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長波短波相互作用方程式のパンルベ特性と可積分性

Painlevé properties and integrability of the long- and short- wave interaction equation

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Abstract

The integrability in the sense of Painlevé property is examined in the long- and short- wave interaction equation. The equation described in a coupled form of the NLS equation with the K-dV equation has only two parameters in the normalized form. When the equation is reduced to the ODE through the traveling wave transformation, it is shown to pass the Painlevé test for three cases of the parameters. On the other hand, for these parameters, when the test is directly applied to the original PDE, it is found that two cases except for one do not pass the test without any restrictions. However, the test is found not to be successful in the nearly integrable region. Furthermore, the possibility of 'finite time integrability' is discussed for a special case of the parameters.

1 Introduction.

In dispersive media, wave interactions play an important role in energy exchange among different two or more wave modes, if resonance conditions with respect to wave frequencies (and wave numbers) or wave velocities are satisfied in these wave modes. The long- and short-wave interaction is one of such interactions and thought of as a special case of the three-wave interaction. [1] That is to say, assuming a single long wave $(\Delta\mathbf{k}, \omega(\Delta k))$ and two short waves $(\mathbf{k} \pm \Delta\mathbf{k}/2, \omega(k \pm \Delta k/2))$, from the resonance condition of the three-wave interaction given as

$$\omega(\Delta k) = \omega(k + \Delta k/2) - \omega(k - \Delta k/2), \tag{1}$$

we can approximately obtain the following resonance condition between long and short waves:

$$\Delta\mathbf{k} \cdot (\partial\omega/\partial\mathbf{k})|_{\mathbf{k}} \simeq \omega(\Delta k), \tag{2}$$

where $k = |\mathbf{k}|$ and $\omega \ll 1$ is assumed for Δk ($\ll k$). The above condition is found to be equivalent to

$$\mathbf{v}_p \cdot \mathbf{v}_g \simeq v_p^2 \text{ or } v_g \cos\psi \simeq v_p, \tag{3}$$

where the phase velocity of the long wave is given by $\mathbf{v}_p = \omega(\Delta k)\Delta\mathbf{k}/\Delta k^2$ and the group velocity of the short wave by $\mathbf{v}_g = (\partial\omega/\partial\mathbf{k})|_{\mathbf{k}}$. Therefore, if the above condition is satisfied, the interaction is possible between the two waves propagating in the different direction by ψ. In particular, this condition is simplified to $v_g \sim v_p$ when both waves propagate in the same direction ($\psi = 0$).

Such a resonance condition can be satisfied in water waves, plasma waves and others in dispersive media. [1]–[6] Although several nonlinear interaction equations have been proposed for these waves, in this article, we deal with the following equation, which is represented in a coupled form of the Nonlinear Schrödinger (NLS) equation with the Korteweg-de Vries (K-dV) equation: [7]

$$\mathrm{i}\, S_t \pm S_{xx} = SL, \quad L_t + \alpha L L_x + \beta L_{xxx} = |S|_x^2, \tag{4}$$

where L and S denote, respectively, the real long wave and the complex amplitude of the envelope of the short wave, while x and t are spatial and temporal coordinates in a frame of reference moving with the phase velocity of the long wave or the group velocity of the short wave.

In the above equation, which is expressed in the normalized form with only two parameters α and β, the parameters and the alternative of the $\pm$ signs in front of S_{xx} depend upon the individual properties of the waves and the media concerned: [7] the gravity and capillary waves in a single layer fluid ($\alpha, \beta \leq 0$ and $+$ sign), the gravity waves in a two-layer fluid ($\beta \leq 0$ and $-$ sign), the ion acoustic and electron plasma waves ($\alpha \geq 0$, $\beta \leq 0$ and $+$sign) and so on. However, since the case of $-$ sign can be formally obtained if t, L and β in eq.(4) are replaced by $-t$, $-L$ and $-\beta$, we will consider only the case of $+$ sign in the followings.

Depending upon the parameters α and β, physical meanings and mathematical properties of this equation can be said as follows: When both α and β are equal to zero, eq.(4) represents the case when the magnitude of the long wave is much less than that of the short wave ($|L| \ll |S|$). For this case, the equation is proved to be integrable or to have the n-soliton solution by means of the inverse scattering transform (IST) method. [8, 9] On the other hand, when both α and β have finite values, the equation represents the case for which the magnitudes of the long and short waves are of the same order ($|L| \sim |S|$). In this case, not only analytic solitary wave (one-soliton) solutions, but also a variety of numerical solitary wave solutions including ones with oscillatory damped tails are found. [10] It is expected, however, that the long time asymptotic wave behavior may become chaotic for general initial waves or soliton interactions, since the equation for $\beta = 1$ is shown to be non-integrable through IST [11]. Additionally, in the Hirota bilinear form for $\alpha = -6\beta$, the n-soliton solution has not been found for $\alpha, \beta \neq 0$. [11, 12] Nevertheless, for the nearly integrable case in the vicinity of $\alpha = \beta = 0$, it is numerically shown that the wave behavior is regular or irregular depending upon initial conditions and values of the parameters. [10]

As is seen in the above, though eq.(4) is shown to be non-integrable for the particular α and β, the integrability has not yet been analytically surveyed for all values of the parameters, in particular, in the nearly integrable region. Therefore, in this article, the integrability of eq.(4) is examined in the (α, β) parameter space by means of the Painlevé test, which is known as one of the useful and practical techniques to test the integrability despite some drawbacks. [13, 14]

The organization of this article is as follows: In section 2, the results of the test are shown for the reduced ordinary differential equation (ODE) through a variable transformation (Painlevé ODE test). In addition, they are confirmed by examining the surface of section for particular parameters. In section 3, for the cases which pass the ODE test, the original partial differential equation (PDE) is directly tested (Painlevé PDE test). And finally, in section 4, we remark the validity of the test in the nearly integrable region and the possibility of the 'finite time integrability'.

2 Painlevé ODE test.

For the Painlevé ODE test, we first reduce eq.(4) to the ODE through the following traveling wave transformation:

$$S = f(\zeta)\exp[i(\lambda/2)(x - Vt)], \quad L = g(\zeta), \quad (\zeta = x - \lambda t) \tag{5}$$

where λ and V are constants. Substituting (5) into eq.(4) and integrating g with respect to ζ, we can easily obtain the reduced ODE

$$f_{\zeta\zeta} + (\lambda/2)(V - \lambda/2)f = fg, \quad \beta g_{\zeta\zeta} + (\alpha/2)g^2 - \lambda g = f^2 - C^2, \tag{6}$$

where we have imposed the boundary conditions: $f \to C$ (const.), $f_\zeta, f_{\zeta\zeta}, g, g_\zeta, g_{\zeta\zeta} \to 0$ as $|\zeta| \to \infty$, and $\lambda = 2V$ for $C \neq 0$.

Making use of the following variable transformation into eq.(6):

$$g \to (2/\beta)^{1/2}g, \quad \zeta \to (\beta/2)^{1/4}\zeta, \tag{7}$$

we can show that our system has Hénon-Heiles Hamiltonian

$$H = (1/2)[f_\zeta^2 + g_\zeta^2] + I(f, g), \tag{8}$$

where

$$I = (\beta/2)^{1/2}(\lambda/4)(V - \lambda/2)f^2 - (2/\beta)^{1/2}(\lambda/4)g^2 - (f^2 - C^2)g/2 + \alpha g^3/(6\beta).$$

Since the Painlevé properties (P-properties) in the above system have been examined by Chang *et al.* [15] for $\beta > 0$ and $C = 0$, it is expected that our ODE has similar singular structures. In fact, it is found that eq.(6) has similar P-properties. [10]

According to the procedure of the test by Ablowitz *et al*, [16] the solutions of eq.(6) are expanded in the following Laurent series:

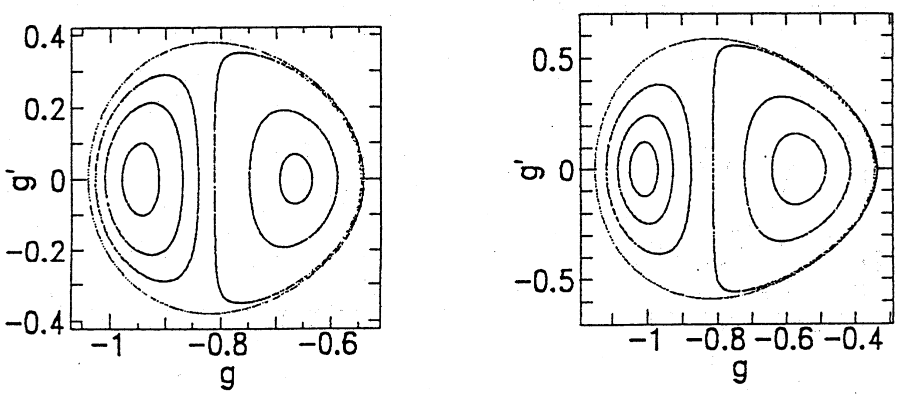
$$f = (\zeta - \zeta_0)^{-a}\sum_{j=0}^{\infty} f_j(\zeta - \zeta_0)^j, \quad g = (\zeta - \zeta_0)^{-b}\sum_{j=0}^{\infty} g_j(\zeta - \zeta_0)^j. \tag{9}$$

Table 1: Painlevé ODE Test

$\alpha = \beta = 0$		$a = 1, b = 2, f_0^2 = -2\lambda, g_0 = 2$ $r = -1, 4$ (general solution) P-property
$\alpha = -6\beta$	Case I	$a = 2, b = 2, f_0^2 = -72\beta, g_0 = 6$ $r = -3, -1, 6, 8$ (singular solution) P-property
	Case II	$a = 1, b = 2, g_0 = 2, f_0$: arbitrary $r = -1, 0, 3, 6$ (general solution) P-property
$\alpha = -\beta$	Case I	$a = 2, c = 2, f_0^2 = 18\beta,\ g_0 = 6$ $r = -1, 2, 3, 6$ (general solution) P-property for $V - \lambda/2 + 2/\beta = 0$ $(C = 0)$ or $V = \lambda = 0$ $(C \neq 0)$
	Case II	$a = -4, b = 2, g_0 = 12, f_0$: arbitrary $r = -7, -1, 0, 6$ (singular solution) P-property

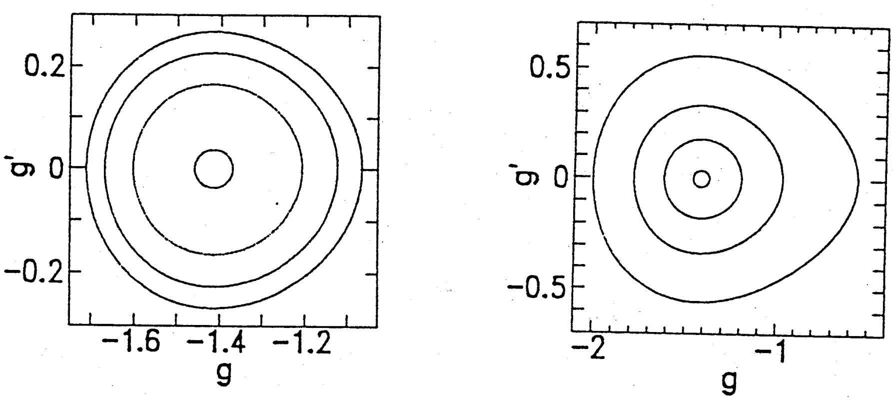
where ζ_0 denotes an arbitrary movable singularity depending upon initial conditions. Substituting the above expression into eq.(6) and equating coefficients of powers of ζ, we can obtain the leading orders a and b for $j = 0$, and the recursion relations with respect to f_j and g_j for $j \geq 1$. From the recursion relations, we can see that the coefficients f_j or/and g_j become arbitrary for particular values of $j = r$, which is called resonances. The resonances for $r = -1$ and 0 are, respectively, corresponding to the arbitrariness of ζ_0 and f_0 (and/or g_0), though negative resonances for $r < -1$ are ignored. [18] For the P-property, these a, b and r are required, at least, to be integers, which means that the solutions should be of the pole type or the single-valued. Then, Table I shows that the candidates for the P-property are limited to three significant cases of α and β. It is found in this table that the case $\alpha = \beta = 0$ has only general solution, while the other cases have both general and singular solutions in pairs. In these solutions, the general solution means that the equation has equal arbitrary parameters to the order of the equation, while the singular solution means that the solution has less arbitrariness than the order of the equation. However, in order for these three candidates to have the P-property, the self-consistency of the resonance must be checked in the recursion relations. Resulting from this, it is finally found that the Case I for $\alpha = -\beta$ has the P-property under the restrictions that either $V - \lambda/2 + 2/\beta = 0$ for $C = 0$ or $V = \lambda = 0$ for $C \neq 0$, while the other cases have P-property without any restrictions.

The results of the ODE test can be confirmed by examining the surfaces of section for the Hénon-Heiles system (8) when $\beta > 0$ and $C = 0$. Although phase trajectories in this system move through the four-dimensional phase space (f, f_ζ, g, g_ζ), we can construct the two- dimensional surface of section (g, g_ζ), by slicing the phase space at $f = 0$ and taking the trajectories with $f_\zeta > 0$ for the fixed total energy $E(= H)$. In the followings, typical examples of the surfaces of section are shown for both integrable and non-integrable cases: Figure 1 shows the sections for the integrable case $\alpha = -6\beta$, where $\alpha = -2, \beta = 1/3, \lambda = 2$ and $V = 1/2$ are taken. Figures (a) and (b), respectively, show the surfaces when E=0.0072 and 0.262. We can see that the closed smooth curves are lying on the surface even if the energy increases, which means that the trajectories move on the tori in the original phase space even in the nonlinear regime. The situation on the regular motion of the trajectories is similar when $\alpha = -\beta$, if the condition $V - \lambda/2 + 2/\beta = 0$ is satisfied. Figure 2 shows this case, where we take $\alpha = -4, \beta = 4, \lambda = 2$ and $V = 1/2$. As is seen in both Figs.(a) for E=0.036 and (b) for E=0.216, even if the energy increases, the smooth curves are retained on the surface, which means that the motion of the trajectories is regular. However, if this condition is not satisfied, the motion of the trajectories are irregular in the nonlinear regime. This is shown in Fig.3, where $\alpha = -1/3, \beta = 1/3, \lambda = 2$ and $V = 1/2$. It is found from both Figs.(a) for E=3.79 and (b) for E=7.79 that the smooth curves are partly replaced by vaguely scattered points, when the energy increases. Furthermore, in Fig.4 for $\alpha = \beta$, we can illustrate one of the examples which do not pass the test and show the large regions of chaotic motion, where we take $\alpha = \beta = 1/3, \lambda = 2$ and $V = 1/2$. Although closed smooth curves are lying on the surface for sufficiently small energy E=0.05 (Fig.(a)), when the energy increases to E=0.2 (Fig.(b)), the curves become vague due to scattering points along them. Finally, when E=0.5, all smooth curves disappear and random spread of points are found all over the surface within the maximum energy shell (Fig.(c)).



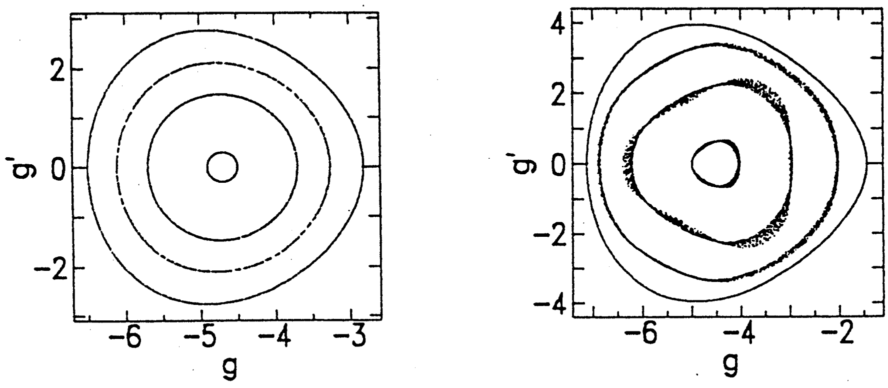
(a) (b)

Figure 1: Surface of section for the integrable case $\alpha = -6\beta$, where $\alpha = -2, \beta = 1/3, \lambda = 2$ and $V = 1/2$: (a)E=0.0072, (b)E=0.262.



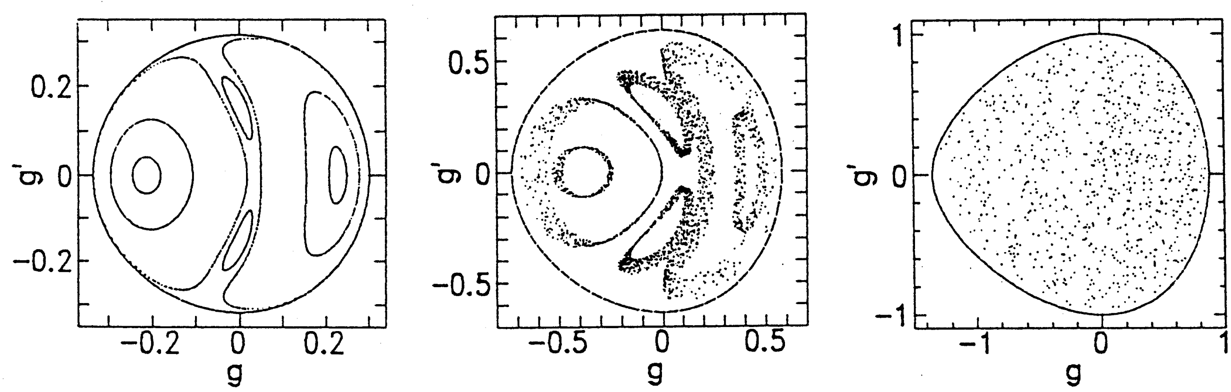
(a) (b)

Figure 2: Surface of section for the integrable case $\alpha = -\beta$ when $V - \lambda/2 + 2/\beta = 0$, where $\alpha = -4, \beta = 4, \lambda = 2$ and $V = 1/2$: (a) E=0.0036, (b)E=0.216.



(a) (b)

Figure 3: Surface of section for the non-integrable case $\alpha = -\beta$ when $V - \lambda/2 + 2/\beta \neq 0$, where $\alpha = -1/3, \beta = 1/3, \lambda = 2$ and $V = 1/2$: (a)E=3.79, (b)E=7.79.



(a) (b) (c)

Figure 4: Surface of section for the non-integrable case $\alpha = \beta$, where $\alpha = \beta = 1/3, \lambda = 2$ and $V = 1/2$: (a)E=0.005, (b)E=0.2, (c) E=0.5.

3 Painlevé PDE test.

It is known that the test in the reduced ODE gives only necessary conditions for the original PDE to be completely integrable. [16] In other words, a given PDE is not completely integrable when the ODE reduced from the PDE does not have the P-property. Therefore, in this section, the integrability of the original PDE is directly examined for the three cases that pass the ODE test in the preceding section.

Let us apply the Painlevé PDE test, whose direct procedure was introduced by Weiss *et al.* [17] In this test, a given partial differential equation is said to have the P- property if the solutions are single-valued in the neighborhood of the arbitrary and analytic (movable) singular manifold. Since the singular manifold for the ODE reduces to the singularity with respect to a single variable, the PDE test may be considered as a straightforward extention of the ODE test with similar procedure. For convenience, rewriting eq.(4) in the following form:

$$\mathrm{i}\, u_t + u_{xx} = uw, \quad -\mathrm{i}\, v_t + v_{xx} = vw, \quad w_t + \alpha w w_x + \beta w_{xxx} = (uv)_x, \tag{10}$$

the solutions are set as

$$u = \phi^{-a} \sum_{j=0}^{\infty} u_j \phi^j, \quad v = \phi^{-b} \sum_{j=0}^{\infty} v_j \phi^j, \quad w = \phi^{-c} \sum_{j=0}^{\infty} w_j \phi^j. \tag{11}$$

Making use of (11) into eq.(10), we can determine the leading order a, b and c and the resonances r like in the ODE test, whose values are integers for the same three cases of α and β as in Table I. The results of the PDE test are shown in Table II, where the case $\alpha = \beta = 0$ have only general solution, while the other two cases have both singular and general solutions. [19] Checking the recursion relations for the self-consistency of the resonances, it is finally found that the case of $\alpha = \beta = 0$ and the Case II for $\alpha = -\beta$ hold the P-property without any restrictions. The latter case, however, is excluded in the present context, since the solutions u and v are regular to vanish closely near the singular manifold $\phi = 0$. Consequently, the significant solution is only w which is nothing but that of the K-dV equation, where the resonances occur for $r = -1, 4, 6$. On the other hand, the other cases have the P-property through the traveling wave transformation like $\phi = x - ct$ (c:const.), that is to say, the P-property is conditional. Thus, only the case of $\alpha = \beta = 0$ is

Table 2: Painlevé PDE Test

$\alpha=\beta=0$		$a=b=1, c=2, u_0v_0=2\phi_x\phi_t$ (u_0 or v_0: arbitrary) $w_0=2\phi_x^2$, $r=-1,0,2,3,4$ (general solution) P-property
$\alpha=-6\beta$	Case I	$a=b=2, c=2, u_0v_0=-72\beta\phi_x^4$, ($u_0$ or v_0: arbitrary) $w_0=6\phi_x^2$, $r=-3,-1,0,4,5,6,8$, (singular solution) Conditional P-property
	Case II	$a=b=1, c=2, w_0=2\phi_x^2$, u_0 and v_0: arbitrary $r=-1,0,0,3,3,4,6$ (general solution) Conditional P-property
$\alpha=-\beta$	Case I	$a=b=2, c=2, u_0v_0=18\beta\phi_x^4$ (u_0 or v_0: arbitrary) $w_0=6\phi_x^2$, $r=-1,0,2,3,4,5,6$ (general solution) Conditional P-property
	Case II	$a=b=-4, c=2, w_0=12\phi_x^2$, u_0 and v_0: arbitrary $r=-7,-7,-1,0,0,4,6$ (singular solution) P-property

completely integrable, which is consistent with the result of IST method. [8, 9]

4 Concluding remarks.

We can see in Table II that the leading orders and some coefficients in the expansions are coincident or adjustable between the completely integrable case $\alpha=\beta=0$ and the case for $\alpha=-6\beta$ (Case II). Although this suggests that these two cases are closely related to each other, the test is found not to be successful in the nearly integrable region $\alpha,\beta\sim 0$ for $\alpha=-6\beta$, since the singular manifold expansions become non-uniformly valid when β tends to zero. This non-uniformity may be due to the small parameter β in the highest order derivative term in eq.(4). Additionally, since there exists one-soliton solutions which are uniformly valid for $\alpha=-6\beta$ including $\alpha=\beta=0$, [10] the usual singular manifold expansions (9) and (11) is not appropriate to examine the integrability in this region.

On the other hand, in the general solution for $\alpha=-6\beta$ (Case II), we should remark that the compatibility condition that permits the P-property is found to be relaxed considerably for a finite time. That is to say, since the significant compatibility condition for the P- property is written as

$$\theta_t - \theta\theta_x = 0, \tag{12}$$

through $\theta = \phi_t/\phi_x$, the general solution of the above wave equation

$$\theta = \Theta(x + t\theta), \tag{13}$$

is analytic for a finite time depending upon initial conditions, where Θ denotes an arbitrary function. Therefore, for a certain class of ϕ which is given by (13) through $\theta = \phi_t/\phi_x$, the compatibility condition (12) can be satisfied for a finite time during which the solution (13) is analytic and arbitrary. This means that the equation holds the P-property for the finite time and is expected to have multi-soliton solutions for the time. As a special case, it is easily seen that the condition (12) is identically satisfied for an infinitely long time under the traveling wave transformation $\phi = x - ct$, which is confirmed by the existence of one-soliton solution. [10] Thus, for $\alpha = -6\beta$, though one soliton state is valid for an infinitely long time, the soliton interactions due to multi-soliton state might be elastic for the finite time, that is to say, the possibility of 'finite time integrability' is expected.

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