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On a Discrete Soliton Equation of Toda-type Related to a Cellular Automaton

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A discrete soliton equation is proposed. Features of this equation are as follows: 1. Its variables are all discrete. Especially, its dependent variable can take any non-negative integer value. 2. It can be transformed into a cellular automaton with two-valued site values. 3. It has multi-soliton solutions. 4. It is a discrete analogue of Toda lattice equation.

Moreover, another type of discrete soliton equation which has the above features 1~3 is proposed.

KEYWORDS : discrete, soliton, Toda lattice, cellular automaton

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Soliton equations belong to a special class of nonlinear equations. It is known that almost every equation in this class is related each other through their algebraic structure. Therefore, we can analyze structures of continuous soliton equations and those of discrete soliton equations from the unified viewpoint. [1,2]

Most soliton equations have variables which are all continuous, partially discrete, or all discrete. Moreover, soliton cellular automata have been proposed in recent years. [3-8] We can consider that a cellular automaton is a difference equation with discrete dependent and independent variables. One of the features of a cellular automaton is that their dependent variable takes on a finite set of discrete values.

In this letter, we propose an equation of which variables are all discrete and which has multi-soliton solutions. Its dependent variable can take any non-negative integer value. Moreover, it can be transformed into a cellular automaton.

At first, we propose a definition of the equation. Let us suppose a discrete variable u_j^t of which value is 0 or positive integer. The subscript j is integer and refers to a space variable. The superscript t is also integer and refers to time. Suppose the following $1+1$ dimensional equation on u_j^t:

$$u_j^{t+1} - 2u_j^t + u_j^{t-1} = F(u_{j+1}^t) - 2F(u_j^t) + F(u_{j-1}^t), \tag{1}$$

where $F(n)$ is defined as follows:

$$F(n) = \begin{cases} 1 & \text{if } n = 0 \\ n & \text{if } n > 0 \end{cases} .$$

Moreover, we apply a boundary condition $u_j^t \to 0$ $(j \to \pm\infty)$ to eq. (1).

From eq. (1), a state at $t+1$ is determined by those at t and at $t-1$. A state at $t-1$ is determined by those at t and at $t+1$. In this sense, eq. (1) is reversible for time. Using eq. (1) and the boundary condition, it is easily shown that a quantity $\sum_{i=-\infty}^{\infty}(u_i^{t+1} - u_i^t)$ is constant for time. Therefore, a quantity $\sum_{i=-\infty}^{\infty} u_i^t$ is a linear function of t.

Figure 1 (a) shows an example of time evolution of a state. The row at time t denotes a state at t, that is, $\cdots u_{j-2}^t u_{j-1}^t u_j^t u_{j+1}^t u_{j+2}^t \cdots$. Note that '.' denotes 0. The quantity $\sum_{i=-\infty}^{\infty} u_i^t$ is always 13 in the example of Fig. 1 (a). Figure 1 (b) shows another example of time evolution. Since $\sum_{i=-\infty}^{\infty} u_i^t = 6 - 2t$ in this example, $\sum_{i=-\infty}^{\infty} u_i^t < 0$ for $t \geq 4$. Actually, a state at $t = 4$

becomes $\cdots 000(-2)000 \cdots$ by eq. (1). Since the function $F(n)$ is defined only for $n \geq 0$, a state at $t = 5$ can not be determined by eq. (1). If the definition of $F(n)$ is extended for $n < 0$, this problem can be avoided. However, various extensions are applicable and we have not been able to analyze the properties of eq. (1) completely. Therefore, let us consider only cases where $u_j^t \geq 0$ for any j and t. In such cases, $\sum_{i=-\infty}^{\infty} u_i^t$ becomes constant for t.

Let us introduce a new variable v_j^t which is related to u_j^t through the following equation:

$$v_{j+1}^t - 2v_j^t + v_{j-1}^t = (u_{j+1}^t - 2u_j^t + u_{j-1}^t) - (u_j^{t+1} - 2u_j^t + u_j^{t-1}). \tag{2}$$

As a boundary condition of v_j^t, we adopt $v_j^t \to 0$ $(j \to \pm\infty)$. From eqs. (1) and (2), the relation

$$v_{j+1}^t - 2v_j^t + v_{j-1}^t = -\delta(u_{j+1}^t) + 2\delta(u_j^t) - \delta(u_{j-1}^t), \tag{3}$$

is derived where $\delta(n)$ is defined as

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}.$$

Then, we get the relation

$$v_j^t = 1 - \delta(u_j^t), \tag{4}$$

from eq. (3) and the boundary condition. Moreover, the following relation is obtained from eq. (2):

$$u_j^t = \sum_{s=0}^{\infty} \sum_{i=0}^{\infty} \left(-v_{j+s-i+1}^{t-s-i-1} + 2v_{j+s-i}^{t-s-i-1} - v_{j+s-i-1}^{t-s-i-1} \right). \tag{5}$$

Using eqs. (3) and (5), a closed form of time evolution equation on v_j^t is obtained. The value of right-hand side of eq. (4) is always 0 or 1. Therefore, eq. (1) can be transformed into a cellular automaton with two-valued site values v_j^t if $u_j^t \geq 0$ for any j and t.

Next, we show some special solutions of eq. (1). As noted above, we consider only solutions of which values are zero or positive for any j and t.

The first type of solutions are 'solitary wave' solutions shown in Fig. 2. In the solutions, the same pattern of values appear cyclically. The solution of Fig. 2 (b) is obtained by reversing the direction of space coordinate of Fig. 2 (a). The solution of Fig. 2 (c) is a static solution. All solutions shown in Fig. 2 can be considered to be an analogue of a solitary wave solution of a continuous soliton equation. The non-zero values at each time is a solitary wave and it moves right (Fig. 2 (a)), moves left (Fig. 2 (b)), does not move (Fig. 2 (c)), as time proceeds.

The mean speed of the wave is $\frac{N}{N+1}$ (Fig. 2 (a)), $-\frac{N}{N+1}$ (Fig. 2 (b)), 0 (Fig. 2 (c)). We call the wave shown in Fig. 2 (a), (b) and (c), 'wave-N', 'wave-($-N$)' and 'wave-0', respectively.

An example of the second type of solutions is shown in Fig. 3. In this figure, three wave-3's exist simultaneously. Unlike the solitary wave solution of continuous soliton equations, non-zero area of wave-n is always finite. Therefore, the same wave-n's can exist simultaneously if they are appropriately arranged. We call this type of solutions 'multiple wave' solutions.

Examples of the third type of solutions are shown in Fig. 4. Two different waves exist initially in every example. Since their speeds are different, they interact each other. It is a remarkable feature of this system that the same waves reappear after the interaction. Due to the nonlinearity of eq. (1), orbits and patterns of waves are shifted through the interaction.

We can consider that solutions as shown in Fig. 4 are analogous to a 2-soliton solution of continuous soliton equations. Both waves of the solution in Fig. 4 are solitons, they preserve their identity, and 'phase shift' occurs through their interaction. Since the pattern of non-zero values included in each wave changes cyclically as time proceeds, the definition of phase shift of this system should be different from that of continuous soliton systems.

We define the phase shift of each wave by a pair of integers (k, s) as follows:

> Consider the interaction of wave-ℓ and wave-m ($\ell \neq m$). If wave-m does not exist initially, wave-ℓ propagates as a solitary wave. A pattern of wave-ℓ in this case coincides with a pattern of wave-ℓ before the interaction of the two waves. Shift this pattern by k in space and by s in time. If the shifted pattern coincides with the pattern of wave-ℓ after the interaction, the phase shift of wave-ℓ is (k, s).

According to this definition, $(k + n\ell, s + n(\ell + 1))$ for any integer n can be a phase shift of wave-ℓ. However, if one of such phase shifts is given, we can easily get the others. Therefore, this indefiniteness is trivial.

Figure 4 (a) shows the interaction of wave-3 and wave-1. Their phase shifts are $(3,3)$ and $(-1,1)$. Figure 4 (b) shows the interaction of wave-1 and wave-(-2). Their phase shifts are $(1,1)$ and $(-1,1)$. We can show a general rule to calculate the phase shift s_ℓ of wave-ℓ and s_m

of wave-m by the interaction of the two waves as follows:

$$\begin{aligned} &1)\ \text{If } \ell > m \geq 0,\ s_\ell = (2m+1, 2m+1) \text{ and } s_m = (-1,1).\\ &2)\ \text{If } \ell < m \leq 0,\ s_\ell = (2m-1, -2m+1) \text{ and } s_m = (1,1).\\ &3)\ \text{If } \ell > 0 \text{ and } m < 0,\ s_\ell = (1,1) \text{ and } s_m = (-1,1).\end{aligned} \quad (6)$$

In order to prove the above rule, we checked all probable cases of the interaction. Since the proof is long, we omit it here.

An example of the fourth type of solutions is shown in Fig. 5. The solution is analogous to multi-soliton solutions of continuous soliton equations. In the figure, wave-6, wave-2, wave-0, wave-(-1) and wave-(-4) interact each other. All waves preserve their identity through the interaction. Though we have not yet proved that all waves preserve their identity for any initial state, there is no exception in our numerical experiments.

We define a phase shift of a wave by a total shift of its pattern from $t \to -\infty$ to $t \to +\infty$. The empirical rule to calculate a phase shift is as follows:

> Consider the interaction of wave-ℓ_i ($i = 1, 2, \cdots, N$). Suppose that s_i denotes a phase shift of wave-ℓ_i. Let $s_{i,j} = (a_{i,j}, b_{i,j})$ ($i \neq j$) denote a phase shift of wave-ℓ_i in the case of the interaction of wave-ℓ_i and wave-ℓ_j. $s_{i,j}$ can be calculated by rule (6). Note that $s_{i,j} = 0$ if $i = j$. Then, $s_i = \sum_{\substack{j=1\\ j\neq i}}^{N} s_{i,j} \equiv (\sum_{\substack{j=1\\ j\neq i}}^{N} a_{i,j}, \sum_{\substack{j=1\\ j\neq i}}^{N} b_{i,j})$.

For example, wave-2 of Fig. 5 interacts with wave-6, wave-0, wave-(-1) and wave-(-4). Then, its phase shift is $(-1,1) + (1,1) + (1,1) + (1,1) = (2,4)$ and it can be confirmed by the observation of the figure.

The behavior of waves described above shows that each wave plays a role of a soliton. There exist waves which move right and those which move left, like solitons of the Toda lattice equation:

$$r^j_{tt} = -\exp(-r^{j+1}) + 2\exp(-r^j) - \exp(-r^{j-1}), \quad (7)$$

where r^j is a function of lattice number j and time t. If we introduce $w^t_j = -u^t_j$, eq. (1) is rewritten as

$$w^{t+1}_j - 2w^t_j + w^{t-1}_j = -F(-w^t_{j+1}) + 2F(-w^t_j) - F(-w^t_{j-1}). \quad (8)$$

The left-hand side of eq. (8) is a second difference of w^t_j on t and that of eq. (7) is a second derived function of r^j on t. The right-hand side of eq. (8) is a second difference of $-F(-w^t_j)$

on j and that of eq. (7) is a second difference of $-\exp(-r^j)$ on j. From this correspondence, we consider that eq. (8) is of Toda-type (type of Toda lattice equation).

Finally, we propose another type of discrete soliton equation. This equation can be transformed into the soliton cellular automaton in ref. 7. The definition of the cellular automaton is as follows. Suppose that v_j^t takes a value of 0 or 1 and $v_j^t \to 0$ ($j \to \pm\infty$). The evolution rule is

$$v_j^{t+1} = \begin{cases} 1 & \text{if } v_j^t = 0 \text{ and } \sum_{i=-\infty}^{j-1} v_i^t > \sum_{i=-\infty}^{j-1} v_i^{t+1} \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

The reference shows that eq. (9) has multi-soliton solutions and any state is constructed from only solitons.

Let us introduce a variable

$$S_j^t = \sum_{i=-\infty}^{j} v_i^t. \tag{10}$$

Note that $v_j^t = S_j^t - S_{j-1}^t$ and $S_j^t \geq S_j^{t+1}$ for any j and t. Then, S_j^t satisfies

$$S_{j+1}^{t+1} - S_j^t = 1 - F(S_{j+1}^t - S_j^{t+1}). \tag{11}$$

If we introduce a variable

$$T_j^t = S_{j+1}^t - S_j^{t+1},$$

T_j^t satisfies

$$T_{j+1}^{t+1} - T_j^t = F(T_j^{t+1}) - F(T_{j+1}^t). \tag{12}$$

By the following transformation of variables;

$$j' = \frac{t-j}{2}, \quad t' = \frac{t+j}{2}, \quad u_{(t-j)/2}^{(t+j-1)/2} = T_j^t,$$

eq. (12) is rewritten as

$$u_{j'}^{t'+1/2} - u_{j'}^{t'-1/2} = F(u_{j'+1/2}^{t'}) - F(u_{j'-1/2}^{t'}). \tag{13}$$

The left-hand side and the right-hand side of eq. (13) have a difference form of first order on t' and on j', respectively. On the other hand, both sides of eq. (1) have a difference form of second order. Therefore, eq. (13) is another type of discrete soliton equation which can be transformed into the cellular automaton (9).

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$t =$ 0	:	· · 1 5 · · 3 4 · · · · ·
1	:	· · · 2 4 · · 4 3 · · · ·
2	:	· · · · 3 3 · · 5 2 · · ·
3	:	· · · · · 4 2 · · 6 1 · ·

(a)

$t =$ 0	:	· · 3 · 3 · ·
1	:	· · 1 2 1 · ·
2	:	· · · 2 · · ·
3	:	· · · · · · ·

(b)

Fig. 1 : Examples of time evolution of a state.
'·' denotes 0.

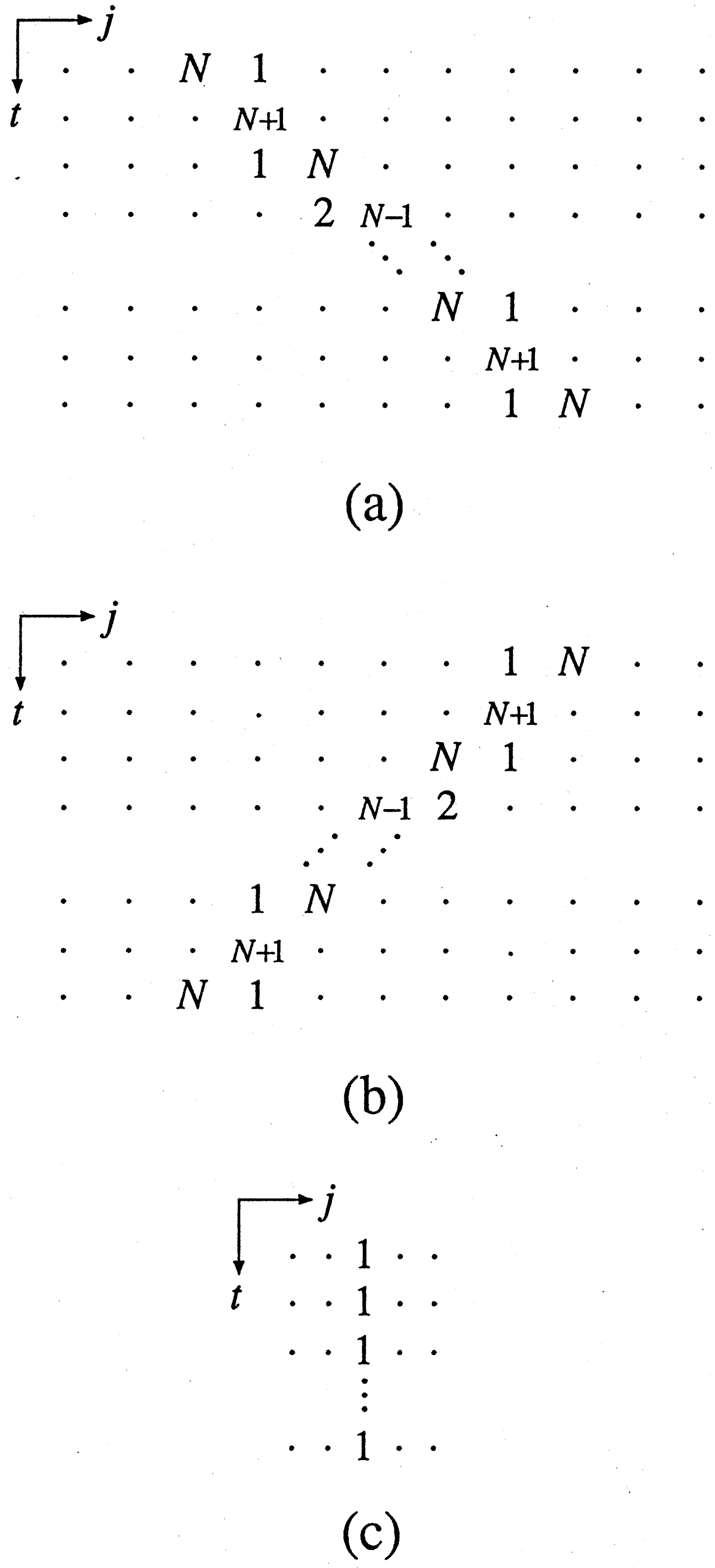

Fig.2 : Solitary wave solutions.
N is a positive integer.

```
  ┌──→ j
  │
  ↓  · · 4 · 3 1 · · 2 2 · · · · · ·
  t  · · 1 3 · 4 · · · 3 1 · · · · ·
     · · · 2 2 1 3 · · · 4 · · · · ·
     · · · · 3 1 2 2 · · 1 3 · · · ·
     · · · · · 4 · 3 1 · · 2 2 · · ·
     · · · · · 1 3 · 4 · · · 3 1 · ·
```

Fig.3 : A multiple wave solution.

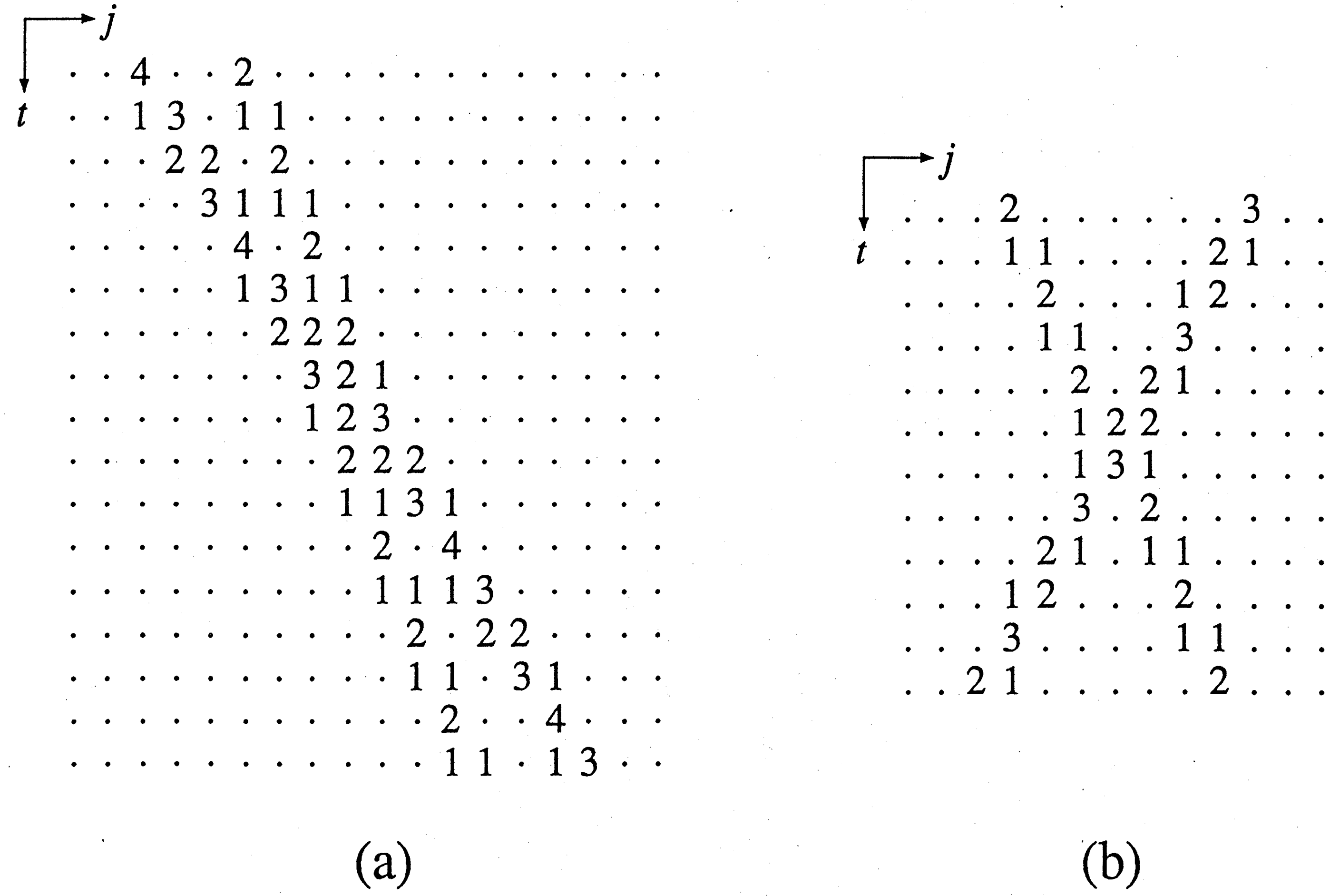

Fig. 4 : Examples of the interaction of two waves.

```
  → j
↓
t
· · · 1 6 · 2 1 · · · · · · 1 · · · · 2 · · 3 2 · · ·
· · · · 2 5 · 3 · · · · · · 1 · · · 1 1 · 2 3 · · · ·
· · · · · · 3 4 1 2 · · · · 1 · · · 2 · 1 4 · · · · ·
· · · · · · · 4 3 2 1 · · · 1 · · 1 1 · 5 · · · · · ·
· · · · · · · · 5 2 3 · · · 1 · · 2 · 4 1 · · · · · ·
· · · · · · · · · 6 2 2 · · 1 · 1 1 3 2 · · · · · · ·
· · · · · · · · · 1 5 3 1 · 1 · 2 2 3 · · · · · · · ·
· · · · · · · · · · 2 4 4 · 1 1 2 4 · · · · · · · · ·
· · · · · · · · · · · 3 4 3 1 3 3 1 · · · · · · · · ·
· · · · · · · · · · · 1 2 5 5 3 2 · · · · · · · · · ·
· · · · · · · · · · · · 2 4 7 4 1 · · · · · · · · · ·
· · · · · · · · · · · · 3 4 3 5 3 · · · · · · · · · ·
· · · · · · · · · · · 2 3 2 2 2 5 2 · · · · · · · · ·
· · · · · · · · · · 1 4 1 1 1 2 1 6 1 · · · · · · · ·
· · · · · · · · · · 5 · 2 · 1 · 3 · 7 · · · · · · · ·
· · · · · · · · · 4 1 1 1 · 1 · 1 2 1 6 · · · · · · ·
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· · · · · · 1 4 · · 2 · · · 1 · · · 1 2 · 4 3 · · · ·
· · · · · · 5 · · 1 1 · · · 1 · · · · 2 1 · 5 2 · · ·
· · · · · 4 1 · · 2 · · · · 1 · · · · · 3 · · 6 1 · ·
· · · · 3 2 · · 1 1 · · · · 1 · · · · · 1 2 · · 7 · ·
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Fig. 5 : An example of the interaction of several waves.