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Stability Analysis of Numerical Solution of Stochastic Differential Equations

三井　斌友
Taketomo MITSUI*
名古屋大学人間情報学研究科
Graduate School of Human Informatics,
Nagoya University, Nagoya 464-01, Japan
e-mail: a41794a@nucc.cc.nagoya-u.ac.jp

1 Introduction

Our present aim is to show a linear stability analysis of numerical solution of stochastic differential equations (SDEs). As in other areas of numerical analysis, *numerical stability* is significant in the case of SDEs which usually require a long (numerical) time-integration. We will take the viewpoint how the analysis for SDEs is similar to as well as different from that for ordinary differential equations (ODEs), for the corresponding theory has been well developed in the ODE case. On the contrary stability analysis for numerical schemes of SDEs is still in a premature stage, although much work has been devoted to it. Here some previous trials for analytical and numerical stability concept in SDEs will be arranged to clear their interrelationships.

1.1 Stochastic initial-value problems

We are concerned with the initial-value problem (IVP) of SDE of Itô-type given as follows.

$$\begin{cases} dX(t) = f(t, X(t))dt + g(t, X(t))dW(t) & (t > 0), \\ X(t) = X_0, & \text{initial condition.} \end{cases} \tag{1.1}$$

Here $W(t)$ stands for the standard Wiener process. Under appropriate assumptions for the functions f and g, the solution $X(t)$, as a random process, of IVP (1.1) is known to exist in the Itô sense. Since many phenomena in science and engineering can be formulated with IVP of SDEs and the problems are often not known to have analytical solutions, numerical solutions turn out to be practically important, and have been developed for the last decade. Its state-of-the-art can be found in *e.g.* [8].

*Joint work with Yoshihiro SAITO and Yoshio KOMORI

1.2 Syntax diagram of stability

For the analysis of numerical stability of differential equations, its distinction and relationship with the stability of the underlying differential equation is considered to be crucial. Hence, after LAMBERT ([12], p38), we introduce a syntax diagram of stability.

Consider IVP of ODEs given by

$$\frac{dy}{dt} = f(t, y) \quad (t > 0), \quad y(0) = y_0, \tag{1.2}$$

and its numerical (discrete variable) solution $\{y_n \ (n = 0, 1, \ldots)\}$ with a fixed stepsize h and step-points $\{t_n; t_n = nh\}$ in the usual manner.

A stability definition can be broken down into the following components:

1. We impose certain conditions C_p on (1.2) which force the exact solution $y(t)$ to display a certain stability property.
2. We apply a numerical method to the problem, assumed to satisfy C_p.
3. We ask what conditions C_m must be imposed on the method in order that the numerical solution displays a stability property analogous to that displayed by the exact solution.

The components and the process can be readily understandable through a diagram shown as in Fig. 1.1.

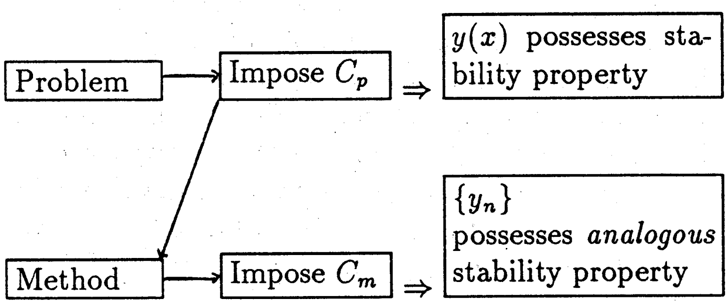

Figure 1.1: Generic syntax diagram of stability

In this way, the syntax diagram for zero-stability of numerical solution can be written ([12]) as in Fig. 1.2.

More important concept is A-stability. When a linear system of ODEs

$$\frac{dy}{dt} = Ay, \tag{1.3}$$

is considered with a d-dimensional matrix A, the asymptotic stability of the solution, *i.e.*, $\| y(t) \| \to 0$ as $t \to \infty$, is expected to hold with a certain norm of vectors. Its counterpart in numerics is *absolute stability*. By introducing the region of absolute stability, $\mathcal{R}$, of a linear multistep or Runge-Kutta method ([3]), the syntax diagram of absolute stability is given by Fig. 1.3. As can be seen, the absolute stability depends on the magnitude of the stepsize h. A method is called A-stable if it is absolutely stable for *any* h. Taking the asymptotic stability of the underlying ODE system into account, A-stability can be said to be a kind of ideal concept of stability in numerical ODEs. However, many barriers are known for A-stable numerical schemes in ODEs ([4]).

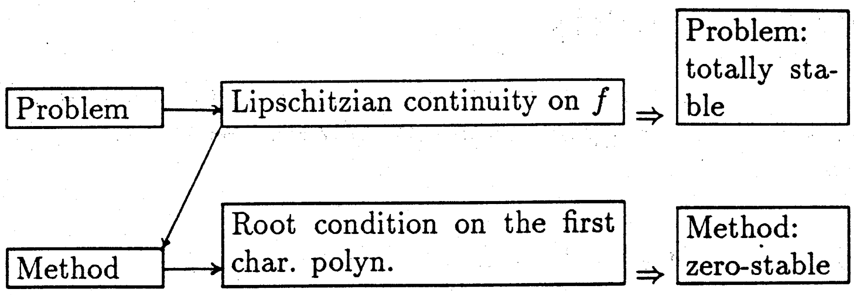

Figure 1.2: Syntax diagram of zero-stability

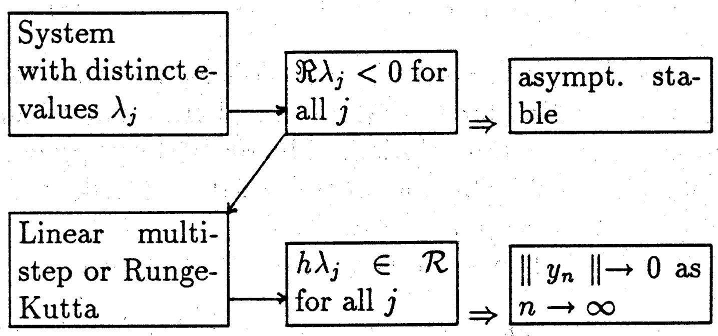

Figure 1.3: Syntax diagram of absolute stability

1.3 Carrying over to SDE

To carry over the usefulness of numerical stability analysis to SDE case, the following questions should be resolved in turn:

Q1. What kind of stability concept is adopted in (analytic) SDE?

Because of its statistical nature, IVP of SDEs is followed by plenty concepts of stability ([5]).

Q2. What is the condition or criterion of stability?

It corresponds to C_p in Fig. 1.1.

Q3. What scheme is to be considered in numerical SDE?

Various numerical schemes have been proposed for SDEs. As we will see later in Section 5, in some cases we must pay attention to the realization means of approximation of the increment of the Wiener process with respect to the numerical stability.

Q4. What is the stability concept in numerical SDE?

Q5. What is the condition or criterion of numerical stability?

It corresponds to C_m in Fig. 1.1.

Q6. How is the analysis confirmed numerically?

2 Stochastic stability

Here we will briefly describe how the stability concept is introduced into SDEs. Then the first trial is given for a syntax diagram of stability. However, we will see a naïve introduction cannot get a success.

2.1 Introduction of stochastic stability

As in (1.1), consider a scalar Itô-type SDE

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW(t) \quad (t > t_0),$$

together with a *nonrandom* initial value $X(t_0) = x_0$. We assume that there exists a unique solution $X(t; t_0, x_0)$ of the equation for $t > t_0$. Some sufficient conditions have been established for the unique existence of the solution. Moreover, we suppose that the equation allows a steady solution $X(t) \equiv 0$. This means the equation $f(t, 0) = g(t, 0) = 0$ holds. A steady solution is often called an *equilibrium position*.

HAS'MINSKII ([5]) gave the following three definitions of stability.

Definition 2.1 *The equilibrium position of the SDE is said to be stochastically stable if for all positive ε and for all t_0 the equality*

$$\lim_{x_0 \to 0} P\left(\sup_{t \geq t_0} |X(t; t_0, x_0)| \geq \varepsilon\right) = 0$$

holds.

Definition 2.2 *The equilibrium position is said to be stochastically asymptotically stable if, in addition to the above condition in Definition 2.1, the equality*

$$\lim_{x_0 \to 0} P\left(\lim_{t \to \infty} |X(t; t_0, x_0)| = 0\right) = 1$$

holds.

Definition 2.3 *The equilibrium position is said to be stochastically asymptotically stable in the large if, moreover to the above two conditions, the equality*

$$P\left(\lim_{t \to \infty} |X(t; t_0, x_0)| = 0\right) = 1, \quad \textit{for all} \quad x_0$$

holds.

Definitions 2.1, 2.2 and 2.3 can be seen as the counterparts of stability, asymptotic stability and asymptotic stability in the large, respectively, in the ODE case. Henceforth they can be a basis of numerical stability consideration.

Actually we can derive a criterion of the asymptotic stochastic stability for the SDE. Assume that the functions f and g are uniformly asymptotically linear with respect to x, that is to say, for certain real constants a and b the equations

$$f(t,x) = ax + \bar{f}(t,x) \quad \text{and} \quad g(t,x) = bx + \bar{g}(t,x)$$

hold with

$$\lim_{|x|\to 0} \frac{|\bar{f}(t,x)| + |\bar{g}(t,x)|}{|x|} = 0$$

uniformly in t. The solution $X(t)$ of the SDE is stochastically asymptotically stable if the condition

$$a - \frac{1}{2}b^2 < 0 \tag{2.1}$$

holds (see [2], p139).

This criterion strongly suggests a possibility of analogous *linear stability* analysis for numerical schemes of SDE to those of ODE. We can consider that the linear parts of f and g are dominant in the asymptotic behaviour of solutions around the equilibrium position.

2.2 Numerical stability along stochastic stability

Following the suggestion in the previous subsection, we introduce a linear test equation (supermartingale equation)

$$dX(t) = \lambda X(t)dt + \mu X(t)dW(t) \quad (t > 0) \tag{2.2}$$

with the initial condition $X(0) = 1$ to the numerical stability analysis. Here λ and μ are *complex numbers*.

Since the exact solution of (2.2) is written as

$$X(t) = \exp\{(\lambda - \frac{1}{2}\mu^2)t + \mu W(t)\},$$

it is quite easy to show that the equilibrium position $X(t) \equiv 0$ is stochastically asymptotically stable if the condition

$$\Re(\lambda - \frac{1}{2}\mu^2) < 0 \tag{2.3}$$

holds. In our mind we employ the condition which can stand for C_p in the syntax diagram.

A typical example of *numerical scheme* is the Euler-Maruyama scheme given as follows. Let h be the stepsize of the variable t, $t_n = nh, (n = 1, 2, \cdots)$ the step-points, and

$$\Delta W_n = W(t_{n+1}) - W(t_n)$$

the increment of the Wiener process at the n-th step-point. For the equation (1.1) the scheme generates a discrete random process $\{X_n\}$ according to the recurrence

$$X_{n+1} = X_n + f(t_n, X_n)h + g(t_n, X_n)\Delta W_n. \tag{2.4}$$

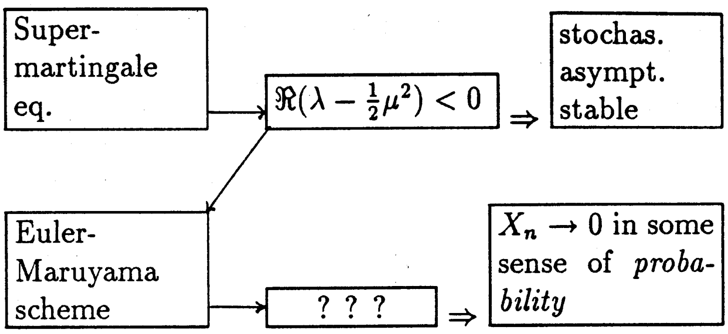

Figure 2.1: A syntax diagram?

The increment ΔW_n will be realized with certain normal random numbers with mean 0 and variance $\sqrt{h}$.

Thus we think of a syntax diagram shown in Fig. 2.1.

However, this syntax diagram *does not* work well. The reason follows. The criterion (2.3) for the stochastic asymptotic stability of (2.2) allows the case $\Re\lambda > 0$. It implies that some sample paths of the solution surely *decrease* to 0, whereas their distributions possibly *increase*. This can be understood through the fact that when $\Re\lambda > 0$ the equation cannot be asymptotically stable even in the ODE sense. Henceforth it is impossible to carry out a numerical scheme *until* all the sample paths of the exact solution diminish to 0 if two conditions $\Re\lambda > 0$ and $\Re(\lambda - \frac{1}{2}\mu^2) < 0$ are valid simultaneously. Since the numerical solution by *e.g.* the Euler-Maruyama scheme would reflect this statistical property, nobody can expect a numerically stable solution.

This investigation implies the necessity of another stability concept for SDEs. That is, we try to answer the question what SDE is having all sample paths whose distribution tends to 0 as $t \to \infty$.

3 MS-stability

Analysis of the previous section suggests an introduction of norm of the SDE solution with respect to the stability concept.

3.1 Asymptotic stability in p-th mean

Return to the general IVP of SDE given in (1.1):

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW(t) \quad (t > t_0), \qquad X(t_0) = x_0.$$

Definition 3.1 *The steady solution $X(t) \equiv 0$ is said to be asymptotically stable in p-th mean if for all positive ε there exists a positive δ which satisfies*

$$\mathbf{E}\left(|X(t)|^p\right) < \varepsilon \quad \textit{for all} \quad t \geq 0 \quad \textit{and} \quad |x_0| < \delta \tag{3.1}$$

and, furthermore if there exists a positive δ_0 satisfying

$$\lim_{t\to\infty} \mathbf{E}\left(|X(t)|^p\right) = 0 \quad \textit{for all} \quad |x_0| < \delta_0 \,. \tag{3.2}$$

Here **E** *menas the mathematical expectation.*

Roughly speaking, by the asymptotic stability in p-th mean we expect the asymptotic diminishing of the solution in the p-th moment.

The case of $p = 2$ is most frequently used and called the *mean-square case.* Thus we introduce the norm of the solution by

$$\|X\| = \{\mathbf{E}|X|^2\}^{\frac{1}{2}}.$$

The necessary and sufficient condition is given in the following.

Lemma 3.1 *The linear test equation (supermartingale equation) (2.2) with the unit initial value is asymptotically stable in the mean-square sense iff the inequality*

$$2\Re\lambda + |\mu|^2 < 0$$

holds.

Proof. For the solution $X(t)$ of (2.2) with the unit initial condition, its mean-square $Y(t) = \mathbf{E}|X(t)|^2$ satisfies an IVP of ODE

$$dY = (2\Re\lambda + |\mu|^2)Ydt \quad (t > 0), \qquad Y(0) = 1.$$

The solution is obviously asymptotically stable when the inequality holds, and vice versa. □

Note that since the inequality $\Re(2\lambda - \mu^2) \leq 2\Re\lambda + |\mu|^2$ is always valid, the asymptotic stability in the mean-square sense implies the stochastic stability.

In the sequel, the stability in the mean-square sense will be abbreviated as MS-stability.

3.2 Numerical MS-stability

For asymptotically MS-stable problems of SDEs, what conditions are imposed to derive numerically asymptotically MS-stable solutions? That is to say, what conditions should be for the numerical solution $\{X_n\}$ of the linear test equation (2.2) to achieve

$$\|X_n\| \to 0 \quad \text{as } n \to \infty.$$

Denote $\mathbf{E}|X_n|^2$ by Y_n. When we apply a numerical scheme to the linear test equation and take the mean-square norm, we obtain a one-step difference equation of the form

$$Y_{n+1} = R(\bar{h}, k)Y_n \tag{3.3}$$

where two scalars $\bar{h}$ and k stand for $h\lambda$ and μ^2/λ, respectively. We can call $R(\bar{h}, k)$ the *stability function* of the scheme, for the MS-stability of the numerical scheme is subject to its magnitude. That is, the equivalence

$$Y_n \to 0 \quad \text{as} \quad n \to \infty \Leftrightarrow |R(\bar{h}, k)| < 1$$

holds.

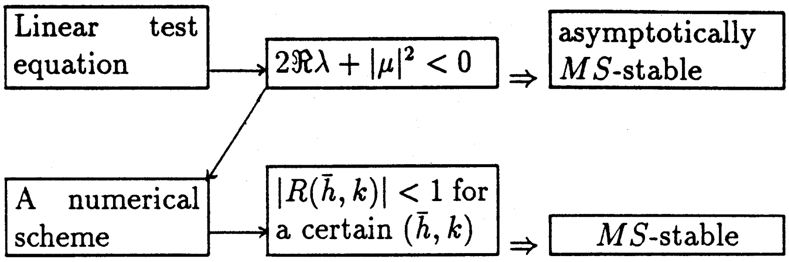

Figure 3.1: Syntax diagram of MS-stability

Definition 3.2 ([14]) *The scheme is said to be MS-stable for $\bar{h}$ and k if its stability function $R(\bar{h}, k)$ is less than unity in magnitude. The set in $\mathbb{C}^2$ given by*

$$\mathcal{R} = \{(\bar{h}, k);\ |R(\bar{h}, k)| < 1 \text{ holds}\}$$

is called the domain of MS-stability of the scheme.

The syntax diagram of MS-stability is in Fig. 3.1.

To compare the stability performance of various numerical schemes, we are to draw their figures. However, the complex values λ and μ yield the pair $(\bar{h}, k)$ in *four* dimensions! We have to restrict ourselves to the case of real values of λ and/or μ for viewing the figures.

In addition, we can say that a numerical scheme is A-stable if it is MS-stable for any h.

3.3 Stability function of some schemes

We will derive the stability function of some numerical schemes known in the literature. Details with figures will appear in [14].

First is the Euler-Maruyama scheme (2.4), whose application to (2.2) implies

$$X_{n+1} = X_n + h\lambda X_n + \mu X_n \Delta W_n.$$

We obtain the stability function as

$$R(\bar{h}, k) = |1 + \bar{h}|^2 + |k\bar{h}|.$$

Fortunately the function depends on $\bar{h}$ and $|k|$, not on k. Therefore we can get the feature of the domain of MS-stability in the three-dimensional space of $(\bar{h}, |k|)$.

Next is the semi-implicit Euler scheme given by

$$X_{n+1} = X_n + \{\alpha f(t_{n+1}, X_{n+1}) + (1 - \alpha) f(t_n, X_n)\} h + g(t_n, X_n) \Delta W_n, \tag{3.4}$$

where α is a parameter representing its implicitness. Note we assume the implicitness only on the drift term f. A calculation leads to the stability function

$$R(\bar{h}, k, \alpha) = \frac{|1 + (1 - \alpha)\bar{h}|^2 + |k\bar{h}|}{|1 - \alpha\bar{h}|^2}.$$

By comparing the regions of MS-stability of the Euler-Maruyama and the semi-implicit Euler schemes under the restriction of real $\bar{h}$ and k we can see that the latter is superior to the former with respect to the stability.

4 Extension to multi-dimensional case

Different from the ODE case, the extension of linear stability analysis from the scalar to the multi-dimensional equation is not straightforward in the SDE case. Recall the syntax diagram of absolute stability of the ODE case shown in Fig. 1.3. There the product of the stepsize h and an eigenvalue of the coefficient matrix discriminates the absolute stability of the numerical solution. This is due to the *linearity* of the numerical schemes.

On the analogy of this, we try to consider the linear multi-dimensional test system of SDEs

$$dX(t) = AX(t)dt + BX(t)dW(t) \quad (t > 0) \tag{4.1}$$

with the initial condition $X(0) = X_0$, where $X \in \mathbb{R}^d, A$ and $B \in \mathbb{R}^{d\times d}$. Furthermore we assume that $W(t)$ is a *scalar*.

Even though, a linear stability analysis is still too difficult for (4.1), because the second moment $Y(t) = \mathrm{E}(X(t)X(t)^T)$ obeys the following IVP of the matrix ODE

$$\frac{dY}{dt} = AY + YA^T + BYB^T \quad (t > 0), \qquad Y(0) = X_0X_0^T. \tag{4.2}$$

This IVP is hard to handle.

4.1 Simultaneously diagonalizable case

For a simpler case, we assume that the matrices A and B in (4.1) are simultaneously diagonalizable. That is to say, there exists a nonsingular matrix $T \in \mathbb{C}^{d\times d}$ satisfying the equations

$$\Lambda = T^{-1}AT \quad \text{and} \quad M = T^{-1}BT,$$

where

$$\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_d) \quad \text{and} \quad M = \mathrm{diag}(\mu_1, \cdots, \mu_d)$$

with $\lambda_j, \mu_j \in \mathbb{C}$.

The transformed second moment $Z(t) = T^{-1}Y(t)T^{-H}$ (hereafter T^H stands for the Hermitian conjugate of T) fulfills the IVP of ODEs

$$\frac{dZ}{dt} = \Lambda Z + Z\Lambda^H + MZM^H, \quad Z(0) = T^{-1}X_0X_0^HT^{-H}. \tag{4.3}$$

Denoting the (i,j)-component of $Z(t)$ by $z_{ij}(t)$, we can show that the asymptotic stability of all the diagonal component $\{z_{ii}(t)\}$ is equivalent to the MS-stability of linear multi-dimensional test equation (4.1). Due to the diagonality of Λ and M, we obtain ODE

$$\frac{dz_{ii}}{dt} = \left(\lambda_i + \overline{\lambda}_i + |\mu_i|^2\right) z_{ii},$$

which yields the criterion of asymptotic stability so that the inequality

$$2\Re\lambda_i + |\mu_i|^2 < 0$$

holds for all i.

Henceforth we can conclude that the MS-stability of linear multi-dimensional test equation (4.2) is *eventually* equivalent to that of the scalar equation

$$dX(t) = \tilde{\lambda}X(t)dt + \tilde{\mu}X(t)dW(t) \ (t > 0) \tag{4.4}$$

where $\tilde{\lambda}$ and $\tilde{\mu}$ are constants so that the equality

$$2\Re\tilde{\lambda} + |\tilde{\mu}|^2 = \max\left\{2\Re\lambda_i + |\mu_i|^2\right\}$$

holds.

4.2 Proposed test equation and an ROW-type scheme

From the viewpoint described above, we have proposed a 2-dimensional linear SDE given in the following to analyse numerical stabilities in the multidimensional case ([9]).

$$d\boldsymbol{X}(t) = A\boldsymbol{X}(t)dt + B\boldsymbol{X}(t)dW(t),$$

with the matrices

$$A = \begin{bmatrix} 0 & 1 \\ \beta & \gamma \end{bmatrix}, \qquad B = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}, \tag{4.5}$$

and the scalar Wiener process $W(t)$. The exact solution of this equation is given analytically as follows. Denoting the time-increment $t - t_0$ and the increment of the Wiener process $W(t,\omega) - W(t_0,\omega)$ by Δ and ΔW, respectively, and introducing the notations

$$p = -\frac{\alpha^2}{2}\Delta + \alpha\Delta W, \quad S_q = \sqrt{\gamma^2 + 4\beta}, \quad \lambda_1 = p + \frac{\gamma\Delta + S_q\Delta}{2}, \quad \lambda_2 = p + \frac{\gamma\Delta - S_q\Delta}{2},$$

$$\Lambda^+ = e^{\lambda_1\Delta} + e^{\lambda_2\Delta}, \quad \Lambda^- = e^{\lambda_1\Delta} - e^{\lambda_2\Delta},$$

we can express the exact solution as

$$\boldsymbol{X}(t) = -\frac{1}{4S_q}\begin{bmatrix} 2\gamma\Lambda^- - 2S_q\Lambda^+ & -4\Lambda^- \\ -4\beta\Lambda^- & -2\gamma\Lambda^- - 2S_q\Lambda^+ \end{bmatrix}\boldsymbol{X}(t_0). \tag{4.6}$$

Note that the solution depends on the increments, not on the intermediate values between t_0 and t.

As an example of linear stability analysis, let us analyse PLATEN's scheme of weak order two ([8], p485). The scheme applied to the scalar linear test equation (4.4) with $\tilde{\lambda}$ and $\tilde{\mu}$ yields the linear recurrence

$$X_{n+1} = \left\{1 + \tilde{\lambda}h + \tilde{\mu}\Delta W_n + \tilde{\lambda}\tilde{\mu}h\Delta W_n + \frac{1}{2}\tilde{\mu}^2\{(\Delta W_n)^2 - h\} + \frac{1}{2}\tilde{\lambda}^2h^2\right\}X_n, \tag{4.7}$$

the multiplying factor of whose right-hand side is denoted by $P(h, \Delta W_n)$. By the definition, $\mathbf{E}|P(h,\cdot)|^2$ will give the region of MS-stability of the scheme, while $\mathbf{E}P(h,\cdot) = 1 + \tilde{\lambda}h + \frac{1}{2}\tilde{\lambda}^2h^2$ leads to the condition of asymptotic stability of the mean-value.

However, we can observe that an application of PLATEN's scheme to the 2-dimensional test equation with the parameter values

$$\alpha = 3, \quad \beta = -93, \quad \gamma = -25$$

and the initial values

$$X^{(1)}(0) = 1, \quad X^{(2)}(0) = 0$$

brings a numerically MS-unstable solution even for the stepsize $h = 2^{-3}$. This can be considered so that the stepsize still falls into the instability region of the mean-value.

Based on the observation, we try to design an ROW-type scheme with suitable stability features. Order conditions of the ROW-type scheme in the weak sense can be derived by using *rooted tree* analysis ([10]). Taking advantage of the result, we obtain a 4-stage second-order scheme of ROW-type, which is initially appicable to the Stratonovich-type SDEs, with A- and other desired stability properties when assuming $\tilde{\mu}$ is real. Details will be found in [11].

5 T-stability

We have described in Sections 3 and 4 the analytical and numerical MS-stability. From the viewpoint of computer implementation, MS-stability may still cause difficulty. The reason follows.

To evaluate the quantity of the expectation

$$Y_n = \mathbf{E}(|X_n|^2)$$

where X_n is an approximating sequence of the solution sample path, in a certain probability X_n *happens to overflow* in computer simulations. This actually violates the evaluation of Y_n.

The above situation suggests an introduction of another stability notion with respect to the approximate sequence of sample path (trajectory). It must take into account the *driving process*, whose way of realization a numerical scheme for SDE requires for the increment ΔW_n of the Wiener process. For example, in the Euler-Maruyama scheme given in (2.4) as

$$X_{n+1} = X_n + f(t_n, X_n)h + g(t_n, X_n)\Delta W_n,$$

ΔW_n which stands for $W(t_{n+1}) - W(t_n)$ can be exactly realized with $\xi_n\sqrt{h}$ where ξ_n is a normal random variable with zero mean and unit variance. More sophisticated schemes need more complicated normal random variables. And these random variables are to be realized through an *approximation* with pseudo-random numbers on computer, for the normal random number requires infinitely many trials.

Therefore, we arrive at the following.

Definition 5.1 *Assume that the inequality $\Re\left(\lambda - \frac{1}{2}\mu^2\right) < 0$ holds for the scalar linear test equation (2.2), that is, the test equation is stochastically asymptotically stable in the large. Denote by $\{X_n\}$ $(n = 1, 2, \ldots)$ the sequece of approximate solutions of the equation by a certain numerical scheme.*

The numerical scheme equipped with a specified driving process said to be T-stable if $|X_n| \to 0$ $(n \to \infty)$ holds for the driving process.

5.1 How to get a criterion

The above definition looks appropriate for numerical simulations. However we meet another problem: A criterion of T-stability depends not only on the scheme but also on the driving process. It causes our analysis more difficult. At present available simple results are only for the Euler-Maruyama scheme with *two- or three-point random variables.*

This approximation means as follows. The increment ΔW_n is approximated with $U_n\sqrt{h}$, where U_n is either of the following probability distributions.

i) Two-point random variables

$$P(U_n = \pm 1) = 1/2$$

ii) Three-point random variables

$$P(U_n = \pm\sqrt{3}) = 1/6, \quad P(U_n = 0) = 2/3$$

Applying the Euler-Maruyama scheme to the scalar test equation (2.2) yields

$$\begin{aligned} X_{n+1} &= (1 + \lambda h + \mu U_n \sqrt{h}) X_n \\ &= \prod_{i=0}^{n} (1 + \lambda h + \mu U_i \sqrt{h}) X_0. \end{aligned} \tag{5.1}$$

Taking the mean with respect to $(n+1)$ time-steps, we obtain an averaged one-step difference equation

$$X_{n+1} = A(h; \lambda, \mu) X_n. \tag{5.2}$$

The quantity $A(h; \lambda, \mu)$ is called the *averaged stability function* of the scheme. Since the equivalence

$$X_n \to 0 \text{ as } n \to \infty \Leftrightarrow |A(h; \lambda, \mu)| < 1$$

holds, we can call the set

$$\mathcal{A} = \{(h; \lambda, \mu);\ |A(h; \lambda, \mu)| < 1\}$$

the region of T-stability of the scheme.

The syntax diagram of T-stability is in Fig. 5.1.

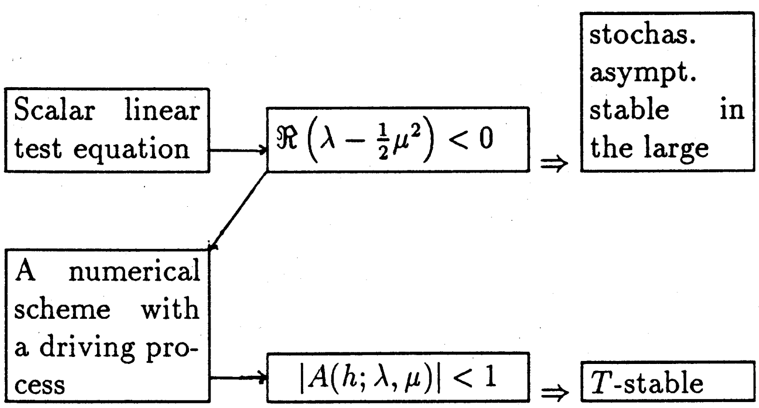

Figure 5.1: Syntax diagram of T-stability

Actual calculation shows the following averaged stability functions.
i) The Euler-Maruyama scheme with the two-point random variables.

$$\begin{aligned} A^2(h;\lambda,\mu) &= (1+\lambda h+\mu\sqrt{h})(1+\lambda h-\mu\sqrt{h}) \\ &= (1+\lambda h)^2-\mu^2 h \end{aligned}$$

ii) The Euler-Maruyama scheme with the three-point random variables.

$$\begin{aligned} A^6(h;\lambda,\mu) &= (1+\lambda h+\mu\sqrt{3h})(1+\lambda h)^4(1+\lambda h-\mu\sqrt{3h}) \\ &= (1+\lambda h)^4\{(1+\lambda h)^2-3\mu^2 h\} \end{aligned}$$

The regions of T-stability of the above cases can be found in [13]. More generally, due to the law of large numbers and utilizing the distribution function of the normal random variable, we obtain the formula

$$\log A(h;\lambda,\mu)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\log|1+\lambda h+\mu\sqrt{h}x|\exp(-x^2/2)dx,$$

which shows the T-stability of the Euler-Maruyama scheme in the ideal case (with the normal random variable as the driving force). As the integral in the right-hand side does not seem to have a closed form of expression, it is still hard to get the region.

As seen, many problems are still remained unsolved in numerical schemes for SDEs.

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