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Donskerのデルタ関数とSDEの離散化による熱核の近似

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Kloeden & Platenの本[Kl-Pl 92]にSDEの解のstrong Ito-Taylor schemeによる近似が論じられている。そこでは主にL^2-近似が考察されているが，本報告における主要な結果はそれが任意の$1<p<\infty$に対しL^p-近似の意味で，さらにSDEの係数が十分滑らかなときはMalliavin解析における$\mathbb{D}^p_\infty$-近似の意味でもなりたつことを示した点である。このことを用いてMalliavin解析の方法を適用すれば，固定された時刻$t>0$において解がMalliavinの意味で非退化のとき，対応する熱核(すなわち解の分布密度)に対するapproximation schemeを具体的に与えることが出来る。この近似の精度はorder γのIto-Taylor approximationに対しては$O(|\pi|^\gamma)$ ($|\pi|$は時間分割のstep)となりL^2-近似の精度と変わらない。

以上が大体研究集会で報告した内容の要旨であるが，

これは中国，武漢科学院の Yaozhong Hu 氏との共同研究であり，共同論文

Yaozhong Hu and Shinzo Watanabe : Donsker's delta functions and approximation of heat kernels by the time discretization method

として発表予定であるので，ここではこれ以上詳しく述べない。ただこの論文中の1つの Lemma は [Kl-Pl 92] において L^2 場合に論じられている Lemma 5.7.3 および Lemma 10.8.1 を自然に拡張乃至精密化したもので L^p-近似を証明する際の key となる。この Lemma はそれ自身，Kloeden-Platen の本を読まれる際の参考として興味あるものと思うので以下にこの Lemma の部分だけ詳しく論じることにした。

Let $W = W_0^r$ be the classical r-dimensional Wiener space:

$$W_0^r = \{w \in C([0,\infty) \to \mathbf{R}^r) | w(0) = 0\}$$

and P be the standard Wiener measure on W_0^r. Then $w(t) = (w^1(t), \cdots, w^r(t))$ for $w \in W$ is a realization of r-dimensional Brownian motion on W. Also we write w_t^i for $w^i(t)$.

For a given finite $T > 0$ and $l = 1, 2, \ldots$, let

$$\Delta = \left\{(s_1, s_2, \ldots, s_l) \in \mathbf{R}^l \mid 0 \leq s_1 < s_2, \ldots < s_l \leq T\right\}$$

and $L^2(\Delta)$ be the usual L^2-space of real square-integrable functions on Δ. Let $f(s_1,\dots,s_l)$ be an $L^2(\Delta)$-valued Wiener functional on W such that $f(s_1,\dots,s_l)$ is $\mathcal{B}_{s_1}$-measurable for each fixed $s_1<\dots<s_l$. Define for a multi-index $\alpha=(\alpha_1,\cdots,\alpha_l)$ with $\alpha_i\in\{0,1,\cdots,r\}$ and $0\le u\le v\le T$,

$$I_\alpha(f)_{u,v}=\int_{u<s_1<\cdots<s_l<v} f(s_1,\cdots,s_l)dw_{s_1}^{\alpha_1}\cdots dw_{s_l}^{\alpha_l}$$

by iterated Itô's stochastic integrals. Set, for $0\le u<s\le T$,

$$\begin{aligned}\|f\|_u(s) &= |f(s)| && \text{if } l=1\\ &= \sup_{u<s_1<\dots<s_{l-1}<s}|f(s_1,\dots,s_{l-1},s)| && \text{if } l>1.\end{aligned}$$

Also we introduce the following notations for multi-indices; for a multi-index $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_l)$ with $\alpha_i\in\{0,1,\cdots,r\}$, we write $l=l(\alpha)$ and set $n(\alpha)=\#\{k;\alpha_k=0\}$. If $l(\alpha)\ge 2$, we set $-\alpha=(\alpha_2,\cdots,\alpha_l)$ and $\alpha-=(\alpha_1,\cdots,\alpha_{l-1})$.

Lemma 0.1. *(1) For $p\ge 1$ and $0\le u<v\le T$,*

$$E\left[\sup_{u\le t\le v}|I_\alpha(f)_{u,t}|^{2p}\right]\le C(u-v)^{p[l(\alpha)+n(\alpha)]-1}\int_u^v E[\|f\|_u(t)^{2p}]dt \tag{0.1}$$

(2) Let $\pi:0=t_0<t_1<\dots<t_n=T$ be a partition of $[0,T]$. Set $|\pi|=\sup_i(t_{i+1}-t_i)$ and $m(s)=m$ if $t_m\le s<t_{m+1}$. Consider the following expectation for each $0\le t\le T$:

$$F_t^\alpha=E\left(\sup_{0\le s\le t}|\sum_{m=0}^{m(s)-1}I_\alpha(f)_{t_m,t_{m+1}}+I_\alpha(f)_{t_{m(s)},s}|^{2p}\right).$$

Then, for $p\ge 1$ and $0\le t\le T$,

$$\begin{aligned}F_t^\alpha &\le C|\pi|^{2p[l(\alpha)-1]}\int_0^t E\left(\|f\|_{t_{m(s)}}(s)^{2p}\right)ds && \text{if } l(\alpha)=n(\alpha),\\ &\le C|\pi|^{p[l(\alpha)+n(\alpha)-1]}\int_0^t E\left(\|f\|_{t_{m(s)}}(s)^{2p}\right)ds && \text{if } l(\alpha)\ne n(\alpha).\end{aligned} \tag{0.2}$$

Here, C are positive constants depending on T,p and α which may vary from lines to lines.

Proof: (1) can be proved by induction on the length $l(\alpha)$ of α: If $l(\alpha)>1$,

$$I_\alpha(f)_{u,t}=\int_u^t I_{\alpha-}(f^s)_{u,s}dw_s^{\alpha_l}$$

where $f^s(s_1,\dots,s_{l-1})=f(s_1,\dots,s_{l-1},s)$. If $\alpha_l=0$, then

$$\begin{aligned}E[\sup_{u\le t\le v}|I_\alpha(f)_{u,t}|^{2p}] &\le E\left\{[\int_u^v|I_{\alpha-}(f^s)_{u,s}|ds]^{2p}\right\}\\ &\le (v-u)^{2p-1}\int_u^v E[|I_{\alpha-}(f^s)_{u,s}|^{2p}]ds\end{aligned}$$

by the Hölder inequality. If $\alpha_l\ne 0$, then

$$\begin{aligned}E[\sup_{u\le t\le v}|I_\alpha(f)_{u,t}|^{2p}] &\le C\cdot E\left\{[\int_u^v|I_{\alpha-}(f^s)_{u,s}|^2ds]^p\right\}\\ &\le C(v-u)^{p-1}\int_u^v E[|I_{\alpha-}(f^s)_{u,s}|^{2p}]ds\end{aligned}$$

by a standard martingale inequality of the Burkholder-Davis-Gundy type for stochastic integrals (cf. [IW89], p.110) and the Hölder inequality. The inequality (0.1) for the case $l(\alpha) = 1$ can be obtained by the same estimates. Then we can conclude the proof by induction if we note the following facts: $l(\alpha-) = l(\alpha) - 1$, $n(\alpha-) = n(\alpha)$ or $n(\alpha) - 1$ according as $\alpha_l \neq 0$ or $\alpha_l = 0$ and $\|f^s\|_u(t) \leq \|f\|_u(s)$ if $t \leq s$.

Next, we prove (2). We note that

$$\Xi(t) := \sum_{m=0}^{m(t)-1} I_\alpha(f)_{t_m,t_{m+1}} + I_\alpha(f)_{t_{m(t)},t} = \int_0^t I_{\alpha-}(f^s)_{t_{m(s)},s} dw_s^{\alpha_l}.$$

If $n(\alpha) = l(\alpha)$, then

$$\begin{aligned} E[\sup_{0\leq s\leq t} |\Xi(s)|^{2p}] &\leq E\left\{\left[\int_0^t |I_{\alpha-}(f^s)_{t_{m(s)},s}|ds\right]^{2p}\right\} \\ &\leq C\cdot\int_0^t E\left\{|I_{\alpha-}(f^s)_{t_{m(s)},s}|^{2p}\right\} ds \\ &\leq C\cdot\{\sum_{m=0}^{m(t)}\int_{t_m}^{t_{m+1}\wedge t} E\left\{|I_{\alpha-}(f^s)_{t_m,s}|^{2p}\right\} ds. \end{aligned}$$

By the estimate (0.1), this is dominated by

$$C|\pi|^{2pl(\alpha-)-1}\cdot\sum_{m=0}^{m(t)}\int_{t_m}^{t_{m+1}\wedge t} ds\int_{t_m}^{s} E\left[\|f^s\|_{t_m}(\tau)^{2p}\right] d\tau \leq C|\pi|^{2pl(\alpha-)}\int_0^t E\left[\|f\|_{t_{m(s)}}(s)^{2p}\right] ds.$$

Since $l(\alpha-) = l(\alpha) - 1$, (0.2) is obtained in this case.

If $n(\alpha) \neq l(\alpha)$, and $\alpha_l \neq 0$, then by the Burkholder-Davis-Gundy inequality applied to stochastic integral $\Xi(s)$, we have

$$\begin{aligned} E[\sup_{0\leq s\leq t} |\Xi(s)|^{2p}] &\leq C\cdot E\left\{\left[\int_0^t |I_{\alpha-}(f^s)_{t_{m(s)},s}|^2 ds\right]^{p}\right\} \\ &\leq C\cdot\int_0^t E\left\{|I_{\alpha-}(f^s)_{t_{m(s)},s}|^{2p}\right\} ds. \end{aligned}$$

By the same estimate as above using (0.1), this is further dominated by

$$C|\pi|^{p[l(\alpha-)+n(\alpha-)]}\int_0^t E\left[\|f\|_{t_{m(s)}}(s)^{2p}\right] ds.$$

Since $\alpha_l \neq 0$, we have $l(\alpha-) + n(\alpha-) = l(\alpha) + n(\alpha) - 1$ and hence (0.2) is obtained.

Finally we consider the case $n(\alpha) \neq l(\alpha)$ and $\alpha_l = 0$. We have

$$\begin{aligned} F_t^\alpha &\leq C\cdot E\left\{\sup_{0\leq s\leq t} |\sum_{m=0}^{m(s)-1} I_\alpha(f)_{t_m,t_{m+1}}|^{2p}\right\} + C\cdot E\left\{\sup_{0\leq s\leq t} |I_\alpha(f)_{t_{m(s)},s}|^{2p}\right\} \\ &:= I_1 + I_2 \end{aligned}$$

and estimate these two terms separately. We first note that

$$I_1 = C\cdot E\left\{\sup_{0\leq k\leq m(t)-1} |\sum_{m=0}^{k} I_\alpha(f)_{t_m,t_{m+1}}|^{2p}\right\}$$

and $S_k = \sum_{m=0}^k I_\alpha(f)_{t_m,t_{m+1}}$ forms a discrete time martingale. Then we can apply the Burkholder-Davis-Gundy inequality for the discrete time martingale (cf. [Ga73]) to obtain that

$$\begin{aligned} I_1 &\leq C\cdot E\left\{\left[\sum_{m=0}^{m(t)-1} |I_\alpha(f)_{t_m,t_{m+1}}|^2\right]^{p}\right\} \\ &= C\cdot E\left\{\left[\sum_{m=0}^{m(t)-1} |\int_{t_m}^{t_{m+1}} I_{\alpha-}(f^s)_{t_m,s} ds|^2\right]^{p}\right\}. \end{aligned}$$

Since

$$|\int_{t_m}^{t_{m+1}} I_{\alpha-}(f^s)_{t_m,s}ds|^2 \le (t_{m+1}-t_m)\cdot \int_{t_m}^{t_{m+1}} |I_{\alpha-}(f^s)_{t_m,s}|^2 ds,$$

this is further dominated by

$$\begin{aligned} & C|\pi|^p \cdot E\left\{\left[\sum_{m=0}^{m(t)-1}\int_{t_m}^{t_{m+1}} |I_{\alpha-}(f^s)_{t_m,s}|^2 ds\right]^p\right\} \\ \le\ & C|\pi|^p \cdot E\left\{\left[\int_0^t |I_{\alpha-}(f^s)_{t_{m(s)},s}|^2 ds\right]^p\right\} \\ \le\ & C|\pi|^p \cdot \int_0^t E\left\{|I_{\alpha-}(f^s)_{t_{m(s)},s}|^{2p}\right\} ds. \end{aligned}$$

Then by the same estimate as above using (0.1), we deduce that the above is dominated by

$$C|\pi|^{p[l(\alpha-)+n(\alpha-)+1]}\int_0^t E(\|f\|_{t_{m(s)}}(s)^{2p})ds.$$

Since $l(\alpha-)+n(\alpha-)+1 = l(\alpha)+n(\alpha)-1$, I_1 is now dominated as desired.

As for I_2, we have,

$$\begin{aligned} E\left[\sup_{0\le s\le t}|I_\alpha(f)_{t_{m(s)},s}|^{2p}\right] &= E\left[\sup_{0\le s\le t}|\int_{m(s)}^s I_{\alpha-}(f^\tau)_{t_{m(s)},\tau}d\tau|^{2p}\right] \\ &\le E\left\{\sup_{0\le s\le t}\left[(s-t_{m(s)})^{2p-1}\int_{m(s)}^s |I_{\alpha-}(f^\tau)_{t_{m(s)},\tau}|^{2p}d\tau\right]\right\} \\ &\le |\pi|^{2p-1}E\left[\sum_{m=0}^{m(t)}\int_{t_m}^{t_{m+1}\wedge t}|I_{\alpha-}(f^s)_{t_m,s}|^{2p}ds\right]. \end{aligned}$$

By using (0.1), this can be dominated by

$$\begin{aligned} & C|\pi|^{2p-1}\cdot|\pi|^{p[l(\alpha-)+n(\alpha-)]}\int_0^t E(\|f\|_{t_{m(s)}}(s)^{2p})ds \\ =\ & C|\pi|^{p-1}\cdot|\pi|^{p[l(\alpha)+n(\alpha)-1]}\int_0^t E(\|f\|_{t_{m(s)}}(s)^{2p})ds. \end{aligned}$$

Since $p \ge 1$, we obtained the desired estimate for I_2 and the proof of (0.2) is now complete.□

References

[Ga73] A. M. Garsia, Martingale inequalities, Mathematics Lecture Note Series, Benjamin, 1973.

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[Kl-Pl92] P. E. Kloeden and E. Platen, Numerical solutions of stochastic differential equations, Springer, 1992.