The Solution of Toda Equation and Dimensional Functions

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Abstract

A relation between the Toda lattice and some fractal set is studied. The solutions of the Toda lattice in Moser's form are derived from the dimension function of one-parameter deformation of the fractal set. The orbits of the Toda lattice and fractal set have information geometrical meaning. They are geodesics on the space of exponential type distribution.

1   Introduction

Among nonlinear integrable systems, the Toda lattice is one of the important examples of finite mass interactions. It has soliton solutions which describes interaction of solitary waves. The soliton solutions of Toda lattice are described exactly by logarithms of determinants of some matrices [1]. For example, Toda lattice equation,

$$\frac{d^2 x_n}{dt^2} = \exp(x_{n-1} - x_n) - \exp(x_n - x_{n+1}) \tag{1}$$

is solved by

$$x_n = \tilde{P}t + \log \det(B(n)B^{-1}(n-1)), \tag{2}$$

where $B(n)$ is some $N \times N$ matrix and N denotes the number of solitons in this solution. The i, j entry of $B(n)$ is

$$\delta_{ij} + \frac{c_i(0)c_j(0)\exp((\sinh\gamma_i + \sinh\gamma_j)t - (\gamma_i + \gamma_j)(n+1))}{1 - \exp(\gamma_i + \gamma_j)}, \tag{3}$$

where c_i and γ_i are arbitrary constants.

The function "log det" has some resemblance to "codim" or "corank". For example, the equation

$$\text{codim}(XY) = \text{codim}X + \text{codim}((X^T)^{\perp} \cup Y) \tag{4}$$

holds for linear operators X and Y. In this equation, the symbol "codim" is employed to denote the corank of matrices or codimension of linear space, X and Y are matrices in the left hand side and linear space of their image in the right hand side.

If the linear space $(X^T)^{\perp}$ is contained in Y, then $(X^T)^{\perp} \cup Y$ is equal to Y. In this case eq.(4) reminds us the following relation:

$$\log\det(XY) = \log\det X + \log\det Y. \tag{5}$$

Therefore if log det has some meaning of dimension, the Toda lattice equation is considered to describe a time evolution of dimension of some set. Since log det is continuous valued, the set should have continuous dimension. This suggests that it could be some fractal set. We show by introducing some time variables that the set proposed by Campbell is just the one.

We introduce the Toda lattice again in Moser's description in §2, Volkmann's theorem on some fractal set in §3, and Campbell's fractal set in §4. Moreover we show that the time evolution of the Toda lattice is related to the definition of Campbell's fractal set. We study geometrical meaning of both time evolutions in §5.

2   Toda Lattice

We treat the Toda lattice of finite mass points, which has Hamiltonian

$$\mathcal{H} = \frac{1}{2}\sum_{k=1}^{n} y_k^2 + \sum_{k=1}^{n-1} e^{x_k - x_{k+1}}, \tag{6}$$

where x_k is the coordinate of position of the k-th mass point and y_k is the velocity $\dot{x}_k$. We set formal variables $x_0 = -\infty$ and $x_{n+1} = +\infty$, which give boundary conditions for the Hamiltonian system. This system describes motions of n mass points under exponential interaction. Flaschka [2] introduced variables a_k and b_k to rewrite the system in a matrix style. The variables a_k and b_k are given by

$$a_k = \frac{1}{2}\exp\left(\frac{x_k - x_{k+1}}{2}\right), \quad b_k = -\frac{1}{2}y_k. \tag{7}$$

The equation of the system is transformed into

$$\frac{dL}{dt} = [B, L], \tag{8}$$

where

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & b_{n-1} & a_{n-1} \\ 0 & \cdots & 0 & a_{n-1} & b_n \end{pmatrix}, \quad B = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 \\ -a_1 & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & a_{n-1} \\ 0 & \cdots & 0 & -a_{n-1} & 0 \end{pmatrix},$$
$$\tag{9}$$

and $[\ ,\ ]$ means commutator. It is more clearly seen in this form that the eigenvalues of the matrix L are constants of motion. Let $\xi_k, k = 1, 2, \ldots, n$ be the eigenvalues of L. Since L is symmetric and a_k's do not become zero, the eigenvalues are all real and can be arranged in order as

$$\xi_1 < \xi_2 < \ldots < \xi_n.$$

The Toda lattice has as many constants of motion as coordinates of mass points. Such system is called completely integrable.

Following Moser [3], we define $f(\xi)$ as

$$f(\xi) = e_n(\xi I - L)^{-1}e_n^T, \tag{10}$$

where $e_n = (0, 0, \ldots, 0, 1)$ and I is the identity matrix. the function $f(\xi)$ has poles at $\xi = \xi_1, \xi_k, \cdots, \xi_n$. Since ξ_k's are simple, $f(\xi)$ is expressed as the following partial fraction form,

$$f(\xi) = \sum_{k=1}^{n} \frac{p_k}{\xi - \xi_k}, \tag{11}$$

where p_k is the residue at ξ_k. It is known that p_k's are all positive and their total sum is equal to one. That is,

$$\sum_{k=1}^{n} p_k = 1.$$

Since the eigenvalues of L are preserved through time evolution, the poles ξ_k are constants of motion and the p_k's evolve by the following equation

$$\dot{p_k} = -2(\xi_k - \sum_{j=1}^{n} \xi_j p_j) p_k. \tag{12}$$

The right hand side of this equation means that p_k's essentially obey linear equations associated with the first term and the total sum of the p_k's is preserved by the second term. Let us replace $-2\xi_k$ with λ_k to unify notations explained in the following sections. The exact solution of an initial value problem is

$$p_k(t) = \frac{q_k e^{\lambda_k t}}{Z(q, \lambda, t)}, \tag{13}$$

$$Z(q, \lambda, t) = \sum_{j=1}^{m} q_j e^{\lambda_j t}, \tag{14}$$

where q_k is the initial value determined by $p_k(0)$, and where $(q_1, q_2, \ldots, q_n)$ and $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ are abbreviated as q and λ respectively.

3   Volkmann's Theorem

In this section we summarize Hausdorff dimension of a certain subset in the interval $[0, 1]$. This subset is employed for the Hausdorff dimension of another subset of Campbell which is treated in the next section.

Let x be a number in $[0, 1]$, which has M-ary expansion

$$x = \sum_{i=1}^{\infty} \frac{b_i(x)}{M^i}, \tag{15}$$

where $b_i(x)$ is in $\{0, 1, \ldots, M - 1\}$. Let $\mathcal{G} = (G_1, G_2, \ldots, G_m)$ be a partition of the M-ary digit set. That is, the non-empty sets G_i's satisfy the following

conditions.

$$\bigcup_{i=1}^{m} G_i = \{0, 1, \ldots, M-1\}, \quad G_i \bigcap G_j = \emptyset \quad \text{for each} \quad i \neq j. \qquad (16)$$

Let $\sharp G_k / M$ be q_k, where $\sharp$ means the cardinality of a set. We count digits in $b_1(x), b_2(x), \ldots, b_n(x)$ which belong to G_i and let the number be $A_i(x, n)$. That is, $\dfrac{A_i(x, n)}{n}$ is the appearance ratio of the digits in G_i for n-truncated M-ary expansion of x.

Let $p = (p_1, p_2, \ldots, p_m)$ be a vector of arbitrary nonnegative numbers which satisfy the following conditions:

$$\sum_{i=1}^{m} p_i = 1. \qquad (17)$$

We define the set $F(p, \mathcal{G})$ in $[0, 1]$.

$$F(p, \mathcal{G}) = \{x \in [0, 1] | \lim_{n \to \infty} \frac{A_i(x, n)}{n} = p_i, \quad \text{for each} \quad i\} \qquad (18)$$

Volkmann proved that the Hausdorff dimension of the set $F(p, \mathcal{G})$ is calculated with relative entropy [5]:

$$\text{codim} F(p, \mathcal{G}) = \sum_{i=1}^{m} p_i \log_M \frac{p_i}{q_i}, \qquad (19)$$

where $\text{codim} F$ means $1 - \dim F$. Hereafter we employ the same symbol codim as in the case of linear subspace. For example, the case $M = 3, G_1 = \{1\}, G_2 = \{0, 2\}, p_1 = 0, p_2 = 1$, the $F(p, \mathcal{G})$ is the Cantor set.

4   Campbell's Theorem

Next we introduce another subset in $[0, 1]$ which depends on arbitrary real numbers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ [4]. The meaning of the notations x, $b_i(x)$, $\mathcal{G}$ and q_i is as above.

Let E be a number which satisfies $\min_i \lambda_i < E < \max_i \lambda_i$. For a partition $\mathcal{G}$, we define a function $Y : \{0, 1, \ldots, M-1\} \to \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ which maps digits $b \in G_i \mapsto \lambda_i$. As before, the digits $b_1(x), b_2(x), \ldots$ is determined by

M-ary expansion of $x \in [0,1]$. The set $H(E,\mathcal{G},\lambda)$ is defined as the set which satisfies the following condition,

$$H(E,\mathcal{G},\lambda) = \{x \in [0,1] | \lim_{n \to \infty} \frac{Y(b_1(x)) + Y(b_2(x)) + \cdots + Y(b_n(x))}{n} \leq E\}.$$
(20)

The set has self-similarity for M-ary expansion.

Let us consider minimum relative entropy problem to get the Hausdorff dimension of the set $H(E,\mathcal{G},\lambda)$. We regard $q = (q_1, q_2, \ldots, q_m)$ and $p = (p_1, p_2, \ldots, p_m)$ as vectors on the hyperplane of unit total sum in m-dimensional linear space. Assume that q is not in the hyper-halfplane $\sum_{i=1}^{m} \lambda_i p_i \leq E$. Then there exists the unique vector which minimizes the relative entropy $\sum_{i=1}^{m} p_i \log_M \frac{p_i}{q_i}$ in the hyper-halfplane. We write $\tilde{p}$ to denote the vector, which is expressed as

$$\tilde{p}_i(t) = \frac{q_i e^{\lambda_i t}}{Z(q,\lambda,t)}, \quad Z(q,\lambda,t) = \sum_{k=1}^{m} q_k e^{\lambda_k t},$$
(21)

where t is uniquely determined by the equation $\dfrac{d \log Z(q,\lambda,t)}{dt} = E$. The uniqueness comes from the convexity of the function $e^{Et} Z(q,\lambda,t)$. The convexity also uniquely determines E from t.

The Hausdorff dimension of the set $H(E(t),\mathcal{G},\lambda)$ is equal to that of $F(\tilde{p},\mathcal{G})$. That is, we have

$$\operatorname{codim} H(E(t),\mathcal{G},\lambda) = \sum_{i=1}^{m} \tilde{p}_i \log_M \frac{\tilde{p}_i}{q_i}.$$
(22)

The set $H(E(t),\mathcal{G},\lambda)$ is also described as,

$$\operatorname{codim} H(E(t),\mathcal{G},\lambda) = tE(t) - \log_M Z(q,\lambda,t).$$
(23)

This is because the set $F(\tilde{p},\mathcal{G})$ is spread widest in $H(E,\mathcal{G},\lambda)$, since $\tilde{p}$ is the closest to q (which is determined by $\mathcal{G}$) in the viewpoint of relative entropy. Hence the Hausdorff dimension of $H(E,\mathcal{G},\lambda)$ is dominated by that of $F(\tilde{p},\mathcal{G})$.

5 Geometry of Time Evolutions

As mentioned above, the partition function $Z(q, \lambda, t)$ appears naturally both in the Toda lattice and the Campbell's fractal set. We study geometrical behavior of $Z(q, \lambda, t)$ and $p(t)$ or $\tilde{p}(t)$ in this section.

Consider t in $H(E(t), \mathcal{G}, \lambda)$ as a time variable. Then time evolution of the set $H(E(t), \mathcal{G}, \lambda)$ and its Hausdorff dimension are related with those of Toda lattice. The same function $Z(q, \lambda, t)$ plays an important role in both cases. In the field of Toda lattice, q is some initial value and λ is constants in motion. In the field of fractal set, q is determined by partition $\mathcal{G}$ and λ is employed to define the set. Obviously the time evolution of $Z(q, \lambda, t)$ is the same in both cases. This is more clearly understood by the geometrical aspect of the vector $p(t)$ (Toda lattice) or $\tilde{p}(t)$ (Campbell's fractal) which determines the time evolution of the system in detail. At first, we note that the behavior of $p(t)$ or $\tilde{p}(t)$ is understood as the motion in the space of probability, since their total sum is equal to one.

An important distribution in probability theory is the exponential type one, which is defined in the discrete variable case by

$$p_k(t_1, t_2, \ldots, t_r) = c(t_1, t_2, \ldots, t_r) q_k e^{t_1 f_1(k) + \cdots + t_r f_r(k)}, \quad k = 1, 2, \ldots, m, \quad (24)$$

where $t_1, t_2, \ldots, t_r$ are parameters of distribution, $c(t_1, t_2, \ldots, t_r)$ is the normalization factor and $f_j(k), q_k$ are arbitrary functions of k. In the single parameter case, the distribution is expressed as

$$p_k(t) = c(t) q_k e^{\lambda_k t}, \quad (25)$$

where λ_k is employed instead of $f_1(k)$. The above p_k is just the same as in the previous sections regarding $c(t)$ as $1/Z(t)$.

The exponential type distributions have geometrical structure which is naturally derived from the property of probability. They have metric g_{ij} and dual connections $\Gamma^{\pm 1}_{ij,k}$ (see for example [6]);

$$g_{ij} = \sum_{l=1}^{m} \frac{\partial \log p_l}{\partial t_i} \frac{\partial \log p_l}{\partial t_j} p_l \quad (26)$$

$$\Gamma^{\pm 1}_{ij,k} = \sum_{l=1}^{m} \left(\frac{\partial \log p_l}{\partial t_i \partial t_j} + \frac{1 \mp 1}{2} \frac{\partial \log p_l}{\partial t_i} \frac{\partial \log p_l}{\partial t_j} \right) \frac{\partial \log p_l}{\partial t_k} p_l \quad (27)$$

The metric g_{ij} is called the Fisher information which satisfies the following condition for arbitrary vector fields X, Y, Z.

$$Z \langle X, Y \rangle = \left\langle \nabla_Z^{+1} X, Y \right\rangle + \left\langle X, \nabla_Z^{-1} Y \right\rangle, \qquad (28)$$

where $\langle , \rangle$ is the inner product associated to the metric and $\nabla^{\pm 1}$ are covariant derivatives associated to the connections. When a space has zero curvature with respect to the Γ^1, the space is called 1-flat. The parameters t_i's in an exponential type distribution are all geodesics associated with the connection Γ^1, so that the exponential type distributions are all 1-flat. Therefore the orbits of the time evolutions of Toda lattice and Campbell's fractal set are both considered to be geodesic flows on the exponential type distribution. By considering higher hierarchies of time evolutions, we obtain 1-flat coordinates on the space of probability, fractal sets and tridiagonal matrices of Toda lattice systems. This flat property is an expression of complete integrability.

6 Concluding Remarks

We have studied the relation between Toda lattice and Campbell's fractal set through dimension and its time evolution. They are identified in view of geometry on probability.

We have treated the continuous valued case in this paper. The discrete valued case is more natural in view of dimension function. Replacing log det with dim, we will find new integrable systems of discrete value. It is an interesting problem to find the discrete analogue.

Acknowledgments

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