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On theories having a small Galois group

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Abstract

Let L be a first order language. Let D be an infinite L-structure and F a definably closed subset of D. Then $\mathrm{Th}(D, F)$ is called small if it has a model (D_1, F_1) such that $\mathrm{Aut}(\mathrm{acl}(F_1)/F_1)$ is small. In this note, we prove the following:

Theorem: Suppose that $\mathrm{Th}(D, F)$ is a small theory with the definable irreducibility property. Then $\mathrm{Aut}(\mathrm{acl}(F)/F) \cong \mathrm{Aut}(\mathrm{acl}(F_1)/F_1)$ for every model (D_1, F_1) of $\mathrm{Th}(D, F)$.

As a corollary we show the following:

Corollary: Let F be a perfect field. Then the absolute Galois group of F is small if and only if the absolute Galois group of F_1 is isomorphic to that of F for any F_1 elementarily equivalent to F.

0. Introduction

Let L be a first order language. Let D be an infinite L-structure and F a definably closed subset of D. By $\mathrm{Aut}(\mathrm{acl}(F)/F)$ we mean a set of permutations of $\mathrm{acl}(F)$ induced by elementary maps which fix F pointwise.

Let us observe the case where F is a pseudo-finite field (see, e.g., [1]) and D is an algebraically closed extension of F. It is seen that F is a perfect field, and so it is definably closed. Then $\mathrm{Aut}(\mathrm{acl}(F)/F)$ coincides with the absolute Galois group of F. It is also known that the absolute Galois group is isomorphic to the profinite completion of the group of integers $\mathbf{Z}$. In this case, $\mathrm{Th}(D, F)$ satisfies the following condition:

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(*) $\mathrm{Aut}(\mathrm{acl}(F)/F) \cong \mathrm{Aut}(\mathrm{acl}(F_1)/F_1)$ for every model (D_1, F_1) of $\mathrm{Th}(D, F)$.

In this note we want to give a criterion for $\mathrm{Th}(D, F)$ to satisfy (*). To state our results, we need some preparations.

In case D is an algebraically closed field we can consider $\mathrm{Aut}(\mathrm{acl}(F)/F)$ as a profinite group. In a general context we can do as well: Let A be a definably closed subset of D such that $F \subset A \subset \mathrm{acl}(F)$. Then we say that A is *normal* over F if it is invariant under $\mathrm{Aut}(\mathrm{acl}(F)/F)$. And we say that A is ***finitely generated*** over F if $A = \mathrm{dcl}(\bar{a}F)$ for some $\bar{a} \in \mathrm{acl}(F)$. Let $\mathcal{A}$ be a family of the subsets of $\mathrm{acl}(F)$ which are finitely generated and normal over F. In the obvious way, $\mathrm{Aut}(\mathrm{acl}(F)/F)$ can be identified with a profinite group:

$$\mathrm{Aut}(\mathrm{acl}(F)/F) \cong \mathrm{projlim}_{A \in \mathcal{A}} \mathrm{Aut}(A/F)$$

Through this isomorphism, the Krull topology is induced on $\mathrm{Aut}(\mathrm{acl}(F)/F)$. A profinite group G is said to be *small* if for any finite groups H there are only finitely many continuous homomorphisms of G into H (see [1, p.185]). In particular the profinite completion of $\mathbf{Z}$ is small.

Here we define $\mathrm{Th}(D, F)$ to be *small* if it has a model (D_1, F_1) such that $\mathrm{Aut}(\mathrm{acl}(F_1)/F_1)$ is small. (Our definition is related to that of Hrushovski. See Remark 2.5). Does $\mathrm{Th}(D, F)$ satisfy the condition (*) if it is small? The answer is No. In general $\mathrm{Th}(D, F)$ does not necessarily satisfy (*), even if it has a model (D_1, F_1) such that $\mathrm{Aut}(\mathrm{acl}(F_1)/F_1)$ is finite. We introduce some property on $\mathrm{Th}(D, F)$, *the definable irreducibility property*, and prove that a small theory with the definable irreducibility property satisfies the condition (*) (Theorem 3.1). As a corollary we give a characterization of a perfect field with a small absolute Galois group (Corollary 3.3).

Notation . We only assume basic knowledge of model theory. For the rest of the paper we fix an infinite L-structure D and a definably closed subset F of D. The type of a over A is denoted by $\mathrm{tp}(a/A)$. We say that p is an algebraic type if it has a finite number of realizations. An element a is algebraic over A if the type $\mathrm{tp}(a/A)$ is an algebraic type. The set of all algebraic elements over A is denoted by $\mathrm{acl}(A)$. If a is the unique realization of $\mathrm{tp}(a/A)$ then we say that a is definable over A. The set of all definable elements over A is denoted by $\mathrm{dcl}(A)$. For any $A \subset B$, $\mathrm{Aut}(B/A)$ means the set of permutations of B induced by elementary maps which fix A pointwise.

Let Th(D, F) be a theory in a language $L \cup \{P\}$, where P is a new unary predicate whose interpretation in (D, F) is F. $L \cup \{P\}$ is denoted by L^* and Th(D, F) by T^*.

1. The Definable Irreducibility Property

1.1. Definition . (i) Let A be a set. Then we say that $\phi(\bar{x}\bar{a}) \in L(A)$ is *A-irreducible* if it is algebraic and isolated over A.
(ii) Let (D_1, F_1) be a model of T^* and $\phi(\bar{x}\bar{a})$ an F_1-irreducible formula with exactly n realizations. Then we say that $\phi(\bar{x}\bar{a})$ has *the definable irreducibility property* (DIP) if there is an L^*-formula $\theta(\bar{y})$ such that for any model (D_2, F_2) of T^* and any realization $\bar{b}$ of θ in F_2, $\phi(\bar{x}\bar{b})$ is an F_2-irreducible formula with n realizations. We denote such a $\theta(\bar{y})$ by $\theta^n_\phi(\bar{y})$.
(iii) We say that T^* has the DIP if any F_1-irreducible formula has the DIP for any model (D_1, F_1) of T^*.

1.2 Example . Let A be a countable set and E an equivalence relation on A with infinitely many four element classes. Let $\{A_n : n < \omega\}$ be an enumeration of the classes of E. For each $n < \omega$, let U_n and V_n be subsets of A such that $A_n = U_n \cup V_n$ and $|U_n| = |V_n| = 2$. Let $\pi : A \to A/E$ be a projection and $F = \{\pi(a) : a \in A\}$. Let $D = (A \cup F, E, \pi, \{U_n\}_{n<\omega}, \{V_n\}_{n<\omega})$. Clearly dcl$(F) = F$. Then $T^* =$ Th(D, F) does not have the DIP: Let $\phi(xy) =$ "$\pi(x) = y$". For every $b \in F$, $\phi(xb)$ is not F-irreducible. On the other hand, for a saturated model (D_1, F_1) of T^*, we can take an element $c \in F_1$ such that $\phi(xc)$ is F_1-irreducible. Then $\phi(xc)$ does not have the DIP.

1.3. Lemma . *If D is an algebraically closed field, then T^* has the DIP.*

Proof . Take any model (D_1, F_1) of T^*. Note that F_1 is a perfect field since it is definably closed. And take an F_1-irreducible formula $\phi(\bar{x}\bar{a})$. Let $\phi(\bar{x}\bar{a})$ have n realizations. Pick a realization $\bar{e}$ of $\phi(\bar{x}\bar{a})$. By the primitive element theorem we can get an element d such that dcl$(dF_1) =$ dcl$(\bar{e}F_1)$. Take an F_1-irreducible formula $\psi(x\bar{b}) \in$ tp(d/F_1). Let $\psi(x\bar{b})$ have exactly m realizations.

First we show that $\psi(x\bar{b})$ has the DIP. By elimination of quantifiers, $\psi(x\bar{b})$ may be identified with a polynomial equation "$p(x, \bar{b}) = 0$" of degree m. Let

X be the set of the general polynomials of degree $< m$. Then we can define $\theta_\psi^m(\bar{z})$ by

$$\bigwedge_{p_1,p_2\in X} \neg\exists\bar{z}_1,\bar{z}_2 \in P\forall x[p(x,\bar{z}) = p_1(x,\bar{z}_1)p_2(x,\bar{z}_2)]$$

Hence $\psi(x\bar{b})$ has the DIP.

We must show that $\phi(x\bar{a})$ has the DIP. Now $\bar{e}$ and d are inter-definable over F_1, and so we can take an L^*-formula $\alpha(\bar{x}x)$ which satisfies

1. $(D_1, F_1) \models \alpha(\bar{e}d)$;
2. For any model (D_2, F_2) of T^*, $(D_2, F_2) \models \alpha(\bar{e}'d')$ implies $\mathrm{dcl}(\bar{e}'F_2) = \mathrm{dcl}(d'F_2)$.

Then set $\theta(\bar{y})$ by

$$\exists^{=n}\bar{x}\phi(\bar{x}\bar{y}) \wedge \exists\bar{z} \in P\forall\bar{x}[\phi(\bar{x}\bar{y}) \wedge \theta_\psi^m(\bar{z}) \to \exists x(\psi(x\bar{z}) \wedge \alpha(\bar{x}x))].$$

The formula $\theta(\bar{y})$ is consistent since it is realized by $\bar{a}$. It is easily seen that $\theta(\bar{y}) = \theta_\phi^n(\bar{y})$. Hence $\phi(x\bar{a})$ has the DIP. This completes the proof of the lemma.

2. Theories with a Small Galois Group

2.1. Definition . We say that T^* has a *small Galois group* (or for short, T^* is small) if it has a model (D_1, F_1) such that $\mathrm{Aut}(\mathrm{acl}(F_1)/F_1)$ is small.

2.2. Example . Let A be an infinite set and E an equivalence relation on A with infinitely many two element classes. Let $\pi : A \to A/E$ be a projection and $F = \{\pi(a) : a \in A\}$. Let $D = (A \cup F, E, \pi)$. It is clear that $\mathrm{dcl}(F) = F$. Take any model (D_1, F_1) of T^* and let $\kappa = |D_1|$. Then it can be seen that $\mathrm{Aut}(\mathrm{acl}(F_1)/F_1)$ is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^\kappa$. Hence $\mathrm{Th}(D, F)$ is not small.

2.3. Lemma . *Let T^* be a theory with the DIP. Let (D_1, F_1) and (D_2, F_2) be models of T^*. Suppose that $A \subset \mathrm{acl}(F_1)$ is finitely generated and normal over F_1. Then there is a $B \subset \mathrm{acl}(F_2)$ which is finitely generated and normal over F_2 such that $\mathrm{Aut}(A/F_1) \cong \mathrm{Aut}(B/F_2)$.*

Proof . Since A is finitely generated over F_1, there is a tuple $\bar{a}$ with $A = \mathrm{dcl}(\bar{a}F_1)$. Let $\{\bar{a}_1, ..., \bar{a}_n\}$ be a set of all conjugates of $\bar{a}$ over F_1. Note that $\bar{a}_i$'s are inter-definable over F_1, since A is normal over F_1. Take an irreducible formula $\phi(\bar{x}_1...\bar{x}_n\bar{c}) \in \mathrm{tp}(\bar{a}_1...\bar{a}_n/F_1)$. Let $\phi(\bar{x}_1...\bar{x}_n\bar{c})$ have m realizations. By the DIP we can get an L^*-formula $\theta^m_\phi(\bar{y})$. For each $g \in \mathrm{Aut}(\mathrm{acl}(F_1)/F_1)$ define $\sigma_g \in S_n$ by

$$\sigma_g(i) = j \Leftrightarrow g(\bar{a}_i) = \bar{a}_j(^\forall i, j \leq n)$$

Let $X = \{\sigma_g \in S_n : g \in \mathrm{Aut}(A/F_1)\}$. Clearly $X \cong \mathrm{Aut}(A/F_1)$. Set a formula $\Sigma(\bar{y})$ by

1. $\theta^m_\phi(\bar{y}) \wedge P(\bar{y})$ and
2. $\bigwedge_{\sigma \in X} \forall \bar{x}_1...\forall \bar{x}_n[\phi(\bar{x}_1...\bar{x}_n\bar{y}) \to \phi(\bar{x}_{\sigma(1)}...\bar{x}_{\sigma(n)}\bar{y})]$.

Then $\Sigma(\bar{y})$ is consistent since it is realized by $\bar{c}$. So we can pick a realization $\bar{d}$ of $\Sigma(\bar{y})$ in (D_2, F_2). By 1, $\phi(\bar{x}_1...\bar{x}_n\bar{d})$ is an F_2-irreducible formula with m realizations. Take a realization $\bar{b}_1...\bar{b}_n$ of $\phi(\bar{x}_1...\bar{x}_n\bar{d})$. Let $B = \mathrm{dcl}(\bar{b}_1F_2)$. Clearly B is finitely generated over F_2. Since $\phi(\bar{x}_1...\bar{x}_n\bar{d})$ is F_2-irreducible, $\bar{b}_i$'s are inter-definable over F_2, and so B is normal over F_2. By 2 we have $X \cong \mathrm{Aut}(B/F_2)$. Hence $\mathrm{Aut}(A/F_1) \cong \mathrm{Aut}(B/F_2)$.

2.4. Lemma . *If T^* is a small theory with the DIP, then* $\mathrm{Aut}(\mathrm{acl}(F_1)/F_1)$ *is small for every model (D_1, F_1) of T^*.*

Proof . Assume otherwise. Then there is a model (D_1, F_1) of T^* such that $\mathrm{Aut}(\mathrm{acl}(F_1)/F_1)$ is not small. So, for some $n < \omega$ there are infinitely many A_i's such that $|\mathrm{Aut}(A_i/F_1)| \leq n$. On the other hand, by our assumption, we can take a model (D_2, F_2) of T^* such that $\mathrm{Aut}(\mathrm{acl}(F_2)/F_2)$ is small. However, using 2.3 we get infinitely many B_i's such that $|\mathrm{Aut}(B_i/F_2)| \leq n$. This contradicts the smallness of $\mathrm{Aut}(\mathrm{acl}(F_2)/F_2)$.

2.5. Remark . In [2] and [3], Hrushovski has defined T^* to be ***bounded*** if $\mathrm{Aut}(\mathrm{acl}(F_1)/F_1)$ is small for every model (D_1, F_1) of T^* (under the stronger assumption than ours). So 2.4 states that every small theory with the DIP is bounded. On the other hand there is a small, unbounded theory (and therefore it does not have the DIP): Let A be a countable set and E an equivalence relation on A with infinitely many two element classes. Let

$\{U_i\}_{i<\omega}$ be an enumeration of all classes of E. For each $i < j < \omega$, let $f_{ij} : U_i \to U_j$ be a bijection. Let $\pi : A \to A/E$ be a projection and $F = \{\pi(a) : a \in A\}$. Set $D = (A \cup F, E, \pi, \{U_i\}_{i<\omega}, \{f_{ij}\}_{i<j<\omega})$. Then, for each model (D_1, F_1) of $\mathrm{Th}(D, F)$ we have

$$\mathrm{Aut}(\mathrm{acl}(F_1)/F_1) \cong (\mathbf{Z}/2\mathbf{Z})^{\kappa+1},$$

where $\kappa = |F_1 - F|$. So $\mathrm{Aut}(\mathrm{acl}(F_1)/F_1)$ is small if κ is finite. Otherwise it is not small.

3. Theorem and Corollary

In this section, we prove that a small theory with the DIP satisfies the condition (*) in the introduction. As a corollary we give a characterization of a perfect field with a small absolute Galois group.

3.1. Theorem . *Let D be an infinite structure and F a definably closed subset of D. Suppose that $\mathrm{Th}(D, F)$ is a small theory with the definable irreducibility property. Then $\mathrm{Aut}(\mathrm{acl}(F_1)/F_1) \cong \mathrm{Aut}(\mathrm{acl}(F)/F)$ for every model (D_1, F_1) of $\mathrm{Th}(D, F)$.*

Proof . Take any models (D_1, F_1) and (D_2, F_2) of $\mathrm{Th}(D, F)$. Let $G = \mathrm{Aut}(\mathrm{acl}(F_1)/F_1)$ and $H = \mathrm{Aut}(\mathrm{acl}(F_2)/F_2)$. We will show that $G \cong H$. Let $\mathcal{A}, \mathcal{B}$ be families of all subsets which are finitely generated and normal over F_1, F_2 respectively. For each $n < \omega$ let

$$A_n = \mathrm{dcl}(\bigcup\{A \in \mathcal{A} : |\mathrm{Aut}(A/F_1)| \leq n\});$$

$$B_n = \mathrm{dcl}(\bigcup\{B \in \mathcal{B} : |\mathrm{Aut}(B/F_2)| \leq n\});$$

$$G_n = \mathrm{Aut}(A_n/F_1); H_n = \mathrm{Aut}(B_n/F_2).$$

By 2.4, G and H are small. So, for each $n < \omega$, A_n and B_n are finitely generated and hence G_n and H_n are finite groups.

First we see that $G_n \cong H_n$ for each $n < \omega$. Fix any $n < \omega$. By 2.3 we can take an element $B \in \mathcal{B}$ such that $\mathrm{Aut}(B/F_2) \cong G_n$. Then we have $B \subset B_n$. Hence there is a homomorphism of G_n onto H_n. By the similar argument, we obtain a homomorphism of H_n onto G_n. Hence G_n and H_n are isomorphic.

Next find isomorphisms $\Phi_n : G_n \to H_n$ $(n < \omega)$ satisfying

$$n < m < \omega \Rightarrow \pi^H_{mn} \circ \Phi_m = \Phi_n \circ \pi^G_{mn},$$

where $\pi^G_{mn} : G_m \to G_n$ and $\pi^H_{mn} : H_m \to H_n$ are canonical projections. In fact we can get such isomorphisms, since the number of isomorphisms of G_n with H_n is at most finite. Using the sequence $(\Phi_n)_{n<\omega}$, we can define an isomorphism of G with H in a natural way.

3.2. Remarks . (i) The theory of the example in 2.5 is small, but does not satisfy the condition (*). This shows that the DIP is necessary for the above theorem.
(ii) Let $\{U_i\}_{i<\omega}$ be a disjoint family of two element sets, and let $A = \bigcup_{i<\omega} U_i$. Let F be an arbitrary set which is distinct from A. Put $D = (A \cup F, \{U_i\}_{i<\omega})$. Then it is clear that $\mathrm{Th}(D, F)$ has the DIP. For each model (D_1, F_1) of $\mathrm{Th}(D, F)$, we have $\mathrm{Aut}(\mathrm{acl}(F_1)/F_1) \cong (\mathbf{Z}/2\mathbf{Z})^\omega$. Hence $\mathrm{Th}(D, F)$ satisfies the condition (*), but is not small.

3.3. Corollary . *Let F be a perfect field. Then the following are equivalent:*
(i) The absolute Galois group of F is small;
(ii) If F_1 is elementarily equivalent to F, then the absolute Galois group of F_1 is isomorphic to that of F.

Proof . Let D be the algebraic closure of F. Let $T^* = \mathrm{Th}(D, F)$. By 1.3 T^* has the DIP. Note that if a field F_1 is elementarily equivalent to F then there is a structure D_1 such that (D_1, F_1) is a model of $\mathrm{Th}(D, F)$. So, by 3.1 we obtain the implication (i)$\to$ (ii). We must show (ii)$\to$ (i). Suppose that $\mathrm{Aut}(D/F)$ is not small. Then, for some $n < \omega$ there is an infinite set $\{A_i\}_{i<\kappa}$ of the finitely generated normal extensions of F such that $|\mathrm{Aut}(A_i/F)| = n$ for each $i < \kappa$. For each $i < \kappa$ let d_i be a primitive element such that $A_i = \mathrm{dcl}(d_i F)$. Let $p(x, \bar{y})$ be a general polynomial of degree n. Then there is a set $\{\bar{a}_i\}_{i<\kappa}$ of n-tuples from F such that d_i is a solution of $p(x, \bar{a}_i)$ for each $i < \kappa$. Take any $\lambda > \kappa$. By compactness there are a model (D_1, F_1) of T^*, a set $\{\bar{b}_i\}_{i<\lambda}$ of n-tuples from F_1 and a set $\{e_i\}_{i<\lambda}$ of elements of D_1 with the following property:

1. $p(x, \bar{b}_i)$ is an F_1-irreducible polynomial with a solution e_i, for each $i < \lambda$;

2. The solutions of $p(x, \bar{b}_i)$ are inter-definable over F_1, for each $i < \lambda$;

3. $\mathrm{dcl}(e_i F_1) \neq \mathrm{dcl}(e_j F_1)$, for each $i, j < \lambda$ with $i \neq j$.

For each $i < \lambda$ let $B_i = \mathrm{dcl}(e_i F_1)$. By 1, 2 and 3, $\{B_i\}_{i<\lambda}$ is the family of the distinct finitely generated normal extensions of F_1 such that $|\mathrm{Aut}(B_i/F_1)| = n$ for each $i < \lambda$. Hence the absolute Galois group of F_1 is not isomorphic to that of F. This completes the proof of the corollary.

Reference

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