic propositional logic. That is, the intuitionistic logic is the unique fixed point of* Δ.

(Predicate) There exists a super-intuitionistic predicate logic $\mathbf{L}$ *satisfying* $\Delta(\mathbf{L}) = \mathbf{L}$ *which is not identical to* $\mathbf{H}_*$.

For example, $\mathbf{H}_* + K$ and $\mathbf{H}_* + W^*$ are fixed points of Δ, where $K \equiv \neg\neg\forall x(p(x) \vee \neg p(x))$ with p being a unary predicate variable. These non-trivial fixed points of Δ have interesting properties from the logical point of view.

Lemma 3.19 *If* $\Delta(\mathbf{L}) = \mathbf{L}$, *then* $\mathbf{L}$ *has the disjunction and existence properties.*

Hence, if $\Delta(\mathbf{L}) = \mathbf{L}$, then $\mathbf{L}$ has both of the disjunction and existence properties and the same propositional fragment as the intuitionistic logic. Now we show the existence of a continuum of such logics. For this purpose, it suffices to show the following lemma, which implies the existence of a continuum of fixed points of Δ.

Lemma 3.20 *We can construct a denumerable sequence $\{X_i \,;\, i < \omega\}$ of axioms such that*
(1) for each subset $S \subseteq \omega$, $\mathbf{H}_* + \{X_i \,;\, i \in S\}$ is a fixed point of Δ.
(2) $\{\mathbf{H}_* + X_i\}_{i<\omega}$ is strongly independent.

The methods in [15] do not provide the axioms satisfying (1) and (2) of Lemma 3.20. The Wajsberg method generates axioms of the form $W_i \vee S$ $(i < \omega)$. Here each W_i $(i < \omega)$ is a special axiom not provable in $\mathbf{C}_*$. Hence, if $\mathbf{H}_* + W_i \vee S$ is a fixed point of Δ, then S is not provable in $\mathbf{H}_* + W_i$ by the disjunction property. Then $\mathbf{H}_* + W_i \vee S = \mathbf{H}_* + S$ for each $i < \omega$. The Jankov method gives axioms each of which is not valid in some Kripke sheaf with finite Kripke base. On the other hand we have

Proposition 3.21 *If $\mathbf{H}_* + S$ is a fixed point of Δ, then S is valid in every Kripke sheaf with finite Kripke base.*

Hence we cannot apply methods in [15]. We shall use our newly modified method.

Lemma 3.22 *For every sentence A, $\mathbf{H}_* + A \supset W^*$ is a fixed point of Δ.*

Since $J(\mathbf{M}_i \,;\, W^*)$ is of the form $A \supset W^*$, the sequence $\{J(\mathbf{M}_i \,;\, W^*) \,;\, i < \omega\}$ satisfies the condition (1) of Lemma 3.20. In Theorem 3.13, we have already proved that this sequence satisfies the condition (2) of Lemma 3.20, and hence, we have

Theorem 3.23 *There exists a continuum of super-intuitionistic predicate logics each of which has both of the disjunction and existence properties and moreover the same propositional fragment as the intuitionistic logic.*

Ferrari and Miglioli [1] constructed a continuum of logics with both of the disjunction and existence properties by highly transcendental manner. In fact, none of their logics have the same propositional fragment as the intuitionistic logic, and are finitely axiomatizable. On the other hand, we have here infinitely many finitely axiomatizable logics with these properties and the same propositional fragment as the intuitionistic logic.

In [15], we presented the following problem:

(Problem 3 in [15]) *How many logics are there between $\mathbf{H}_*$ and $\mathbf{H}_* + K$?*

Unfortunately, our method developed here cannot be applied to this problem, either. In fact, K is valid in every pendulum with finite Kripke base. Hence, the above problem still remains open. We know only that there exist at least infinitely many logics between $\mathbf{H}_*$ and $\mathbf{H}_* + K$.

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Nobu-Yuki SUZUKI
Department of Computer and Information Sciences
Faculty of Engineering
Ibaraki University
Nakanarusawa, Hitachi 316
JAPAN