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Singular Invariant Hyperfunctions on the spaces of symmetric matrices $Sym_n(\mathbb{R})$, and, of Complex and Quaternion Hermitian matrices $Her_n(\mathbb{C})$, $Her_n(\mathbb{H})$

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Japanese Abstract

本講演をおこなったとき，筆者は，まず黒板を使って問題を説明し，ついで OHP 原稿を利用して実際の計算結果を提示した．実際に，これらの計算結果を板書しながら説明することは，時間的にも不可能であると判断したためである．この原稿では，講演中に使用した OHP 原稿を，ほとんど手を加えないまま，コメントをあいだに挟んだだけで提示することにした．これで，講演の内容はおわかりいただけると思う．Introduction だけは新たに書き加えた．

1 Introduction — Abstract in English

Let $P(x)$ be a real-valued homogeneous polynomial on an m-dimensional real vector space $\boldsymbol{V} := \mathbb{R}^m$. Consider the subgroup $\boldsymbol{G}$ of $\mathbf{GL}(\boldsymbol{V})$,

$$\boldsymbol{G} := \{g \in \mathbf{GL}(\boldsymbol{V}); P(g\cdot x) = \nu(g)P(x)\}. \tag{1}$$

where $\nu(g)$ is a constant depending only on $g \in \boldsymbol{G}$. Let $\boldsymbol{S} := \{x \in \mathbb{R}^m; P(0) = 0\}$ and let $\boldsymbol{V}_0 \cup \boldsymbol{V}_1 \cdots \boldsymbol{V}_l$ be the connected component decomposition of the set $\boldsymbol{V} - \boldsymbol{S}$.

In order to construct a $\boldsymbol{G}$-invariant hyperfunction, which is automatically a tempered distribution, we take a complex power of the polynomial function $P(x)$. We define the functions $|P(x)|_i^s$ $(i = 0, 1, \ldots, l)$ with a holomorphic parameter $s \in \mathbb{C}$ on $\boldsymbol{V}$ by

$$|P(x)|_i^s := \begin{cases} |P(x)|^s & ,\text{if } x \in \boldsymbol{V}_i, \\ 0 & ,\text{if } x \notin \boldsymbol{V}_i. \end{cases} \tag{2}$$

Let $\mathcal{S}(\boldsymbol{V})$ be the space of rapidly decreasing smooth functions on $\boldsymbol{V}$. The integral

$$Z_i(f,s) := \int |P(x)|_i^s f(x)dx \tag{3}$$

is absolutely convergent for $f(x) \in \mathcal{S}(\boldsymbol{V})$ if the real part $\Re(s)$ of s is sufficiently large. Thus we can regard $|P(x)|_i^s$ as a tempered distribution on $\boldsymbol{V}$ with a holomorphic parameter $s \in \mathbb{C}$ when $\Re(s)$ is large. It is well known that $Z_i(f,s)$ is meromorphically extended to the whole complex plane $s \in \mathbb{C}$. The possible poles of $Z_i(f,s)$ appear in points of negative rational numbers. Let $C_0^\infty(\boldsymbol{V}-\boldsymbol{S})$ be the space of C^∞-functions on $\boldsymbol{V}-\boldsymbol{S}$. If $f(x) \in C_0^\infty(\boldsymbol{V}-\boldsymbol{S})$, then $Z_i(f,s)$ is absolutely convergent for any $s \in \mathbb{C}$. This means that $Z_i(f,s)$ is an entire function in $s \in \mathbb{C}$.

Suppose that $Z_i(f,s)$ has a pole of order n_0 at $s = s_0$. The Laurent expansion of $Z_i(f,s)$ at $s = s_0$ is written as

$$\frac{Z_i^{(s_0,-n_0)}(f)}{(s-s_0)^{n_0}} + \frac{Z_i^{(s_0,-n_0+1)}(f)}{(s-s_0)^{n_0-1}} + \cdots + \frac{Z_i^{(s_0,-1)}(f)}{(s-s_0)} + Z_i^{(s_0,0)}(f) + Z_i^{(s_0,1)}(f)(s-s_0) + \cdots \tag{4}$$

where the coefficients $Z_i^{(s_0,k)}(f)$ $(i = 0, 1, \ldots, l, s_0 \in \mathbb{Q}_{<0}, k \in \mathbb{Z}$ and $k > -n_0)$ are tempered distributions. When $f(x) \in C_0^\infty(\boldsymbol{V}-\boldsymbol{S})$, then $Z_i(f,s)$ is holomorphic at $s = s_0$. Thus $Z_i^{(s_0,k)}(f) = 0$ for negative integers k. That is to say, the support of the tempered distributions $f \mapsto Z_i^{(s_0,k)}(f)$ is contained in $\boldsymbol{S}$ if k is negative. We say that such a tempered distribution *singular distributions*.

If we can calculate the location and the exact orders of $Z_i(f,s)$'s poles, then we can construct singular distributions supported in $\boldsymbol{S}$ as negative-order coefficients of the Laurent expansions of $Z_i(f,s)$. In addition, the coefficients hold the invariance with respect to the action of the group $\boldsymbol{G}$.

The location of poles and the possible maximal orders of poles are determined by computing the divisors of the b-functions of $P(x)^s$. But the following problems can not be solved only by calculating the b-function; 1) determining the *exact order* of poles and 2) obtaining the *exact support* of the singular distributions appearing in the coefficients of the Laurent expansion.

In the presentation, the author gave a complete answer for these problems in the following cases.

1. $\boldsymbol{V} = Sym_n(\mathbb{R}) :=$ the space of $n \times n$ real symmetric matrices, $P(x) = \det(x)$ for $x \in Sym_n(\mathbb{R})$, $\boldsymbol{G} = \mathbf{GL}_n(\mathbb{R})$, $g \cdot x = gx^tg$ for $g \in \boldsymbol{G}$ and $x \in \boldsymbol{V}$.

2. $\boldsymbol{V} = Her_n(\mathbb{C}) :=$ the space of $n \times n$ complex Hermitian matrices, $P(x) = \det(x)$ for $x \in Her_n(\mathbb{C})$, $\boldsymbol{G} = \mathbf{GL}_n(\mathbb{C})$, $g \cdot x = gx^t\bar{g}$ for $g \in \boldsymbol{G}$ and $x \in \boldsymbol{V}$.

3. $\boldsymbol{V} = Her_n(\mathbb{H}) :=$ the space of $n \times n$ quaternion Hermitian matrices, $P(x) = \det(x)$ for $x \in Her_n(\mathbb{H})$, $\boldsymbol{G} = \mathbf{GL}_n(\mathbb{H})$, $g \cdot x = gx^t\bar{g}$ for $g \in \boldsymbol{G}$ and $x \in \boldsymbol{V}$.

Here, ${}^tg$ and $\bar{g}$ stand for the transposed and conjugate matrices of g , respectively.

The contents of this note is the same as the lecture at July 31,1995 in the main hall of RIMS, Kyoto University. The author arranges the original OHP slides shown at the presentation and reprinted here with slight modification and some comments.

2 Slides and comments

The purpose of this lecture is to construct a suitable basis of the space of singular invariant hyperfunctions on $\boldsymbol{V}$. The basis consists of the coefficients of the Laurent expansion of $|\det(x)|^s$, the complex power of the determinant function. We estimate the exact order of the poles of $|\det(x)|^s$ and give the exact support of the negative-order coefficients of the Laurent expansion of $|\det(x)|^s$ at its poles.

Similar results are obtained by Blind [Bli94].

Let $\boldsymbol{V} := Sym_n(\mathbb{R})$ be the space of $n \times n$ symmetric matrices over the real field $\mathbb{R}$, and let $\mathbf{GL}_n(\mathbb{R})$ (reap. $\mathbf{SL}_n(\mathbb{R})$) be the general (resp. special) linear group over $\mathbb{R}$. Then the real algebraic group $\boldsymbol{G} := \mathbf{GL}_n(\mathbb{R})$ operates on the vector space $\boldsymbol{V}$ by

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$$g : x \longmapsto g \cdot x \cdot {}^t g,$$

with $x \in \boldsymbol{V}$ and $g \in \boldsymbol{G}$. We say that a hyperfunction $f(x)$ on $\boldsymbol{V}$ is *singular* if the support of $f(x)$ is contained in the set $\boldsymbol{S} := \{x \in \boldsymbol{V};\ \det(x) = 0\}$. We call $\boldsymbol{S}$ a *singular set* of $\boldsymbol{V}$. In addition, if $f(x)$ is $\mathbf{SL}_n(\mathbb{R})$-invariant, i.e., $f(g \cdot x) = f(x)$ for all $g \in \mathbf{SL}_n(\mathbb{R})$, we call $f(x)$ a *singular invariant* hyperfunction on $\boldsymbol{V}$.

Let $P(x) := \det(x)$. Then $P(x)$ is an irreducible polynomial on $\boldsymbol{V}$, and is relatively invariant with respect to the action of $\boldsymbol{G}$ corresponding to the character $\det(g)^2$,i.e., $P(g \cdot x) = \det(g)^2 P(x)$. The non-singular subset $\boldsymbol{V} - \boldsymbol{S}$ decomposes into $(n+1)$ open $\boldsymbol{G}$-orbits

$$\boldsymbol{V}_i := \{x \in Sym_n(\mathbb{R}); \mathrm{sgn} = (n-i, i)\}. \tag{5}$$

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with $i = 0, 1, \ldots, n$. Here, $\mathrm{sgn}(x)$ for $x \in Sym_n(\mathbb{R})$ stands for the signature of the quadratic form $q_x(\vec{v}) := {}^t\vec{v} \cdot x \cdot \vec{v}$ on $\vec{v} \in \mathbb{R}^n$. We let for a complex number $s \in \mathbb{C}$,

$$|P(x)|_i^s := \begin{cases} |P(x)|^s & ,\text{if } x \in \boldsymbol{V}_i, \\ 0 & ,\text{if } x \notin \boldsymbol{V}_i. \end{cases} \tag{6}$$

Let $\mathcal{S}(\boldsymbol{V})$ be the space of rapidly decreasing functions on $\boldsymbol{V}$. For $f(x) \in \mathcal{S}(\boldsymbol{V})$, the integral

$$Z_i(f,s) := \int_{\boldsymbol{V}} |P(x)|_i^s f(x) dx, \tag{7}$$

is convergent if the real part of s is sufficiently large and is holomorphically extended to the whole complex plane. Thus we can regard $|P(x)|_i^s$ as a tempered distribution with a meromorphic parameter $s \in \mathbb{C}$.

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We consider a linear combination of $|P(x)|_i^s$

$$P^{[\vec{a},s]}(x) := \sum_{i=0}^{n} a_i |P(x)|_i^s, \tag{8}$$

with $s \in \mathbb{C}$ and $\vec{a} := (a_0, a_1, \ldots, a_n) \in \mathbb{C}^{n+1}$. Then $P^{[\vec{a},s]}(x)$ is a hyperfunction with a meromorphic parameter $s \in \mathbb{C}$, and depends on $\vec{a} \in \mathbb{C}^{n+1}$ linearly.

The following theorem is well known (see for example [Mur90]).

Theorem 2.1. *1. $P^{[\vec{a},s]}(x)$ is holomorphic with respect to $s \in \mathbb{C}$ except for the poles at $s = -\frac{k+1}{2}$ with $k = 1, 2, \ldots$.*

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2. The possibly highest order of $P^{[\vec{a},s]}(x)$ at $s = -\frac{k+1}{2}$ is given by

$$\begin{cases} \lfloor \frac{k+1}{2} \rfloor & ,(k = 1, 2. \ldots, n-1), \\ \lfloor \frac{n}{2} \rfloor & ,(k = n, n+1. \ldots, \text{ and } k+n \text{ is odd}), \\ \lfloor \frac{n+1}{2} \rfloor & ,(k = n, n+1. \ldots, \text{ and } k+n \text{ is even}). \end{cases} \tag{9}$$

Here, $\lfloor x \rfloor$ means the floor of $x \in \mathbb{R}$, i.e., the largest integer less than x.

Any negative-order coefficient of a Laurent expansion of $P^{[\vec{a},s]}(x)$ is a singular invariant hyperfunction since the integral

$$\int f(x) P^{[\vec{a},s]}(x) dx = \sum_{i=0}^{n} Z_i(f, s) \tag{10}$$

is an entire function with respect to $s \in \mathbb{C}$ if $f(x) \in C_0^\infty(\boldsymbol{V} - \boldsymbol{S})$.

Conversely, we have the following proposition.

Slide 5 **Proposition 2.2 ([Mur88],[Mur90]).** *Any singular invariant hyperfunction on $\boldsymbol{V}$ is given as a linear combination of some negative-order coefficients of Laurent expansions of $P^{[\vec{a},s]}(x)$ at various poles and for some $\vec{a} \in \mathbb{C}^{n+1}$.*

Proof. The prehomogeneous vector space

$$(\boldsymbol{G}, \boldsymbol{V}) := (\mathbf{GL}_n(\mathbb{R}), Sym_n(\mathbb{R}))$$

satisfies sufficient conditions stated in [Mur88] and [Mur90]. One is the finite-orbit condition and the other is that the dimension of the space of relatively invariant hyperfunctions coincides with the number of open orbits. □

The vector space V decomposes into a finite number of G-orbits;

$$V := \bigsqcup_{\substack{0 \le i \le n \\ 0 \le j \le n-i}} S_i^j \tag{11}$$

where

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$$S_i^j := \{x \in Sym_n(\mathbb{R});\ \operatorname{sgn}(x) = (n-i-j, j)\} \tag{12}$$

with integers $0 \le i \le n$ and $0 \le j \le n-i$. A G-orbit in S is called a *singular orbit.* The subset $S_i := \{x \in V;\ \operatorname{rank}(x) = n - i\}$ is the set of elements of rank $(n-i)$. It is easily seen that $S := \bigsqcup_{1 \le i \le n} S_i$ and $S_i = \bigsqcup_{0 \le j \le n-i} S_i^j$.

Each singular orbit is a stratum which not only is a G-orbit but is an $\mathbf{SL}_n(\mathbb{R})$-orbit. The strata $\{S_i^j\}_{1 \le i \le n, 0 \le j \le n-i}$ have the following closure inclusion relation

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$$\overline{S_i^j} \supset S_{i+1}^{j-1} \cup S_{i+1}^j, \tag{13}$$

where $\overline{S_i^j}$ means the *closure* of the stratum S_i^j.

The support of a singular invariant hyperfunction is a closed set consisting of a union of some strata S_i^j. Since the support is a closed G-invariant subset, we can express the support of a singular invariant hyperfunction as a closure of a union of the highest rank strata, which is easily rewritten by a union of singular orbits.

We naturally ask the following questions.

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Problem 2.1. What are the principal parts of the Laurent expansion of $P^{[\vec{a},s]}(x)$ at poles ? What are their exact orders of poles ? What are the supports of negative-order coefficients of a Laurent expansion of $P^{[\vec{a},s]}(x)$ at poles ?

In order to determine the exact order of $P^{[\vec{a},s]}(x)$ at $s = s_0$, we introduce the coefficient vectors

$$\vec{d}^{(k)}[s_0] := (d_0^{(k)}[s_0], d_1^{(k)}[s_0], \ldots, d_{n-k}^{(k)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^{n-k+1}$$

with $k = 0, 1, \ldots, n$. Here, $(\mathbb{C}^{n+1})^*$ means the dual vector space of $\mathbb{C}^{n+1}$. Each element of $\vec{d}^{(k)}[s_0]$ is a linear form on $\vec{a} \in \mathbb{C}^{n+1}$,i.e., a linear map from $\mathbb{C}$ to $\mathbb{C}^{n+1}$,

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$$d_i^{(k)}[s_0] : \mathbb{C}^{n+1} \ni \vec{a} \longmapsto \langle d_i^{(k)}[s_0], \vec{a} \rangle \in \mathbb{C}.$$

We denote

$$\langle \vec{d}^{(k)}[s_0], \vec{a} \rangle = (\langle d_0^{(k)}[s_0], \vec{a} \rangle, \langle d_1^{(k)}[s_0], \vec{a} \rangle, \ldots, \langle d_{n-k}^{(k)}[s_0], \vec{a} \rangle) \in \mathbb{C}^{n-k+1}.$$

Definition 2.1 (Coefficient vectors $\vec{d}^{(k)}[s_0]$). We define the *coefficient vectors* $\vec{d}^{(k)}[s_0]$ for $(k = 0, 1, \ldots, n)$ by induction on k in the following way.

1. First, we set

$$\vec{d}^{(0)}[s_0] := (d_0^{(0)}[s_0], d_1^{(0)}[s_0], \ldots, d_n^{(0)}[s_0])$$

   such that $\langle d_i^{(0)}[s_0], \vec{a} \rangle := a_i$ for $i = 0, 1, \ldots, n$.

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2. Next, we define $\vec{d}^{(1)}[s_0]$ and $\vec{d}^{(2)}[s_0]$ by

$$\vec{d}^{(1)}[s_0] := (d_0^{(1)}[s_0], d_1^{(1)}[s_0], \ldots, d_{n-1}^{(1)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^n,$$

   with $d_j^{(1)}[s_0] := d_j^{(0)}[s_0] + \epsilon[s_0] d_{j+1}^{(0)}[s_0]$, and

$$\vec{d}^{(2)}[s_0] := (d_0^{(2)}[s_0], d_1^{(2)}[s_0], \ldots, d_{n-2}^{(2)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^{n-1},$$

with $d_j^{(2)}[s_0] := d_j^{(0)}[s_0] + d_{j+2}^{(0)}[s_0]$. Here,

$$\epsilon[s_0] := \begin{cases} 1 & \text{, (if } s_0 \text{ is a half-integer),} \\ (-1)^{s_0+1} & \text{, (if } s_0 \text{ is an integer).} \end{cases}$$

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3. Lastly, by induction on k, we define all the coefficient vectors $\vec{d}^{(k)}[s_0]$ for $k = 0, 1, \ldots, n$ by

$$\vec{d}^{(2l+1)}[s_0] := (d_0^{(2l+1)}[s_0], d_1^{(2l+1)}[s_0], \ldots, d_{n-2l-1}^{(2l+1)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^{n-2l},$$

with $d_j^{(2l+1)}[s_0] := d_j^{(2l-1)}[s_0] - d_{j+2}^{(2l-1)}[s_0]$, and

$$\vec{d}^{(2l)}[s_0] := (d_0^{(2l)}[s_0], d_1^{(2l)}[s_0], \ldots, d_{n-2l}^{(2l)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^{n-2l+1},$$

with $a_j^{(2l)}[s_0] := d_j^{(2l-2)}[s_0] + d_{j+2}^{(2l-2)}[s_0]$.

Using the above mentioned vectors $\vec{d}^{(k)}[s_0]$, we can determine the exact orders of $P^{[\vec{a},s]}(x)$ at poles.

Theorem 2.3. *The exact order of the poles of $P^{[\vec{a},s]}(x)$ is computed by the following algorithm.*

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1. *At $s = -\frac{2m+1}{2}(m = 1, 2, \ldots)$, the coefficient vectors $\vec{d}^{(k)}[-\frac{2m+1}{2}]$ are defined in Definition 2.1. The exact order $P^{[\vec{a},s]}(x)$ at $s = -\frac{2m+1}{2}(m = 1, 2, \ldots)$ is given in terms of the coefficient vector $\vec{d}^{(2k)}[-\frac{2m+1}{2}]$.*
   - *If $1 \leq m \leq \frac{n}{2}$, then $P^{[\vec{a},s]}(x)$ has a possible pole of order less than m.*
     - *If $\langle \vec{d}^{(2)}[-\frac{2m+1}{2}], \vec{a} \rangle = 0$, then $P^{[\vec{a},s]}(x)$ is holomorphic.*
     - *If $\langle \vec{d}^{(4)}[-\frac{2m+1}{2}], \vec{a} \rangle = 0$ and $\langle \vec{d}^{(2)}[-\frac{2m+1}{2}], \vec{a} \rangle \neq 0$, then $P^{[\vec{a},s]}(x)$ has a pole of order 1.*
     - *Generally, for integers p in $1 \leq p < m$, if $\langle \vec{d}^{(2p+2)}[-\frac{2m+1}{2}], \vec{a} \rangle = 0$ and $\langle \vec{d}^{(2p)}[-\frac{2m+1}{2}], \vec{a} \rangle \neq 0$, then $P^{[\vec{a},s]}(x)$ has a pole of order p.*

- *Lastly, if* $\langle \vec{d}^{(2m)}[-\frac{2m+1}{2}], \vec{a}\rangle \neq 0$ *, then* $P^{[\vec{a},s]}(x)$ *has a pole of order* m.

- *If* $m > \frac{n}{2}$, *then* $P^{[\vec{a},s]}(x)$ *has a possible pole of order less than*

$$n' := \begin{cases} \frac{n-1}{2} & , (\text{if } n \text{ is odd}), \\ \frac{n}{2} & , (\text{if } n \text{ is even}). \end{cases}$$

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  - *If* $\langle \vec{d}^{(2)}[-\frac{2m+1}{2}], \vec{a}\rangle = 0$, *then* $P^{[\vec{a},s]}(x)$ *is holomorphic.*
  - *If* $\langle \vec{d}^{(4)}[-\frac{2m+1}{2}], \vec{a}\rangle = 0$ *and* $\langle \vec{d}^{(2)}[-\frac{2m+1}{2}], \vec{a}\rangle \neq 0$, *then* $P^{[\vec{a},s]}(x)$ *has a pole of order* 1.
  - *Generally, for integers* p *in* $1 \le p < n'$, *if* $\langle \vec{d}^{(2p+2)}[-\frac{2m+1}{2}], \vec{a}\rangle = 0$ *and* $\langle \vec{d}^{(2p)}[-\frac{2m+1}{2}], \vec{a}\rangle \neq 0$, *then* $P^{[\vec{a},s]}(x)$ *has a pole of order* p.
  - *Lastly,* $P^{[\vec{a},s]}(x)$ *has a pole of order* n' *if* $\langle \vec{d}^{(n-1)}[-\frac{2m+1}{2}], \vec{a}\rangle \neq 0$ *(*n *is odd) or* $\langle \vec{d}^{(n)}[-\frac{2m+1}{2}], \vec{a}\rangle \neq 0$ *(*n *is even).*

2. *At* $s = -m(m = 1, 2, \dots)$, *the coefficient vectors* $\vec{d}^{(k)}[-m]$ *are defined in Definition 2.1 with* $\epsilon[-m] = (-1)^{-m+1}$. *We obtain the exact order at* $s = -m(m = 1, 2, \dots)$ *in terms of the coefficient vectors* $\vec{d}^{(2k+1)}[-m]$.

   - *If* $1 \le m \le \frac{n}{2}$, *then* $P^{[\vec{a},s]}(x)$ *has a possible pole of order less than* m.

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     - *If* $\langle \vec{d}^{(1)}[-m], \vec{a}\rangle = 0$, *then* $P^{[\vec{a},s]}(x)$ *is holomorphic.*
     - *If* $\langle \vec{d}^{(3)}[-m], \vec{a}\rangle = 0$ *and* $\langle \vec{d}^{(1)}[-m], \vec{a}\rangle \neq 0$ *, then* $P^{[\vec{a},s]}(x)$ *has a pole of order* 1.
     - *Generally, for integers* p *in* $1 \le p < m$, *if* $\langle \vec{d}^{(2p+1)}[-m], \vec{a}\rangle = 0$ *and* $\langle \vec{d}^{(2p-1)}[-m], \vec{a}\rangle \neq 0$ *, then* $P^{[\vec{a},s]}(x)$ *has a pole of order* p.
     - *Lastly, if* $\langle \vec{d}^{(2m-1)}[-m], \vec{a}\rangle \neq 0$ *, then* $P^{[\vec{a},s]}(x)$ *has a pole of order* m.

- *If $m > \frac{n}{2}$, then $P^{[\vec{a},s]}(x)$ has a possible pole of order less than*

$$n' := \begin{cases} \frac{n+1}{2} & \text{, (if } n \text{ is odd),} \\ \frac{n}{2} & \text{, (if } n \text{ is even).} \end{cases}$$

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  - *If $\langle \vec{d}^{(1)}[-m], \vec{a} \rangle = 0$, then $P^{[\vec{a},s]}(x)$ is holomorphic.*
  - *If $\langle \vec{d}^{(3)}[-m], \vec{a} \rangle = 0$ and $\langle \vec{d}^{(1)}[-m], \vec{a} \rangle \neq 0$, then $P^{[\vec{a},s]}(x)$ has a pole of order 1.*
  - *Generally, for integers p in $1 \leq p < n'$, if $\langle \vec{d}^{(2p+1)}[-m], \vec{a} \rangle = 0$ and $\langle \vec{d}^{(2p-1)}[-m], \vec{a} \rangle \neq 0$, then $P^{[\vec{a},s]}(x)$ has a pole of order p.*
  - *Lastly, $P^{[\vec{a},s]}(x)$ has a pole of order n' if $\langle \vec{d}^{(n)}[-m], \vec{a} \rangle \neq 0$ (n is odd) or $\langle \vec{d}^{(n-1)}[-m], \vec{a} \rangle \neq 0$ (n is even).*

The exact support of $P^{[\vec{a},s]}(x)$ is given in the following theorem.

Theorem 2.4 (Support of the singular invariant hyperfunctions). *Let*

$$P^{[\vec{a},s]}(x) = \sum_{-\infty<j<\infty} P_j^{[\vec{a},-\frac{k+1}{2}]}(x)(s + \frac{k+1}{2})^j \tag{14}$$

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be the Laurent expansion of $P^{[\vec{a},s]}(x)$ at $s = -\frac{k+1}{2}$. The support of the coefficients $P_j^{[\vec{a},-\frac{k+1}{2}]}(x)$ is contained in S if $j < 0$.

1. *The support of $P_{-j}^{[\vec{a},-\frac{2m+1}{2}]}(x)$ for ($j = 1, 2, \dots$) is contained in the closure $\overline{S_{2j}}$. More precisely, it is given by*

$$\mathrm{Supp}(P_{-j}^{[\vec{a},-\frac{2m+1}{2}]}(x)) = \overline{(\bigcup_{p\in\{0\leq p\leq n-2j;\langle d_p^{(2j)}[-\frac{2m+1}{2}],\vec{a}\rangle\neq 0\}} S_{2j}^p)}. \tag{15}$$

2. *The support of* $P_{-j}^{[\vec{a},-m]}(x)$ *for* $(j = 1, 2, \dots)$ *is contained in the closure* $\overline{S_{2j-1}}$. *More precisely, it is given by*

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$$\mathrm{Supp}(P_{-j}^{[\vec{a},-m]}(x)) = \overline{(\bigcup_{p\in\{0\le p\le n-2j+1;\langle d_p^{(2j-1)}[-m],\vec{a}\rangle\neq 0\}} S_{2j-1}^{p})}. \tag{16}$$

Example 2.1.

1. $\sum_{i=0}^{n}(-1)^i|P(x)|_i^s$ is holomorphic at $s = -2k+1 (k = 1, 2, \dots)$.
   $\sum_{i=0}^{n}|P(x)|_i^s$ is holomorphic at $s = -2k (k = 1, 2, \dots)$.

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2. $|P(x)|_0^s$ has poles of order just given by (9).

Remark 2.1. Solving the relations given in Theorem 2.3, we can construct infinitely many $\mathbf{SL}_n(\mathbb{R})$-invariant hyperfunctions whose support coincides with the closure of one orbit S_i^j.

we have constructed a suitable basis of the space of singular invariant hyperfunctions on the space of the $n \times n$ real symmetric matrices $\boldsymbol{V} := Sym_n(\mathbb{R})$, and we have given their exact support. Next we shall deal with the same problem on other similar spaces — the space of complex, quaternion and octanion Hermitian matrices.

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Let $\boldsymbol{V} := Her_n(\mathbb{C})$ be the space of $n \times n$ Hermitian matrices over the complex field $\mathbb{C}$, and let $\mathbf{GL}_n(\mathbb{C})$ (reap. $\mathbf{SL}_n(\mathbb{C})$) be the general (resp. special) linear group over $\mathbb{C}$. Then the real algebraic group $\boldsymbol{G} := \mathbf{GL}_n(\mathbb{C})$ operates on the vector space $\boldsymbol{V}$ by

$$g : x \longmapsto g \cdot x \cdot {}^t\overline{g}, \tag{17}$$

with $x \in \boldsymbol{V}$ and $g \in \boldsymbol{G}$. Here, $\overline{g}$ means the complex conjugate matrix of g and ${}^tg$ is the transposition of g.

In the same way, by putting $\boldsymbol{V} := Her_n(\mathbb{H})$ to be the space of $n \times n$ Hermitian matrices over the Hamilton's quaternion field $\mathbb{H}$, and by putting $\mathbf{GL}_n(\mathbb{H})$ (reap. $\mathbf{SL}_n(\mathbb{H})$) to be the general (resp. special) linear group over $\mathbb{H}$, we can consider the same situation. The group $\boldsymbol{G} := \mathbf{GL}_n(\mathbb{H})$ acts on $\boldsymbol{V}$ in the same manner as (17) where $\overline{g}$ means the quaternion conjugate matrix of g.

We consider the complex case (resp. the quaternion case).

Let $P(x) := \det(x)$. Then $P(x)$ is an irreducible polynomial on $\boldsymbol{V}$, and is relatively invariant with respect to the action of $\boldsymbol{G}$ corresponding to the character $|\det(g)|^2$,i.e., $P(g \cdot x) = |\det(g)|^2 P(x)$. The non-singular subset $\boldsymbol{V} - \boldsymbol{S}$ decomposes into $(n+1)$ open $\boldsymbol{G}$-orbits

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$$\boldsymbol{V}_i := \{x \in Her_n(\mathbb{C}); \mathrm{sgn} = (2(n-i), 2i)\}. \tag{18}$$

in the complex case, and

$$\boldsymbol{V}_i := \{x \in Her_n(\mathbb{C}); \mathrm{sgn} = (4(n-i), 4i)\}. \tag{19}$$

in the quaternion case, with $i = 0, 1, \ldots, n$. Here, $\mathrm{sgn}(x)$ for $x \in Her_n(\mathbb{C})$ (resp. $x \in Her_n(\mathbb{C})$) stands for the signature of the quadratic form $q_x(\vec{v}) := {}^t\vec{v} \cdot x \cdot \vec{v}$ on $\vec{v} \in \mathbb{C}^n$ (resp. $\vec{v} \in \mathbb{C}^n$).

We let for a complex number $s \in \mathbb{C}$,

$$|P(x)|_i^s := \begin{cases} |P(x)|^s & ,\text{if } x \in \boldsymbol{V}_i, \\ 0 & ,\text{if } x \notin \boldsymbol{V}_i. \end{cases} \tag{20}$$

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We consider a linear combination of $|P(x)|_i^s$

$$P^{[\vec{a},s]}(x) := \sum_{i=0}^{n} a_i |P(x)|_i^s, \tag{21}$$

with $s \in \mathbb{C}$ and $\vec{a} := (a_0, a_1, \ldots, a_n) \in \mathbb{C}^{n+1}$. Then $P^{[\vec{a},s]}(x)$ is a hyperfunction with a meromorphic parameter $s \in \mathbb{C}$.

Theorem 2.5.
(In the complex case.)

1. *$P^{[\vec{a},s]}(x)$ is holomorphic with respect to $s \in \mathbb{C}$ except for the poles at $s = -k$ with $k = 1, 2, \ldots$.*

2. *The possibly highest order of $P^{[\vec{a},s]}(x)$ at $s = -k$ is given by*

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$$\begin{cases} k & , (k = 1, 2. \ldots, n-1), \\ n & , (k = n, n+1. \ldots,). \end{cases} \tag{22}$$

(In the quaternion case.)

1. *$P^{[\vec{a},s]}(x)$ is holomorphic with respect to $s \in \mathbb{C}$ except for the poles at $s = -k$ with $k = 1, 2, \ldots$.*

2. *The possibly highest order of $P^{[\vec{a},s]}(x)$ at $s = -k$ is given by*

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$$\begin{cases} \lfloor \frac{k+1}{2} \rfloor & , (k = 1, 2. \ldots, 2n-1), \\ n & , (k = 2n, 2n+1. \ldots,). \end{cases} \tag{23}$$

We define the the coefficient vectors $\vec{d}^{(k)}[s_0]$ in the same way as the case of symmetric matrix space.

$$\vec{d}^{(k)}[s_0] := (d_0^{(k)}[s_0], d_1^{(k)}[s_0], \ldots, d_{n-k}^{(k)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^{n-k+1}$$

with $k = 0, 1, \ldots, n$. Here, $(\mathbb{C}^{n+1})^*$ means the dual vector space of $\mathbb{C}^{n+1}$. Each element of $\vec{d}^{(k)}[s_0]$ is a linear form on $\vec{a} \in \mathbb{C}^{n+1}$,i.e., a linear map from $\mathbb{C}$ to $\mathbb{C}^{n+1}$,

$$d_i^{(k)}[s_0] : \mathbb{C}^{n+1} \ni \vec{a} \longmapsto \langle d_i^{(k)}[s_0], \vec{a} \rangle \in \mathbb{C}.$$

We denote

$$\langle \vec{d}^{(k)}[s_0], \vec{a} \rangle = (\langle d_0^{(k)}[s_0], \vec{a} \rangle, \langle d_1^{(k)}[s_0], \vec{a} \rangle, \ldots, \langle d_{n-k}^{(k)}[s_0], \vec{a} \rangle) \in \mathbb{C}^{n-k+1}.$$

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Definition 2.2 (Coefficient vectors $\vec{d}^{(k)}[s_0]$). We define the *coefficient vectors* $\vec{d}^{(k)}[s_0]$ $(k = 0, 1, \ldots, n)$ by induction on k in the following way. Here, for an integer s_0, we set

$$\epsilon[s_0] := (-1)^{s_0+1}$$

1. First, we set

$$\vec{d}^{(0)}[s_0] := (d_0^{(0)}[s_0], d_1^{(0)}[s_0], \ldots, d_n^{(0)}[s_0])$$

such that $\langle d_i^{(0)}[s_0], \vec{a} \rangle := a_i$ for $i = 0, 1, \ldots, n$.

2. Next, we define $\vec{d}^{(1)}[s_0]$ by

$$\vec{d}^{(1)}[s_0] := (d_0^{(1)}[s_0], d_1^{(1)}[s_0], \ldots, d_{n-1}^{(1)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^n,$$

with $d_j^{(1)}[s_0] := d_j^{(0)}[s_0] + \epsilon[s_0] d_{j+1}^{(0)}[s_0]$.

3. Lastly, by induction on k, we define all the coefficient vectors $\vec{d}^{(k)}[s_0]$ for $k = 0, 1, \dots, n$ by

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$$\vec{d}^{(k)}[s_0] := (d_0^{(k)}[s_0], d_1^{(k)}[s_0], \dots, d_{n-k}^{(k)}[s_0]) \in ((\mathbb{C}^{n+1})^*)^{n-k+1},$$

with $d_j^{(k)}[s_0] := d_j^{(k-1)}[s_0] + \epsilon[s_0]d_{j+1}^{(k-1)}[s_0]$.

Using the above mentioned vectors $\vec{d}^{(k)}[s_0]$, we can determine the exact orders of $P^{[\vec{a},s]}(x)$ at poles.

Theorem 2.6. *The exact order of the poles of $P^{[\vec{a},s]}(x)$ is computed by the following algorithm.*

At $s = -m$ $(m = 1, 2, \dots)$, the coefficient vectors $\vec{a}^{(k)}$ are defined in the way as Definition 2.2.

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1. *(In the complex case.) The exact order $P^{[\vec{a},s]}(x)$ at $s = -m$ $(m = 1, 2, \dots)$ is computed by the following algorithm.*
   - *If $1 \leq m \leq n$, then $P^{[\vec{a},s]}(x)$ has a possible pole of order less than m.*
     - *If $\langle \vec{d}^{(1)}[-m], \vec{a} \rangle = 0$, then $P^{[\vec{a},s]}(x)$ is holomorphic.*
     - *If $\langle \vec{d}^{(2)}[-m], \vec{a} \rangle = 0$ and $\langle \vec{d}^{(1)}[-m], \vec{a} \rangle \neq 0$, then $P^{[\vec{a},s]}(x)$ has a pole of order 1.*
     - *Generally, for integers p in $1 \leq p < m$, if $\langle \vec{d}^{(p+1)}[-m], \vec{a} \rangle = 0$ and $\langle \vec{d}^{(p)}[-m], \vec{a} \rangle \neq 0$, then $P^{[\vec{a},s]}(x)$ has a pole of order p.*

– *Lastly, if* $\langle \vec{d}^{(m)}[-m], \vec{a}\rangle \neq 0$, *then* $P^{[\vec{a},s]}(x)$ *has a pole of order* m.

- *If* $m > n$, *then* $P^{[\vec{a},s]}(x)$ *has a possible pole of order less than* n.
  – *If* $\langle \vec{d}^{(1)}[-m], \vec{a}\rangle = 0$, *then* $P^{[\vec{a},s]}(x)$ *is holomorphic.*
  – *If* $\langle \vec{d}^{(2)}[-m], \vec{a}\rangle = 0$ *and* $\langle \vec{d}^{(1)}[-m], \vec{a}\rangle \neq 0$, *then* $P^{[\vec{a},s]}(x)$ *has a pole of order* 1.
  – *Generally, for integers* p *in* $1 \le p < n$, *if* $\langle \vec{d}^{(p+1)}[-m], \vec{a}\rangle = 0$ *and* $\langle \vec{d}^{(p)}[-m], \vec{a}\rangle \neq 0$, *then* $P^{[\vec{a},s]}(x)$ *has a pole of order* p.

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  – *Lastly,* $P^{[\vec{a},s]}(x)$ *has a pole of order* n *if* $\langle \vec{d}^{(n)}[-m], \vec{a}\rangle \neq 0$.

2. *(In the quaternion case.) The exact order* $P^{[\vec{a},s]}(x)$ *at* $s = -m$ $(m = 1, 2, \dots)$ *is computed by the following algorithm.*

- *If* $1 \le m \le 2n - 1$, *then* $P^{[\vec{a},s]}(x)$ *has a possible pole of order less than* $\lfloor \frac{m+1}{2} \rfloor$.
  – *If* $\langle \vec{d}^{(1)}[-m], \vec{a}\rangle = 0$, *then* $P^{[\vec{a},s]}(x)$ *is holomorphic.*
  – *If* $\langle \vec{d}^{(2)}[-m], \vec{a}\rangle = 0$ *and* $\langle \vec{d}^{(1)}[-m], \vec{a}\rangle \neq 0$, *then* $P^{[\vec{a},s]}(x)$ *has a pole of order* 1.
  – *Generally, for integers* p *in* $1 \le p < \lfloor \frac{m+1}{2} \rfloor$, *if* $\langle \vec{d}^{(p+1)}[-m], \vec{a}\rangle = 0$ *and* $\langle \vec{d}^{(p)}[-m], \vec{a}\rangle \neq 0$, *then* $P^{[\vec{a},s]}(x)$ *has a pole of order* p.
  – *Lastly, if* $\langle \vec{d}^{(\lfloor \frac{m+1}{2} \rfloor)}[-m], \vec{a}\rangle \neq 0$, *then* $P^{[\vec{a},s]}(x)$ *has a pole of order* $\lfloor \frac{m+1}{2} \rfloor$.

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- *If* $m > 2n$, *then* $P^{[\vec{a},s]}(x)$ *has a possible pole of order less than* n.
  – *If* $\langle \vec{d}^{(1)}[-m], \vec{a}\rangle = 0$, *then* $P^{[\vec{a},s]}(x)$ *is holomorphic.*
  – *If* $\langle \vec{d}^{(2)}[-m], \vec{a}\rangle = 0$ *and* $\langle \vec{d}^{(1)}[-m], \vec{a}\rangle \neq 0$, *then* $P^{[\vec{a},s]}(x)$ *has a pole of order* 1.
  – *Generally, for integers* p *in* $1 \le p < n$, *if* $\langle \vec{d}^{(p+1)}[-m], \vec{a}\rangle = 0$ *and* $\langle \vec{d}^{(p)}[-m], \vec{a}\rangle \neq 0$, *then* $P^{[\vec{a},s]}(x)$ *has a pole of order* p.
  – *Lastly,* $P^{[\vec{a},s]}(x)$ *has a pole of order* n *if* $\langle \vec{d}^{(n)}[-m], \vec{a}\rangle \neq 0$.

$$V := \bigsqcup_{\substack{0 \le i \le n \\ 0 \le j \le n-i}} S_i^j \tag{24}$$

where

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$$S_i^j := \{x \in Her_n(\mathbb{C});\ \mathrm{sgn}(x) = (2(n-i-j), 2j)\} \tag{25}$$

in the complex case, or

$$S_i^j := \{x \in Her_n(\mathbb{H});\ \mathrm{sgn}(x) = (4(n-i-j), 4j)\} \tag{26}$$

in the quaternion case, with integers $0 \le i \le n$ and $0 \le j \le n-i$.

The subset $S_i := \{x \in V;\ \mathrm{rank}(x) = n-i\}$ is the set of elements of rank $(n-i)$. It is easily seen that $S := \bigsqcup_{1 \le i \le n} S_i$ and $S_i = \bigsqcup_{0 \le j \le n-i} S_i^j$. Each singular orbit is a stratum which not only is a G-orbit but is an $\mathbf{SL}_n(\mathbb{C})$-orbit in the complex case and but is an $\mathbf{SL}_n(\mathbb{H})$-orbit in the quaternion case. The strata S_i^j $(1 \le i \le n, 0 \le j \le n-i)$ have the closure inclusion relation

$$\overline{S_i^j} \supset S_{i+1}^{j-1} \cup S_{i+1}^j. \tag{27}$$

The support of a singular invariant hyperfunction is a closed set consisting of a union of some strata S_i^j. Since the support is a closed G-invariant subset, we can express the support of a singular invariant hyperfunction as a closure of a union of the highest rank strata, which is easily rewritten by a union of singular orbits. The exact support of the Laurent coefficients of $P^{[\vec{a},s]}(x)$ is given by the following theorem.

Theorem 2.7 (Support of the singular invariant hyperfunctions). *Let*

$$P^{[\vec{a},s]}(x) = \sum_{-\infty<j<\infty} P_j^{[\vec{a},-m]}(x)(s+m)^j \tag{28}$$

be the Laurent expansion of $P^{[\vec{a},s]}(x)$ at $s=-m$. The support of the coefficients $P_j^{[\vec{a},-m]}(x)$ is contained in S if $j<0$. At

Slide 30 *$s=-m$ $(m=1,2,\dots)$, the coefficient vectors $\vec{a}^{(k)}$ are defined in the way as Definition 2.2 with $\epsilon=+1$ when m is odd, or with $\epsilon=-1$ when m is even. In both the complex case and the quaternion case, the support of $P_{-j}^{[\vec{a},-m]}(x)$ $(j=1,2,\dots)$ is contained in the closure $\overline{S_j}$. More precisely, it is given by*

$$\mathrm{Supp}(P_{-j}^{[\vec{a},-m]}(x)) = \overline{(\bigcup_{p\in\{0\le p\le n-j;\langle d_p^{(j)}[-m],\vec{a}\rangle\neq 0\}} S_j^p)}. \tag{29}$$

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