数理解析研究所講究録
933 巻 1995 年 85-104

MINOR SUMMATION FORMULA AND APPLICATIONS, DISCRETE FOURIER TRANSFORMS

MASAO ISHIKAWA AND MASATO WAKAYAMA
石川雅雄　　　若山正人
鳥取大学教育学部　九州大学数理学研究科

ABSTRACT. The aims of the paper are as follows: (1) to prove miscellaneous identities such as pfaffians version of Plücker relations, Lewis-Caroll's formula from the minor summation formula of pfaffians; (2) as an application we give some identities which are considered as special generalizations of Littlewood's formulas. Further in Appendix we give another proof of a minor summation formula of Pfaffians by means of the lattice path method from a combinatorial aspect.

0. INTRODUCTION

Our minor summation formula of pfaffians is viewed as a formula for providing some sort of Fourier transforms for discrete type as well as the Cauchy-Binet formula for matrices. In this situation, the kernel functions of Fourier transforms are represented by a certain series of minor-determinants or subpfaffian indexed by partitions of a suitable matrix instead of the usual exponential functions as well as test functions. We think this point of view is somewhat new. Of course, as we have developed in [IOW], Littelewood's formulas provide information about the irreducible decompositions (= non-commutative Fourier series expansions) of several representations of classical groups. But our viewpoint has more sophisticated sense. Actually, in this paper we develop certain miscellaneous identities of pfaffians and, as a first step, give Fourier expansion's formulas of certain functions like elliptic thetas with special emphasis from this view point.

1. MINOR SUMMATION FORMULA

Let $\mathfrak{S}_n$ be the permutation group of the index set $[n] \stackrel{\text{def}}{=} \{1,2,\dots,n\}$ and, for each permutation $\sigma \in \mathfrak{S}_n$, let $\operatorname{sgn}\sigma$ stand for $(-1)^{\ell(\sigma)}$ where $\ell(\sigma)$ is the number of inversions in σ.

Let $n = 2s$ be even. Let H be the subgroup of $\mathfrak{S}_n$ generated by the elements $(2i-1, 2i)$ for $1 \le i \le s$ and $(2i-1, 2i+1)(2i, 2i+2)$ for $1 \le i < s$. We set a subset $\mathfrak{F}_n$ of $\mathfrak{S}_n$ to be

$$\mathfrak{F}_n = \left\{ \sigma = (\sigma(1),\dots,\sigma(n)) \in \mathfrak{S}_n \,\middle|\, \begin{array}{l} \sigma(2i-1) < \sigma(2i)\ (1 \le i \le s) \\ \sigma(2i-1) < \sigma(2i+1)\ (1 \le i \le s-1) \end{array} \right\}.$$

For each $\pi \in \mathfrak{S}_n$, $H\pi \cap \mathfrak{F}_n$ has a unique element σ. Let $n = 2s$ be an even integer and $B = (b_{ik})_{1 \le i < k \le n}$ be an n by n upper triangular matrix whose entries β_{ik} are in a commutative ring.

The *pfaffian* of B is by definition

$$\operatorname{pf}(B) = \sum_{\sigma \in \mathfrak{F}_n} \operatorname{sgn}\sigma\, b_{\sigma(1)\sigma(2)} \cdots b_{\sigma(n-1)\sigma(n)}. \tag{1.1}$$

When n is a positive integer and N is a positive integer or ∞ such that $n \le N$, let $[n, N]$ denote the totally ordered set $\{n, n+1, \dots, N\}$. Especially we abbreviate $[1, N]$ to $[N]$. Note that, when $N = \infty$, $[N]$ stands for the set of all positive integers $\mathbb{P}$. When r is a positive integer with $r \le N - n + 1$, let $[n, N]_r$ denote the set of all r-tuples $\boldsymbol{i} = (i_1, \dots, i_r)$ such that $i_k \in [n, N]$ and $i_1 < \cdots < i_r$.

Let n and N be positive integers or ∞. An n by N matrix $A = (a_{ij})$ is an array of entries a_{ij} for $(i,j) \in [n] \times [N]$. An n by n matrix $A = (a_{ij})$ is said to be *skew-symmetric* if its entries satisfy $a_{ij} = -a_{ji}$ for $(i,j) \in [n] \times [n]$.

We sometimes regard an upper triangular matrix $A = (a_{ij})_{1\le i<j\le n}$ as an skew-symmetric matrix by the obvious way. When $\boldsymbol{i} = (i_1,\dots,i_r) \in [n]_r$ and $\boldsymbol{j} = (j_1,\dots,j_r) \in [N]_r$, let $A^{\boldsymbol{i}}_{\boldsymbol{j}} = A^{i_1\dots i_r}_{j_1\dots j_r}$ denote the submatrix of A with the entries $a_{i_k j_\ell}$ for $1 \le k,\ell \le r$. When $\boldsymbol{i}$ is $[n]$ itself with the ordinary order, we abbreviate $A^{[n]}_{\boldsymbol{j}}$ to $A_{\boldsymbol{j}}$ for simplicity. We use the similar abbreviation in the case $\boldsymbol{j} = [N]$.

A summation formula of minors, where the sum extends to all columns, weighted by the subpfaffians of a given skew-symmetric matrix, is established in [IW1].

We describe the theorem here which corresponds to the case of $q = 1$ of Theorem 1 in [IW1] and we call it as the minor summation formula of pfaffians.

Theorem 1.1. *Let n be an even integer, N be a positive integer or ∞ such that $n \le N$. Let $T = (t_{ik})$ be any n by N rectangular matrix. Let $B = (b_{ik})$ be any N by N skew-symmetric matrix. Then*

$$\sum_{\boldsymbol{k}\in[N]_n} \mathrm{pf}(B^{\boldsymbol{k}}_{\boldsymbol{k}})\det(T_{\boldsymbol{k}}) = \mathrm{pf}(Q), \tag{1.2}$$

where Q is the skew-symmetric matrix defined by $Q = TB^tT$, i.e.

$$Q_{ij} = \sum_{1\le k<\ell\le N} b_{k\ell}\det(T^{ij}_{k\ell}), \qquad (1 \le i,j \le m). \tag{1.3}$$

From this theorem, we obtain the so-called Cauchy-Binet formula [IOW]: Let n be a positive integer and N be a positive integer or ∞, and suppose $n \le N$.

$$\sum_{\boldsymbol{k}\in[N]_n} \det(X_{\boldsymbol{k}})\det(Y_{\boldsymbol{k}}) = \det\left(X^tY\right), \tag{1.4}$$

for any matrices $X = (x_{ik})_{1\le i\le n, 1\le k\le N}$ and $Y = (y_{ik})_{1\le i\le n, 1\le k\le N}$.

Moreover if we take $n = N = 2l$ then $\mathrm{pf}(B)\det(T) = \mathrm{pf}(TB^tT)$. This means that every determinant can be represented by a pfaffian of the *same* degree. Actually if we choose $B = K_l(b_1,\dots,,b_l)$, where

$$K_l(b_1,\dots,,b_l) = \begin{pmatrix} 0 & b_1 & \dots & 0 & 0 \\ -b_1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & b_l \\ 0 & 0 & \dots & -b_l & 0 \end{pmatrix},$$

then $\det(T) = \mathrm{pf}(TB^tT)$ because $\mathrm{pf}(B) = 1$. On the other hand, by the successive use of this relation we see

$$\det(S)\det(T) = \det(S)\det(T)\,\mathrm{pf}(B) = \det(S)\,\mathrm{pf}(TB^tT) = \mathrm{pf}(STB^tT^tS) = \det(ST).$$

Further it is well-known that every skew symmetric matrix is block diagonalizable, *i.e.* we see that $TB^tT = K_l(b_1,\dots,b_l)$ for some T. Then we observe

$$\mathrm{pf}(B)^2 = \mathrm{pf}(TB^tT)^2 = \mathrm{pf}(K_l(b_1,\dots,b_l))^2 = (b_1\cdots b_l)^2 = \det(B). \tag{1.5}$$

This implies that a square of pfaffian equals the determinant for any skew symmetric matrix. Although the following formula is well-known (cf. [Ste], [IW1]) and directly derived from the very definition of pfaffians, we give here it as the corollary of above theorem.

Corollary 1.1. *Let A and B be m by m skew symmetric matrices. Put $s = [\frac{m}{2}]$, the integer part of $\frac{m}{2}$. Then*

$$\mathrm{pf}(A+B) = \sum_{t=0}^{s}\sum_{\boldsymbol{i}\in m_{2t}} (-1)^{|\boldsymbol{i}|-t}\,\mathrm{pf}(A^{\boldsymbol{i}}_{\boldsymbol{i}})\,\mathrm{pf}(B^{\boldsymbol{i}^c}_{\boldsymbol{i}^c}), \tag{1.6}$$

where we denote by $\boldsymbol{i}^c$ *the complementary set of* $\boldsymbol{i}$ *in* $[m]$ *which is arranged in the increasing order, and* $|\boldsymbol{i}| = i_1 + \cdots + i_{2t}$ *for* $\boldsymbol{i} = (i_1, \ldots, i_{2t})$.

In particular we have the expansion formula of pfaffian with respect to any column (row): For any i, j *we have*

$$\delta_{ij}\,\mathrm{pf}(A) = \sum_{k=1}^{m} (-1)^{k+j-1} a_{kj}\,\mathrm{pf}(A^{ki}), \tag{1.7}$$

$$\delta_{ij}\,\mathrm{pf}(A) = \sum_{k=1}^{m} (-1)^{i+k-1} a_{ik}\,\mathrm{pf}(A^{jk}), \tag{1.8}$$

where A^{ij} *stands for the* $(m-2)$ *by* $(m-2)$ *skew symmetric matrix which is obtained from* A *by removing both the* i, j*-th rows and* i, j*-th columns for* $1 \le i \ne j \le m$.

Proof: Let I_m be an identity matrix of degree m. It is clear that

$$(I_m \quad I_m) \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I_m \\ I_m \end{pmatrix} = A + B.$$

Hence by the minor summation formula we see

$$\begin{aligned} \mathrm{pf}(A+B) &= \mathrm{pf}\left((I_m \quad I_m) \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} {}^t(I_m \quad I_m) \right) \\ &= \sum_{\boldsymbol{k} \in [2m]_m} \mathrm{pf} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}_{\boldsymbol{k}}^{\boldsymbol{k}} \det(I_m \quad I_m)_{\boldsymbol{k}}. \end{aligned}$$

The only indices $\boldsymbol{k}$ in $[2m]_m$ for which $\det(I_m I_m)_{\boldsymbol{k}}$ does not vanish is of the form $\boldsymbol{k} = (\boldsymbol{i}, (m, m, \ldots, m) + \boldsymbol{i}^c)$ for $\boldsymbol{i} \in I_s^m$ and in this case we have $\det(I_m I_m)_{\boldsymbol{k}} = (-1)^{\sigma(\boldsymbol{i}, \boldsymbol{i}^c)}$, where $\sigma(\boldsymbol{i}, \boldsymbol{i}^c)$ means the number of inversions of $\boldsymbol{i}$ via $\boldsymbol{i}^c$. Further, if s is even, then we have

$$\mathrm{pf} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}_{\boldsymbol{k}}^{\boldsymbol{k}} = \mathrm{pf} \begin{pmatrix} A_{\boldsymbol{i}}^{\boldsymbol{i}} & 0 \\ 0 & B_{\boldsymbol{i}^c}^{\boldsymbol{i}^c} \end{pmatrix} = \mathrm{pf}(A_{\boldsymbol{i}}^{\boldsymbol{i}})\,\mathrm{pf}(B_{\boldsymbol{i}^c}^{\boldsymbol{i}^c}).$$

This pfaffian vanishes obviously in the case s is odd. Hence we see

$$\begin{aligned} \mathrm{pf}(A+B) &= \sum_{\boldsymbol{k} \in [2m]_m} \sum_{\boldsymbol{k} = (\boldsymbol{i}, (m,m,\ldots,m) + \boldsymbol{i}^c)} \mathrm{pf}(A_{\boldsymbol{i}}^{\boldsymbol{i}})\,\mathrm{pf}(B_{\boldsymbol{i}^c}^{\boldsymbol{i}^c}) (-1)^{\sigma(\boldsymbol{i}, \boldsymbol{i}^c)} \\ &= \sum_{t=0}^{[m/2]} \sum_{\boldsymbol{i} \in [m]_{2t}} (-1)^{|\boldsymbol{i}| - t}\,\mathrm{pf}(A_{\boldsymbol{i}}^{\boldsymbol{i}})\,\mathrm{pf}(B_{\boldsymbol{i}^c}^{\boldsymbol{i}^c}), \end{aligned}$$

because $\sigma(\boldsymbol{i}, \boldsymbol{i}^c) = |\boldsymbol{i}| - t$ for $\boldsymbol{i} \in [m]_{2t}$.

The latter assertion can be proved by applying the previous result to the following form of the decomposition of a skew symmetric matrix A with respect to the i-th row and column;

$$A = \begin{pmatrix} 0 & * & 0 \\ * & 0 & * \\ 0 & * & 0 \end{pmatrix} + \begin{pmatrix} * & 0 & * \\ 0 & 0 & 0 \\ * & 0 & * \end{pmatrix}.$$

This completes the proof. □

The following formula is a generalization of a theorem by Stembridge [Ste] and can be proved by the above corollary.

Theorem 1.2. *Suppose m, r are positive integers and n is a positive integer or ∞ such that $m+r$ is even and $0 \le m-r \le n$. Let $T = (t_{ik})_{1\le i\le m, 1\le k\le n+r}$ be any m by $(n+r)$ matrix. Let $H = (t_{ik})_{1\le i\le m, 1\le k\le n}$ be the submatrix of T composed of the first r columns, and $G = (t_{i\,r+k})_{1\le i\le m, 1\le k\le r}$ be the submatrix of T composed of the last n columns. Let B be any n by n skew symmetric matrix. Then we have*

$$(1.9)\qquad \sum_{\boldsymbol{k}\in[r+1,r+n]_{m-r}} \mathrm{pf}(B_{\boldsymbol{k}}^{\boldsymbol{k}}) \det(T_{[r]\cup\boldsymbol{k}}) = \mathrm{pf}\begin{pmatrix} Q & HJ_r \\ -J_r{}^tH & O_r \end{pmatrix},$$

where Q is the m by m skew symmetric matrix given by $Q = GB^tG$, i.e.

$$(1.10)\qquad Q_{ij} = \sum_{1\le k<\ell\le n} \beta_{k\ell} \det T_{k+r,\ell+r}^{ij},$$

and $[r]\cup\boldsymbol{k}$ denote the m-tuple of $[r] = (1,\dots,r)$ and $\boldsymbol{k} = (k_1,\dots,k_{m-r}) \in [r+1, r+n]_{m-r}$.

Let J_r denote the square matrix of size r whose (i,j)-entry is 1 if $i = r-j$, and 0 otherwise. Let I_r denote the identity matrix of size r, and let O_r denote the square zero matrix of size r. The following theorem is Theorem 2 of [IW1] and a minor summation formula, where the sum extends to all columns with some fixed columns. One can see that Theorem 1.1 is obviously a special case of the following theorem. The proof is done by a successive uses of the formula (1.6) and the minor summation formula.

Theorem 1.3. *Let $m \le n$ and T. Let $A = (a_{ik})_{1\le i,k\le m}$ and $B = (b_{ik})_{1\le i,k\le n}$ be arbitrary skew symmetric matrices. Then*

$$(1.11)\qquad \begin{aligned} \sum_{t=0}^{[\frac{m}{2}]} z^t \sum_{\substack{\boldsymbol{i}\in[m]_{2t} \\ \boldsymbol{k}\in[n]_{2t}}} \mathrm{pf}(A_{\boldsymbol{i}}^{\boldsymbol{i}}) \mathrm{pf}(B_{\boldsymbol{k}}^{\boldsymbol{k}}) \det(T_{\boldsymbol{k}}^{\boldsymbol{i}}) &= \mathrm{pf}\begin{pmatrix} -J_mAJ_m & J_m \\ -J_m & zQ \end{pmatrix} \\ &= (-1)^{\frac{m(m-1)}{2}} \mathrm{pf}\begin{pmatrix} -A & I_m \\ -I_m & zQ \end{pmatrix}, \end{aligned}$$

where z is a spectra parameter and $Q = TB^tT$, i.e.

$$(1.12)\qquad Q_{ij} = \sum_{1\le k<l\le n} b_{kl} \det(T_{kl}^{ij}), \qquad (1 \le i,j \le m).$$

We also have

Corollary 1.2. *Assume $m \le n$. Let $T = (t_{ik})$ be as in Theorem 1.3. Let $A = (a_{ik})_{0\le i,k\le m}$ and $B = (b_{ik})_{0\le i,k\le n}$ be skew symmetric matrices of size $(m+1)$ and $(n+1)$, respectively. Then*

$$(1.13)\qquad \begin{aligned} &\sum_{\substack{0\le r\le m \\ r\,:\,\text{even}}} z^r \sum_{\substack{\boldsymbol{i}\in[m]_r \\ \boldsymbol{k}\in[n]_r}} \mathrm{pf}(A_{\boldsymbol{i}}^{\boldsymbol{i}}) \mathrm{pf}(B_{\boldsymbol{k}}^{\boldsymbol{k}}) \det(T_{\boldsymbol{k}}^{\boldsymbol{i}}) + \sum_{\substack{0\le r\le m \\ r\,:\,\text{odd}}} z^r \sum_{\substack{\boldsymbol{i}\in[m]_r^m \\ \boldsymbol{k}\in[n]_r^n}} \mathrm{pf}(A_{0\boldsymbol{i}}^{0\boldsymbol{i}}) \mathrm{pf}(B_{0\boldsymbol{k}}^{0\boldsymbol{k}}) \det(T_{\boldsymbol{k}}^{\boldsymbol{i}}) \\ &= \mathrm{pf}\begin{pmatrix} -J_{m+1}AJ_{m+1} & J_{m+1} \\ -J_{m+1} & \widehat{Q} \end{pmatrix} = (-1)^{\frac{m(m-1)}{2}} \mathrm{pf}\begin{pmatrix} -A & I_{m+1} \\ -I_{m+1} & \widehat{Q} \end{pmatrix}, \end{aligned}$$

where $\widehat{Q} = (\widehat{Q}_{ij})$ is given by

$$(1.14)\qquad \widehat{Q}_{ij} = \begin{cases} 0, & \text{if } i = j = 0, \\ z\sum_{1\le k\le n} b_{0k}t_{jk}, & \text{if } i = 0 \text{ and } 1 \le j \le m, \\ z\sum_{1\le k\le n} b_{k0}t_{jk}, & \text{if } j = 0 \text{ and } 1 \le i \le m, \\ z^2\sum_{1\le k<l\le n} b_{kl}\det(T_{kl}^{ij}), & \text{if } 1 \le i,j \le m. \end{cases}$$

2. The Lewis-Caroll formula and the Plücker relation

In this section we provide a Pfaffian version of Lewis-Caroll's formula and Plücker's relation. The latter relation is also treated in [DW], and in [Kn] it is called the (generalized) basic identity. First of all we recall the so-called Lewis-Caroll's formula, or known as Jacobi's formula among for minor determinants. We give a simple proof for completeness. We only use Cramer's formula to provide it. In this section we write $A_{\boldsymbol{i}}$ for $A^{\boldsymbol{i}}_{\boldsymbol{i}}$ for short and we expect that it doesn't cause confusions since we only treat square matrices in this section.

Proposition 2.1. *Let A be an n by n matrix and $\widetilde{A}$ be the matrix of its cofactors. Let $r \le n$ and $\boldsymbol{j}, \boldsymbol{k} \in [n]_r$. Then*

$$\det \widetilde{A}_{\boldsymbol{jk}} = (\det A)^{r-1} \det A_{\boldsymbol{j}^c \boldsymbol{k}^c}, \tag{2.1}$$

where $\boldsymbol{j}^c, \boldsymbol{k}^c \in I^n_{n-r}$ stand for the complementary tuples of $\boldsymbol{j}, \boldsymbol{k}$, respectively.

Proof: We can assume that A is non-singular because both sides of the identity are polynomials in the entries of A. And it is enough to prove this in the case of $\boldsymbol{j} = \boldsymbol{k} = (n-r+1, \dots, n)$. Put $A_{11} = A_{\boldsymbol{j}^c \boldsymbol{k}^c}$, $A_{12} = A_{\boldsymbol{j}^c \boldsymbol{k}}$, $A_{21} = A_{\boldsymbol{j} \boldsymbol{k}^c}$, $A_{22} = A_{\boldsymbol{jk}}$. Then

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Further we can assume that A_{11} is non-singular. Then there exists a matrix $P = \begin{pmatrix} I & * \\ O & I \end{pmatrix}$ such that

$$AP = \begin{pmatrix} A_{11} & O \\ A_{21} & B_{22} \end{pmatrix},$$

where I and O stand for the identity matrix and the zero matrix, respectively. From this identity, we have

$$(AP)^{-1} = \begin{pmatrix} A_{11}^{-1} & O \\ * & B_{22}^{-1} \end{pmatrix}.$$

It follows that

$$A^{-1} = P(AP)^{-1} = \begin{pmatrix} I & * \\ O & I \end{pmatrix} \begin{pmatrix} A_{11}^{-1} & O \\ * & B_{22}^{-1} \end{pmatrix} = \begin{pmatrix} * & * \\ * & B_{22}^{-1} \end{pmatrix}.$$

Thus we have $(A^{-1})_{\boldsymbol{jk}} = B_{22}^{-1}$. Since $\widetilde{A} = |A| A^{-1}$, it follows that $\widetilde{A}_{\boldsymbol{jk}} = |A| B_{22}^{-1}$. The preceding identity gives us $|A_{11}||B_{22}| = |A|$, and these identities show

$$|\widetilde{A}_{\boldsymbol{jk}}| = |A|^r |B_{22}^{-1}| = |A|^{r-1} |A_{11}|.$$

This proves the proposition. □

Example. We give here a few examples of Lewis-Caroll's formula for low degree's matrices.

$$\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = a_{11} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}. \tag{2.1.1}$$

We give one more;

$$\begin{aligned} &\begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} \\ &+ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} = a_{11} \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}. \end{aligned} \tag{2.1.2}$$

Let m be an even integer and A be an m by m skew symmetric matrix. Assume that $\mathrm{pf}(A)$ is nonzero, that is A is non-singular. For $1 \le i \ne j \le m$, recall that A^{ij} is the $(m-2)$ by $(m-2)$ skew symmetric matrix which is obtained from A by removing both the i, j-th rows and i, j-th columns.

Define a skew symmetric matrix $\widehat{A} = (\gamma(i,j))$ by

$$\gamma(i,j) = (-1)^{i+j-1}\,\mathrm{pf}(A^{ij}) \tag{2.2}$$

for $1 \le i < j \le m$. Let $\Delta(i,j) = (-1)^{i+j} \det A^{ij}$ denote the (i,j)-cofactor of A. If we multiply the both sides of (1.7) by $\mathrm{pf}(A)$ and use a basic relation between determinants and pfaffians; $\det A = [\mathrm{pf}(A)]^2$ which we proved in §1 (for a combinatorial proof, see for e.g. [Ste]), we obtain

$$\sum_{i=1}^{m} a_{ij}\gamma(i,k)\ \mathrm{pf}(A) = \delta_{jk}\ [\mathrm{pf}(A)]^2 = \delta_{jk}\ \det A. \tag{2.3}$$

Comparing this equation with the ordinary expansion of $\det A$ as polynomials in a_{ij}'s, we obtain the following relation between $\Delta(i,j)$ and $\gamma(i,j)$:

$$\Delta(i,j) = \gamma(i,j)\ \mathrm{pf}(A). \tag{2.4}$$

The following result is considered as a pfaffian version of Lewis-Caroll's formula.

Theorem 2.1. *Let m be an even integer and A be an m by m skew symmetric matrix. Let $\widehat{A} = (\gamma(i,j))$. Then, for any $\boldsymbol{j} \in [m]_{2t}$, we have*

$$\mathrm{pf}\left[(\widehat{A})_{\boldsymbol{j}}\right] = [\mathrm{pf}(A)]^{t-1}\,\mathrm{pf}(A_{\boldsymbol{j}^c}). \tag{2.5}$$

Proof: Let $\widetilde{A} = \Delta(i,j)$ denote the matrix of the cofactors of A. From (2.4) we have $\widetilde{A} = \mathrm{pf}(A)\,\widehat{A}$, thus $\widetilde{A}_{\boldsymbol{j}} = \mathrm{pf}(A)\,(\widehat{A})_{\boldsymbol{j}}$. It follows that

$$\left|\widetilde{A}_{\boldsymbol{j}}\right| = [\mathrm{pf}(A)]^{2t}\,\left|(\widehat{A})_{\boldsymbol{j}}\right| = |A|^{t}\,\left|(\widehat{A})_{\boldsymbol{j}}\right|.$$

On the other hand, Proposition 2.1 implies that $\left|\widetilde{A}_{\boldsymbol{j}}\right| = |A|^{2t-1}|A_{\boldsymbol{j}^c}|$. Comparing these two identities, we obtain

$$\left|\widehat{A}_{\boldsymbol{j}}\right| = |A|^{t-1}|A_{\boldsymbol{j}^c}|.$$

By taking the square root of both sides of this identity, we obtain

$$\mathrm{pf}\left(\widehat{A}_{\boldsymbol{j}}\right) = \pm\,[\mathrm{pf}(A)]^{t-1}\,\mathrm{pf}\left(A_{\boldsymbol{j}^c}\right).$$

To finish the proof we have to determine the sign. By substituting

$$S = \begin{pmatrix} 0 & 1 & \dots & 1 \\ -1 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & 0 \end{pmatrix} \tag{2.6}$$

in the both sides of the above identity, we can verify that the positive branch is correct because it is easily to check $\mathrm{pf}(S) = 1$. This proves the lemma. □

Example. For $m = 6, t = 1$ and $\boldsymbol{j} = (1,2,3,4)$ in the above theorem, we see

$$\gamma(1,2)\gamma(3,4) - \gamma(2,3)\gamma(1,4) + \gamma(1,3)\gamma(2,4) = \mathrm{pf}(A)\,\mathrm{pf}(A_{(5,6)}).$$

Hence by definition, we see that this turns out to be

$$\begin{aligned} \mathrm{pf}(A_{(3,4,5,6)})\,\mathrm{pf}(A_{(1,2,5,6)}) - \mathrm{pf}(A_{(1,4,5,6)})\,\mathrm{pf}(A_{(2,3,5,6)}) + \mathrm{pf}(A_{(2,4,5,6)})\,\mathrm{pf}(A_{(1,3,5,6)}) \\ = \mathrm{pf}(A)\,\mathrm{pf}(A_{(5,6)}), \end{aligned} \tag{2.7}$$

that is, in more familiar form we see

$$\mathrm{pf}\begin{pmatrix} 0 & a_{34} & a_{35} & a_{36} \\ -a_{34} & 0 & a_{45} & a_{46} \\ -a_{35} & -a_{45} & 0 & a_{56} \\ -a_{36} & -a_{46} & -a_{56} & 0 \end{pmatrix} \mathrm{pf}\begin{pmatrix} 0 & a_{12} & a_{15} & a_{16} \\ -a_{12} & 0 & a_{25} & a_{26} \\ -a_{15} & -a_{25} & 0 & a_{56} \\ -a_{16} & -a_{26} & -a_{56} & 0 \end{pmatrix}$$
$$-\mathrm{pf}\begin{pmatrix} 0 & a_{14} & a_{15} & a_{16} \\ -a_{14} & 0 & a_{45} & a_{46} \\ -a_{15} & -a_{45} & 0 & a_{56} \\ -a_{16} & -a_{46} & -a_{56} & 0 \end{pmatrix} \mathrm{pf}\begin{pmatrix} 0 & a_{23} & a_{25} & a_{26} \\ -a_{23} & 0 & a_{35} & a_{36} \\ -a_{25} & -a_{35} & 0 & a_{56} \\ -a_{26} & -a_{36} & -a_{56} & 0 \end{pmatrix}$$
$$+\mathrm{pf}\begin{pmatrix} 0 & a_{24} & a_{25} & a_{26} \\ -a_{24} & 0 & a_{45} & a_{46} \\ -a_{25} & -a_{45} & 0 & a_{56} \\ -a_{26} & -a_{46} & -a_{56} & 0 \end{pmatrix} \mathrm{pf}\begin{pmatrix} 0 & a_{13} & a_{15} & a_{16} \\ -a_{13} & 0 & a_{35} & a_{36} \\ -a_{15} & -a_{35} & 0 & a_{56} \\ -a_{16} & -a_{36} & -a_{56} & 0 \end{pmatrix}$$
$$= \mathrm{pf}\begin{pmatrix} 0 & a_{56} \\ -a_{56} & 0 \end{pmatrix} \mathrm{pf}\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ -a_{12} & 0 & a_{23} & a_{24} & a_{25} & a_{26} \\ -a_{13} & -a_{23} & 0 & a_{34} & a_{35} & a_{36} \\ -a_{14} & -a_{24} & -a_{34} & 0 & a_{45} & a_{46} \\ -a_{15} & -a_{25} & -a_{35} & -a_{45} & 0 & a_{56} \\ -a_{16} & -a_{26} & -a_{36} & -a_{46} & -a_{56} & 0 \end{pmatrix}.$$

We next state a pfaffian version of Plücker relations (or known as Grassmann-Plücker relations for determinants) which is an algebraic identity of degree two describing the relations among several subpfaffians. This identity is proved in the book [Hi] and a recent paper [DW] in the framework of an exterior algebra.

Theorem 2.2. *Suppose n, m are odd integers. Let A be an $(m+n)\times(m+n)$ skew symmetric matrices of odd degrees. Fix a sequence of integers $\boldsymbol{i}=(i_1,i_2,\dots,i_m)$ in $[m+n]^m$. Put the complement of $\boldsymbol{i}$ by $\boldsymbol{i}^c=(k_1,k_2,\dots,k_n)\in[m+n]^m$ in $[m+n]$. Then the following relation holds.*

$$\sum_{j=1}^{n}(-1)^{j-1}\,\mathrm{pf}((A_{\boldsymbol{i}})_{\check{\boldsymbol{i}}_j})\,\mathrm{pf}(A_{i_j\cup\boldsymbol{i}^c})=\sum_{j=1}^{m}(-1)^{j-1}\,\mathrm{pf}(A_{\boldsymbol{i}\cup k_j})\,\mathrm{pf}((A_{\boldsymbol{i}^c})_{\check{\boldsymbol{k}}_j}). \tag{2.8}$$

Here the notations $\check{\boldsymbol{i}}_j$ means a taking i_j off from the index $\boldsymbol{i}$ and $i_j\cup\boldsymbol{i}^c$ stands for $\{i_j\}\cup\boldsymbol{i}^c$.

Proof: We only use the expansion formula of pfaffian given in Corollary 1.1. In fact, if we expand $\mathrm{pf}(A_{i_j\cup\boldsymbol{i}^c})$ with respect to the first i_j at the left hand side and expand also $\mathrm{pf}(A_{\boldsymbol{i}\cup k_j})$ with respect to the last k_j at the right one, and finally compare it, then it is immediately to see the desired equality. □

For convenience, we use a notation $A(i_1,i_2,\dots,i_{2k})$ instead of $A_{(i_1,i_2,\dots,i_{2k})}$ for a matrix A. Then the following assertion, which is called by the basic identity in [Kn] is a special consequence of the above formula.

Corollary 2.2. *Let A be a skew symmetric matrix of degree N. Fix an index $\boldsymbol{i}=(i_1,i_2,\dots,i_{2k})$ in $[N]^{2k}$. Take an integer l which satisfies $2k+2l\le N$. Then*

$$\begin{aligned}&\mathrm{pf}(A(1,2,\dots,2l))\,\mathrm{pf}(A(i_1,i_2,\dots,i_{2k},1,\dots,2l))\\ &=\sum_{j=1}^{2k-1}(-1)^{j-1}\,\mathrm{pf}(A(i_1,1,2,\dots,2l,i_{j+1}))\,\mathrm{pf}(A(i_2,\dots,\hat{i}_{j+1},\dots,i_{2k},1,\dots,2l))\end{aligned} \tag{2.9}$$

Proof: Put $m=2l+1$, $n=2k+2l-1$ and

$$\begin{aligned}&i_1=1_1, i_2=1, i_3=2,\dots,i_{2l+1}=2l,\\ &k_1=i_2, k_2=i_3,\dots,k_{2k-1}=i_{2k}, k_{2k}=1, k_{2k+1}=2,\dots,k_{2k+2l-1}=2l,\end{aligned}$$

in Theorem 2.2. Then, since each of terms in the left hand side's summation vanish except for the case $j=1\,(i_j=i_1)$, the desired identity immediately follows from the identity of Theorem 2.2. □

Remark. Assume $l = 2$. If we take the special choice of an index $\boldsymbol{i} = (3, 4, \ldots, 2k-4)$ with $2k+4 = N$, then the identity in this corollary is nothing but the identity in Theorem 2.1 for $t = 2$, that is, this basic identity partially covers the Lewis-Caroll identity. Consequently these two identities seem to be located at the transversely directions for each other.

3. Fourier expansion of the elliptic theta

In this section we investigate certain formulas involving the Chebyshev polynomials and the characters of the classical groups. It is also possible to derive these formulas from Cauchy's identity. We also show that the Fourier expansion formulas of Jacobi's elliptic theta-functions are obtained as a corollary of our formula.

First we recall the Chebyshev polynomials of the first and second kinds. Though there are several ways to define the Chebyshev polynomials, here we adopt the way to define them by means of determinants. Put

$$u_{ij} = \begin{cases} 2a & \text{if } i = j, \\ b & \text{if } i = j+1, \\ 1 & \text{if } j = i+1, \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

for $i, j \geq 1$. Let $U^{(n)}$ be the n by n matrix whose (i,j)-entry is given by u_{ij}, and put $u_n(a,b) = \det U^{(n-1)}$ for $n \geq 1$. For example, the first few terms are given by $u_1(a,b) = 1$, $u_2(a,b) = 2a$, $u_3(a,b) = 4a^2 - b$, $u_4(a,b) = 8a^3 - 4ab$. If we expand the determinant $\det U^{(n)}$ with respect to the first row, then we see that the polynomials $u_n(a,b)$ satisfy the recursion formula

$$u_{n+1}(a,b) - 2au_n(a,b) + bu_{n-1}(a,b) = 0 \tag{3.2}$$

for $n \geq 2$. For the integers $n \leq 0$, we define $u_n(a,b)$ as the above recursion formula always holds. The generating function of $u_n(a,b)$ is given by

$$\sum_{n=0}^{\infty} u_{n+1}(a,b)x^n = \frac{1}{1 - 2ax + bx^2}. \tag{3.3}$$

This can be seen from the above recursion formula and the first few terms of $u_n(a,b)$.

If we substitute $b = 1$ into $u_n(a,b)$, then $u_n(a,1)$ are called the Chebyshev polynomials of the second kind, and denoted by $U_n(a)$. We also define t_{ij} by

$$t_{ij} = \begin{cases} a & \text{if } i = j = 1, \\ 2a & \text{if } i = j \geq 2, \\ 1 & \text{if } i = j+1 \text{ or } j = i+1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Let $T^{(n)}$ be the n by n matrix whose (i,j)-entry is t_{ij}. The Chebyshev polynomials of the first kind are by definition $T_n(a) = \det T^{(n)}$. By the same argument as above, we see that the polynomials $T_n(a)$ satisfy the same recurrence formula with $U_n(a)$, i.e.

$$T_{n+1}(a) - 2aT_n(a) + T_{n-1}(a) = 0. \tag{3.5}$$

The first few polynomials are as follows. $T_0(a) = 1$, $T_1(a) = a$, $T_2(a) = 2a^2 - 1$, $T_3(a) = 4a^3 - 3a$, $T_4(a) = 8a^4 - 7a^2 + 1$. We also define $T_n(a)$ for $n < 0$ as the above recurrence formula always holds. The pairs $(T_n(a), U_n(a))$ satisfy the recurrence formula

$$\begin{cases} T_{n+1}(a) = aT_n(a) + (a^2-1)U_n(a) \\ U_{n+1}(a) = T_n(a) + aU_n(a) \end{cases}$$

This can be seen since the first few terms satisfy these equation.

Next we prepare some preliminaries and notation. Let us denote by $\mathbb{N}$ the set of nonnegative integers, and by $\mathbb{Z}$ the set of integers. We use the notation $[i, j] = \{i, i+1, \dots, j\}$ for $i, j \in \mathbb{Z}$ satisfying $i \le j$. A *partition* is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers with finite sum. Sometimes we use a notation which indicates the number of times each integer occurs as a part: $\lambda = (1^{m_1} 2^{m_2} \dots)$ means that exactly m_i of the parts of λ are equal to i. In particular, we use the notation $(r^n) = \underbrace{(r, r, \dots, r)}_{n\text{-times}}$. Also a partition $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ defined by $\lambda'_i = \#\{j : \lambda_j \ge i\}$ is called the conjugate partition of λ. The *length* $l(\lambda)$ of a partition λ is the number of non-zero terms of λ.

For a partition λ, we denote by $r(\lambda)$ (resp. $c(\lambda)$) the number of rows (resp. columns) of odd length in λ. We say also that λ is *even* (resp. *transposed-even*) if $r(\lambda) = 0$ (resp. $c(\lambda) = 0$). Let $n(\lambda) = \sum_{i \ge 1}(i-1)\lambda_i = \sum_{i \ge 1} \binom{\lambda'_i}{2}$. For each cell $x = (i, j)$ in λ, the *hook-length* of λ at x is defined to be $h(x) = \lambda_i - j + \lambda'_j - i + 1$.

For a partition λ, we put $p(\lambda) = \#\{i : \lambda_i \ge i\}$, which is a number of nodes on the main diagonal of λ and define

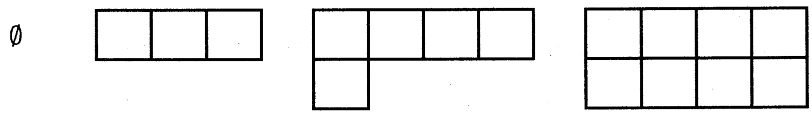
$$\alpha_j = \lambda_j - j, \quad \beta_j = \lambda'_j - j \quad \text{for } 1 \le j \le p(\lambda).$$

Then $\alpha_1 > \cdots > \alpha_{p(\lambda)} \ge 0$ and $\beta_1 > \cdots > \beta_{p(\lambda)} \ge 0$. We write $\lambda = (\alpha|\beta)$ and call this the *Frobenius notation* of λ.

We denote by $\Gamma_{r,n}$ the set of all partitions of the form $\lambda = (\beta_1 + r, \dots, \beta_p + r | \beta_1, \dots, \beta_p)$ with length $\le n$. For example,

$$\Gamma_{2,2} = \{\emptyset, (3) = (2|0), (4, 1) = (3|1), (4, 4) = (32|10)\},$$

and these partitions are depicted by the following diagrams;



If a is a nonnegative integer which doesn't coincide with any of α_i's, then let $q(\alpha, a)$ denote the number of α_i's which are bigger than a. For example, $\lambda = (5441)$ is the partition of 14 and $p(\lambda) = 3$. This partition is denoted by $\lambda = (421|310)$ in the Frobenius notation. If $\alpha = (310)$ then $q(\alpha, 2) = 1$ and $(\alpha + 1|\alpha) = (421|310)$.

Let $\lambda = (\alpha_1, \dots, \alpha_r | \beta_1, \dots, \beta_r)$ be a partition expressed in the Frobenius notation. Let a and b be nonnegative integers such that $a \ne \alpha_1, \dots, \alpha_r$ and $b \ne \beta_1, \dots, \beta_r$. There are some k and l such that $\alpha_k > a > \alpha_{k+1}$ and $\beta_l > b > \beta_{l+1}$. The partition $\lambda \uplus (a|b)$ is defined by

$$\lambda \uplus (a|b) = (\alpha_1, \dots, \alpha_k, a, \alpha_{k+1}, \alpha_r | \beta_1, \dots, \beta_l, b, \beta_{l+1}, \dots, \beta_r). \tag{3.6}$$

For example, $(421|310) \uplus (0|2) = (4210|3210)$.

A *half-partition* of length n is a non-increasing sequence $\lambda = (\lambda_1, \dots, \lambda_n)$ of non-negative half-integers $\lambda_i \in \mathbb{N} + \frac{1}{2}$. Then we can write $\lambda = \mu + (\frac{1}{2})^n$, where μ is a partition of length $\le n$. If there is no confusion, we simply write $\lambda = \mu + \frac{1}{2}$.

If λ is a partition of length $\le n$ or a half-partition of length n, then we put

$$J(\lambda) = \{\lambda_1 + n - 1, \lambda_2 + n - 2, \dots, \lambda_n\}.$$

Conversely, for a subset $J = \{j_1 < \cdots < j_n\}$ of $\mathbb{N}$ or $\mathbb{N} + \frac{1}{2}$, let $\lambda(J)$ be the partition or half-partition defined by the equations

$$\lambda_i = j_{n+1-i} - n + i.$$

We now recall Weyl's character formula. Let

$$T^{X(n)} = (T^{X(n)}_{ik})_{i=1,\dots,n} \qquad (X = A, B, C, D+, D-, D)$$

be the n-rowed matrix defined by

$$
\begin{aligned}
T_{ik}^{A(n)} &= x_i^k \quad \text{for } k \in \mathbb{N}, \\
T_{ik}^{B(n)} &= x_i^{k+1/2} - x_i^{-k-1/2} \quad \text{for } k \in \frac{1}{2}\mathbb{N}, \\
T_{ik}^{C(n)} &= x_i^{k+1} - x_i^{-k-1} \quad \text{for } k \in \mathbb{N}, \\
T_{ik}^{D^+(n)} &= x_i^k + x_i^{-k} \quad \text{for } k \in \frac{1}{2}\mathbb{N}, \\
T_{ik}^{D^-(n)} &= x_i^k - x_i^{-k} \quad \text{for } k \in \frac{1}{2}\mathbb{N},
\end{aligned} \tag{3.7}
$$

and

$$
T_{ik}^{D(n)} = \begin{cases} 1 & \text{if } k = 0 \\ x_i^k - x_i^{-k} & \text{if } k \geq 1. \end{cases}
$$

Then Weyl's character formula can be written in the following form.

Proposition 3.1. *For a partition or a half partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, we have*

$$
\begin{aligned}
\lambda_{X(n)} &= \frac{\det(T_{J(\lambda)}^{X(n)})}{\det(T_{J(\emptyset)}^{X(n)})} \quad \textit{for } X = A, B, C, \\
\lambda_{D(n)}^{\pm} &= \frac{\det(T_{J(\lambda)}^{D^+(n)}) \pm \det(T_{J(\lambda)}^{D^-(n)})}{\det(T_{J(\emptyset)}^{D(n)})}. \quad \square
\end{aligned} \tag{3.8}
$$

Furthermore, Weyl's denominator formula (or the Vandermonde determinant), gives the following explicit description of the denominator of a character given in the above proposition.

Proposition 3.2. *For each series, we have*

$$
\begin{aligned}
\det(T_{J(\emptyset)}^{A(n)}) &= \prod_{1\leq i<j\leq n} (x_j - x_i), \\
\det(T_{J(\emptyset)}^{B(n)}) &= (-1)^{\frac{n(n+1)}{2}} (x_1 \ldots x_n)^{-n+\frac{1}{2}} \prod_{i=1}^{n} (1 - x_i) \prod_{1\leq i<j\leq n} (x_j - x_i)(1 - x_i x_j), \\
\det(T_{J(\emptyset)}^{C(n)}) &= (-1)^{n(n+1)/2} (x_1 \ldots x_n)^{-n} \prod_{i=1}^{n} (1 - x_i^2) \prod_{1\leq i<j\leq n} (x_j - x_i)(1 - x_i x_j), \\
\det(T_{J(\emptyset)}^{D(n)}) &= (-1)^{n(n-1)/2} (x_1 \ldots x_n)^{-n+1} \prod_{1\leq i<j\leq n} (x_j - x_i)(1 - x_i x_j). \quad \square
\end{aligned} \tag{3.9}
$$

Now we prove a fact which is simple, but seems interesting by its corollary.

Proposition 3.3.

$$
\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} u_{k+1}(a,b) b^l s_{(k+l,l)}(x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{1}{1 - 2a x_i + b x_i^2} \tag{3.10}
$$

$$
\sum_{k=0}^{n} \sum_{l=0}^{n-k} u_{k+1}(a,b) b^l s_{(2^l 1^k)}(x_1, \ldots, x_n) = \prod_{i=1}^{n} (1 + 2a x_i + b x_i^2) \tag{3.11}
$$

Proof. First we prove the second identity. Take n-rowed matrices $T = (x_i^{j-1})$ and

$$
S = \begin{pmatrix} 1 & 2a & b & 0 & 0 & 0 & \ldots & \\ 0 & 1 & 2a & b & 0 & 0 & \ldots & \\ \vdots & & \ddots & \ddots & \ddots & & & \\ 0 & \ldots & & 1 & 2a & b & 0 & \ldots \end{pmatrix}.
$$

If $J = \{j_1 < \cdots < j_n\}$ is an index set of columns and $\lambda = \lambda(J)$ is the corresponding partition, then $\det S_J$ vanishes unless $j_n \le n+2$. It is easy to see

$$\det S_J = \begin{cases} u_{\lambda'_1-\lambda'_2} b^{\lambda'_2} & \text{if } l(\lambda') \le 2 \\ 0 & \text{otherwise,} \end{cases}$$

where λ' is the conjugate partition λ. On the other hand, we have

$$S^tT = \operatorname{diag}(1+2ax_1+x_1^2, \ldots, 1+2ax_n+x_n^2)(x_i^{j-1})_{1\le i,j\le n}.$$

This proves the second identity. The first identity is derived from the second one. □

One remarkable fact is that we can prove the Fourier expansion formula of Jacobi's elliptic Theta-functions as a corollary of the above proposition. We use the following notation.

$$(3.12)\qquad \begin{aligned} (a;q)_\infty &= \prod_{n=1}^{\infty}(1-aq^{n-1}), \\ (a;q)_n &= \frac{(a;q)_\infty}{(aq^n;q)_\infty} = \prod_{k=1}^{n}(1-aq^{k-1}). \end{aligned}$$

The symbols $(a;q)_\infty$ and $(a;q)_n$ are abbreviated to $(a)_\infty$ and $(a)_n$ respectively when the second variable is assumed to be q.

Lemma 3.1. *Let n be a nonnegative integer.*

$$(3.13)\qquad \sum_{k=0}^{\infty}\frac{q^{k(k+n)}}{(q)_k(q)_{k+n}} = \frac{1}{(q)_\infty}$$

$$(3.14)\qquad \sum_{k=0}^{\infty}\frac{q^{k(k+n)}}{(q)_k(q)_{k+n+1}} = \frac{1+q^{n+1}}{(q)_\infty}$$

Proof. Note that $\frac{1}{(q)_\infty}$ is the generating function of all partitions. The first identity can be shown by considering a rectangle contained in a partition. Let λ be a partition and let r be the maximum integer such that the rectangle of shape $r\times(r+n)$ is contained in λ. We denote this r by $r_n(\lambda)$. Then the generating function of all partitions such that $r_n(\lambda)=k$ is given by $\frac{q^{k(k+n)}}{(q)_k(q)_{k+n}}$. Thus, by summing over all k, we obtain the generating function of all partitions. The second identity is derived from the first one as follows.

$$\begin{aligned} \sum_{k=0}^{\infty}\frac{q^{k(k+n)}}{(q)_k(q)_{k+n+1}} &= \sum_{k=0}^{\infty}\frac{q^{k(k+n+1)}}{(q)_k(q)_{k+n+1}} + \sum_{k=1}^{\infty}\frac{q^{k(k+n)}(1-q^k)}{(q)_k(q)_{k+n+1}} \\ &= \frac{1}{(q)_\infty} + q^{n+1}\sum_{k=1}^{\infty}\frac{q^{(k-1)(k+n+1)}}{(q)_{k-1}(q)_{k+n+1}} \\ &= \frac{1+q^{n+1}}{(q)_\infty}. \end{aligned}$$

This proves the lemma. □

Corollary 3.1. *Let $q = e^{i\pi\tau}$ ($\Im\tau > 0$).*

(3.15)
$$\vartheta_1(v,\tau) = 2\sum_{k=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin(2n+1)\pi v = 2q^{\frac{1}{4}}Q_0\sin\pi v\prod_{n=1}^{\infty}\left(1-2q^{2n}\cos 2\pi v + q^{4n}\right)$$

(3.16)
$$\vartheta_2(v,\tau) = 2\sum_{k=0}^{\infty} q^{(n+\frac{1}{2})^2}\cos(2n+1)\pi v = 2q^{\frac{1}{4}}Q_0\sin\pi v\prod_{n=1}^{\infty}\left(1+2q^{2n}\cos 2\pi v + q^{4n}\right)$$

(3.17)
$$\vartheta_3(v,\tau) = 1+2\sum_{k=1}^{\infty} q^{n^2}\cos 2n\pi v = Q_0\prod_{n=1}^{\infty}\left(1+2q^{2n-1}\cos 2\pi v + q^{4n-1}\right)$$

(3.18)
$$\vartheta_4(v,\tau) = 1+2\sum_{k=1}^{\infty}(-1)^n q^{n^2}\cos 2n\pi v = Q_0\prod_{n=1}^{\infty}\left(1-2q^{2n-1}\cos 2\pi v + q^{4n-1}\right)$$

Proof. In the second identity of Proposition 3.1 we put $b = 1$ and $n \to \infty$, then we obtain

$$\sum_{n=0}^{\infty} U_{k+1}(a)\sum_{k=0}^{\infty} s_{(2^k1^n)}(x) = \prod_{n=1}^{\infty}\left(1+2ax_n + x_n^2\right).$$

Here $s_{(2^k1^n)}(x)$ stands for the infinite variable Schur function $s_{(2^k1^n)}(x_1, x_2, \dots)$. Substituting $a = \cos 2\pi v$ into the above identity yields

$$\sum_{n=0}^{\infty}\sin 2(n+1)\pi v\sum_{k=0}^{\infty} s_{(2^k1^n)}(x) = \sin 2\pi v\prod_{n=1}^{\infty}\left(1+2x_n\cos 2\pi v + x_n^2\right) \tag{3.19}$$

because of $U_{k+1}(\cos 2\pi v) = \frac{\sin 2(n+1)\pi v}{\sin 2\pi v}$. Further we specialize $x_n = q^{2n}$ $(n = 1, 2, \dots)$ in this identity, then we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\sin 2(n+1)\pi v\sum_{k=0}^{\infty} q^{2(2k+n)} s_{(2^k1^n)}(1, q^2, q^4, \dots)\\ = \sin 2\pi v\prod_{n=1}^{\infty}\left(1+q^{2n}\cos 2(n+1)\pi v + q^{4n}\right)\end{aligned}$$

Recall that by the specialization $x_n = q^{n-1}$ of the Schur function $s_\lambda(x)$ with $\lambda = (2^k1^n)$ we have

$$s_{(2^k1^n)}(1, q, q^2, \dots) = \frac{q^{n(\lambda)}}{\prod_{\lambda\in\lambda}(1-q^{h(x)})} = \frac{q^{\binom{k}{2}+\binom{k+n}{2}}}{(q)_k(q)_{k+n+1}} \tag{3.20}$$

where $n(\lambda) = \sum_{i=1}^{\infty}(i-1)\lambda_i$ and $h(x) = \lambda_i + \lambda'_j - i - j + 1$ for $x = (i,j) \in \lambda$. (See [Ma], p.44 Ex.1.) It follows that

$$\sum_{k=0}^{\infty} q^{2k+n} s_{2^k1^n}(1, q, q^2, \dots) = q^{\binom{n+1}{2}}(1-q^{n+1})\sum_{k=0}^{\infty}\frac{q^{k(k+n+1)}}{(q)_k(q)_{k+n+1}} = \frac{q^{\binom{n+1}{2}}(1-q^{n+1})}{(q)_\infty}$$

Combining the above identities, we obtain

$$\sum_{n=0}^{\infty}\sin 2(n+1)\pi v\, q^{n(n+1)}(1-q^{2n+2}) = \sin 2\pi v\prod_{n=1}^{\infty}\left(1+q^{2n}\cos 2\pi v + q^{4n}\right)$$

The left-hand of this identity is equal to

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sin 2(n+1)\pi v\, q^{n(n+1)} - \sum_{n=0}^{\infty} \sin 2(n+1)\pi v\, q^{(n+1)(n+2)} \\
&= \sin 2\pi v + \sum_{n=1}^{\infty} q^{n(n+1)} \{\sin 2(n+1)\pi v - \sin 2n\pi v\} \\
&= 2 \sin \pi v \cos \pi v + 2 \sum_{n=1}^{\infty} q^{n(n+1)} \cos(2n+1)\pi v \sin \pi v.
\end{aligned}$$

This proves the identity we desire. The identity on ϑ_1 can be proved by a parallel way with substituting $a = -\cos 2\pi v$. Next we prove the identity on ϑ_3. We substitute $x_n = q^{2n-1}$ $(n = 1, 2, \dots)$ into (3.21), then we obtain

$$\sum_{n=0}^{\infty} \sin 2(n+1)\pi v \sum_{k=0}^{\infty} q^{2k+n} s_{(2^k 1^n)}(1, q^2, q^4, \dots) = \sin 2\pi v \prod_{n=1}^{\infty} \left(1 + q^{2n-1} \cos 2\pi v + q^{4n-2}\right) \tag{3.21}$$

By the similar reasoning as above we obtain

$$\sum_{k=0}^{\infty} q^k s_{(2^k 1^n)}(1, q, q^2, \dots) = q^{\binom{n}{2}} (1 - q^{n+1}) \sum_{k=0}^{\infty} \frac{q^{k(k+n)}}{(q)_k (q)_{k(k+n+1)}} = \frac{q^{\binom{n}{2}} (1 - q^{2n+2})}{(q)_\infty} \tag{3.22}$$

Combining (3.22) and (3.23), we obtain

$$\sum_{n=0}^{\infty} q^{n^2} (1 - q^{4n+4}) \sin 2(n+1)\pi v = Q_0 \sin 2\pi v \prod_{n=1}^{\infty} \left(1 + q^{2n-1} \cos 2\pi v + q^{4n-2}\right)$$

The left-side of this identity is equal to

$$\begin{aligned}
&\sum_{n=0}^{\infty} q^{n^2} \sin 2(n+1)\pi v - \sum_{n=0}^{\infty} q^{(n+2)^2} \sin 2(n+1)\pi v \\
&= \sin 2\pi v + q \sin 4\pi v + \sum_{n=2}^{\infty} q^{n^2} \{\sin 2(n+1)\pi v - \sin 2(n-1)\pi v\} \\
&= \sin 2\pi v + 2q \sin 2\pi v \sin 2\pi v + \sum_{n=2}^{\infty} q^{n^2} \{\cos 2\pi v \sin 2\pi v\}
\end{aligned}$$

This proves the identity. The identity on ϑ_4 is also obtained by a parallel reasoning by substituting $a = -\cos 2\pi v$. This completes the proof. □

The following formulas are B, C, D types of Proposition 3.3.

Proposition 3.4. *Let $n \in \mathbb{N}$ and let $X = B, C, D\pm$. Then*

$$\sum_{k=0}^{\infty} U_{k+1}(a) \sum_{l=0}^{\infty} \left((m+1)^l m^k (m-1)^{n-k-l}\right)_{X(n)} = (m^n)_{X(n)} \prod_{i=1}^{n} (x_i + 2a + x_i^{-1}). \tag{3.23}$$

Here $m \in \frac{1}{2}\mathbb{N}$ if $X = B, D\pm$, and $m \in \mathbb{N}$ if $X = C$.

Proof. Let $T_n^{\pm}(\alpha) = \left(T_{ik}^{\pm}(\alpha)\right)$ be the n-rowed matrix whose entries are given by

$$T_{ik}^{\pm}(\alpha) = x_i^{k+\alpha} \pm x_i^{-k-\alpha} \qquad \text{for } k \in \frac{1}{2}\mathbb{N}.$$

Let S_n be the n-rowed matrix defined by

$$S_n = \begin{pmatrix} 1 & 2a & 1 & 0 & 0 & 0 & \dots & \\ 0 & 1 & 2a & 1 & 0 & 0 & \dots & \\ \vdots & & \ddots & \ddots & \ddots & & & \\ 0 & \dots & & 1 & 2a & 1 & 0 & \dots \end{pmatrix}.$$

Since

$$x_i^{k-1+m+\alpha} \pm x_i^{-k+1-m-\alpha} + 2a\left(x_i^{k+m+\alpha} \pm x_i^{-k-m-\alpha}\right) + x_i^{k+1+m+\alpha} \pm x_i^{-k-1-m-\alpha}$$
$$= \left(x_i + 2a + x_i^{-1}\right)\left(x_i^{k+m+\alpha} \pm x_i^{-k-m-\alpha}\right),$$

we have

$$\det\left(T_n^{\pm}(m-1+\alpha)^t S_n\right) = \prod_{i=1}^{n}\left(x_i + 2a + x_i^{-1}\right)\det\left(x_i^{j+m+\alpha} \pm x_i^{-j-m-\alpha}\right)_{1\le i,j\le n}$$

On the other hand, one applies Binet-Cauchy formula to the left-hand side of the above identity to derive

$$\sum_{0\le j_1<j_2<\cdots<j_n\le n+1} \det S_{n\{j_1,\dots,j_n\}} \det T^{\pm}(\alpha)_{n\{j_1+m-1,\dots,j_n+m-1\}}$$
$$= \prod_{i=1}^{n}\left(x_i + 2a + x_i^{-1}\right)\det\left(T_n^{\pm}(\alpha)_{\{m,m+1,\dots,m+n-1\}}\right).$$

Let

$$\psi_{\alpha,n}^{\pm}(\lambda) = \det\left(T_n^{\pm}(\alpha)_{J(\lambda)}\right).$$

Then the above identity means

$$\sum_{k=0}^{n}\sum_{l=0}^{n-k} U_{k+1}(a)\psi_{\alpha,n}^{\pm}((m+1)^l m^k (m-1)^{n-k-l}) = \prod_{i=1}^{n}\left(x_i + 2a + x_i^{-1}\right)\psi_{\alpha,n}^{\pm}(m^k)$$

Put $\alpha = \frac{1}{2}$ (resp. $\alpha = 1$ or $\alpha = 0$) to obtain the formulas for $X = B$ (resp. $X = C$ or $X = D\pm$). The identities we desire are easily derived from this identity and the details are left to the reader. □

Proposition 3.5.

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} U_{k+1}(a)t^{k+2l}\left((k+m)(l+m)m^{n-2}\right)_{X(n)}$$
$$= \prod_{i=1}^{n}\frac{1}{(x_i^{-1} - 2at + t^2 x_i)(x_i - 2at + t^2 x_i^{-1})}$$
$$\times \sum_{k=0}^{n}\sum_{l=0}^{n-k}(-1)^{k+2l}t^{k+2l}U_{k+1}(a)\left((m+1)^{n-k}m^k(m-1)^l\right)_{X(n)}$$

Proof. Let $T_n^{\pm}(\alpha)$ be as before and let S_n' and S_n'' be the n-rowed matrices defined by

$$S_n' = \begin{pmatrix} 1 & 2at^{-1} & t^{-2} & 0 & 0 & 0 & \dots & \\ 0 & 1 & 2at^{-1} & t^{-2} & 0 & 0 & \dots & \\ \vdots & & \ddots & \ddots & \ddots & & & \\ 0 & \dots & & 1 & 2at^{-1} & t^{-2} & 0 & \dots \end{pmatrix}$$

$$S_n'' = \begin{pmatrix} U_1(a) & U_2(a)t & U_3(a)t^2 & U_4(a)t^4 & \dots \\ 0 & U_1(a)t & U_2(a)t^2 & U_3(a)t^4 & \dots \\ \vdots & \ddots & \ddots & & \\ 0 & \dots & 0 & U_1(a)t^{n-1} & \dots \end{pmatrix}.$$

The (i,j)-entry of $T_n^{\pm}(m+\alpha)^t S_n''$ is equal to

$$\begin{aligned}
&\sum_{k=0}^{\infty} U_{k+1}(a) t^{k+i-1}(x_j^{k+i-1+m+\alpha} \pm x_j^{-k-i+1--m\alpha}) \\
&= \frac{t^{i-1}x_j^{i-1+m+\alpha}}{1-2atx_j+t^2x_j^2} \pm \frac{t^{i-1}x_j^{-i+1-m-\alpha}}{1-2atx_j^{-1}+t^2x_j^{-2}} \\
&= t^{i-1}\frac{(x_j^{i-1+m+\alpha} \pm x_j^{-i+1-m-\alpha}) - 2at(x_j^{i-2+m+\alpha} \pm x_j^{-i+2-m-\alpha}) + t^2(x_j^{i-3+m+\alpha} \pm x_j^{-i+3-m-\alpha})}{(1-2atx_j+t^2x_j^2)(1-2atx_j^{-1}+t^2x_j^{-2})} \\
&= t^{i-1}\frac{(T_n^{\pm}(\alpha-2)S_n')_{ij}}{(1-2atx_j+t^2x_j^2)(1-2atx_j^{-1}+t^2x_j^{-2})}
\end{aligned}$$

for $1 \le i,j \le n$. Let $\psi_{\alpha,n}^{\pm}(\lambda)$ be as in the preceding proposition. We use Binet-Cauchy formula to obtain

$$\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} U_{k+1}(a)\psi_{\alpha,n}^{\pm}((k+l+m)(k+l+m)m^{n-2}) \\
&= \frac{1}{(1-2atx_j+t^2x_j^2)(1-2atx_j^{-1}+t^2x_j^{-2})} \\
&\times \sum_{k=0}^{n}\sum_{l=0}^{n-k}(-1)^{k+2l}t^{k+2l}U_{k+1}(a)\psi_{\alpha,n}^{\pm}(m+1)^{n-k-l}m^k(m-1)^l).
\end{aligned}$$

4. Littlewood Type Formulas

In this section we consider Littlewood type formulas concerning the Schur polynomials. These results can be extended to the characters of other classical groups, but we don't have enough space to state them.

The following lemma is the key lemma to evaluate the pfaffian we treat.

Lemma 4.1. *Let m be a positive integer and put*

$$Q_m(x,y) = \frac{(x^m-y^m)^2}{x-y}\frac{(1-t^m x^m y^m)^2}{1-txy}. \tag{4.1}$$

Then

$$\operatorname{pf}\left[Q_m(x_i,x_j)\right]_{1\le i,j\le 2m} = \prod_{1\le i<j\le 2m}(x_i-x_j)(1-tx_ix_j). \tag{4.2}$$

We fix $T = (x_i^{4m+d-2-j})_{1\le i\le 2m, 0\le j\le 4m+d-2}$ in this section.

Let m be a positive integer and let $B = (\beta_{kl})_{0\le k,l\le m-1}$ be an skew-symmetric matrix of size m in the ordinary means. Set $\mathbf{b}_i$ to be the i-th row vector of B for $0 \le i \le m-1$. The matrix B is said to be *(row-)symmetrically proportional* if the $(m-1-k)$-th row is proportional to the k-th. That is to say, there is some c_k such that $\mathbf{b}_{m-1-k} = c_k\mathbf{b}_k$ or $\mathbf{b}_k = c_k\mathbf{b}_{m-1-k}$ for each $0 \le k \le [\frac{m}{2}]-1$. Further B is called *row-symmetric* if the $\mathbf{b}_{m-1-k} = \mathbf{b}_k$ for $0 \le i \le [\frac{m}{2}]-1$, and B is called *row-antisymmetric* if the $\mathbf{b}_{m-1-k} = -\mathbf{b}_k$ for $0 \le k \le [\frac{m+1}{2}]-1$. This notion has importance since it makes us possible to find all the subpfaffians $\operatorname{pf}(B_{j_1\dots j_m})$ of B. From now on we assume that B is always supposed to be skew-symmetric matrix.

Let $P(x) = a_0 + a_1x + \cdots + a_dx^d$ be a polynomial of degree d. $P(x)$ is said to be *symmetric* if $a_i = a_{n-i}$ for $0 \le i \le [\frac{d}{2}]$, and $P(x)$ is said to be *antisymmetric* if $a_i = -a_{n-i}$ for $0 \le i \le [\frac{d+1}{2}]$,

Lemma 4.2. *Let $P(x)$ be a polynomial of degree d. Let $B = (\beta_{kl})_{0\le k,l\le 4m+d-2}$ be the skew-symmetric matrix of size $(4m+d-1)$ which satisfy*

$$\sum_{0\le k<l\le 4m+d-2} \beta_{kl} \begin{vmatrix} x^k & x^l \\ y^k & y^l \end{vmatrix} = -P(x)P(y)Q(x,y). \tag{4.3}$$

The matrix B becomes (row-)symmetrically proportional for all m if and only if $P(x)$ is symmetric or antisymmetric. Further, if the polynomial $P(x)$ is symmetric then B becomes row-symmetric, on the other hand, if $P(x)$ is antisymmetric then B becomes row-antisymmetric.

From now we apply Theorem 1.1 to this T and B given by (4.3). Basically it is possible to find some sort of formula for each skew-symmetric matrix of the form (4.3) if it is row-symmetric or row-antisymmetric. Here we investigate each formula for small d. When $d = 0$, we obtain the following formula (4.4) from this argument. If $d = 1$ and $P(x)$ is antisymmetric, we obtain the following formula (4.5). It is easy to see that the case of $d = 1$ and $P(x)$ being symmetric reduces to this case. If $d = 2$ and $P(x)$ is antisymmetric, then we obtain the formula (4.6).

$$\sum_{\lambda=(\alpha|\alpha+1)} (-1)^{\frac{|\lambda|}{2}} s_\lambda(x_1,\dots,x_m) = \prod_{1\le i<j\le m} (1-x_ix_j), \tag{4.4}$$

$$\sum_{\lambda=(\alpha|\alpha)} (-1)^{\frac{|\lambda|}{2}+p(\lambda)} s_\lambda(x_1,\dots,x_m) = \prod_{i=1}^{m}(1-x_i) \prod_{1\le i<j\le m} (1-x_ix_j), \tag{4.5}$$

$$\sum_{\lambda=(\alpha+1|\alpha)} (-1)^{\frac{|\lambda|}{2}} s_\lambda(x_1,\dots,x_m) = \prod_{1\le i\le j\le m} (1-x_ix_j). \tag{4.6}$$

These formulas are usually called the Littlewood formulas. We obtain further identities of this type by considering the polynomials $P(x)$ of higher degree. If we assume $d = 2$ and $P(x)$ is symmetric, then we obtain the following theorem.

Theorem 4.1. *Let m be a positive integer. Then*

$$\begin{aligned} &\sum_{\lambda=(\alpha+1|\alpha)} (-1)^{\frac{|\lambda|}{2}+p(\lambda)} s_\lambda(x_1,\dots,x_m) \\ &+2\sum_{k=1}^{m} T_k(a) \sum_{\substack{\lambda=(\alpha+1|\alpha)\\ \alpha\not\ni k-1}} (-1)^{\frac{|\lambda|}{2}+q(\lambda,k-1)} s_{\lambda\Cup(0|k-1)}(x_1,\dots,x_m) \\ &= \prod_{i=1}^{m}(1+2ax_i+x_i^2) \prod_{1\le i<j\le m} (1-x_ix_j). \end{aligned} \tag{4.7}$$

If we put $x_i = q^{2i}$ in this formula and we use the q-expansion formula of Jacobi's theta function ϑ_3, we obtain the following corollary.

Corollary 4.1.

$$\sum_{\lambda=(\alpha+1|\alpha)} (-1)^{\frac{|\lambda|}{2}+p(\lambda)} q^{\frac{|\lambda|}{2}+n(\lambda)} \prod_{x\in\lambda} \frac{1}{1-q^{h(x)}} = \frac{\prod_{r=2}^{\infty}(1-q^r)^{[\frac{r}{2}]}}{\prod_{r=1}^{\infty}(1-q^r)}. \tag{4.8}$$

Let m be a nonnegative integer.

$$\sum_{\lambda=(\alpha+1|\alpha)} (-1)^{\frac{|\lambda|}{2}+q(\alpha,m)} q^{\frac{|\lambda|}{2}+n(\lambda\Cup(0|m))} \prod_{x\in\lambda\Cup(0|m)} \frac{1}{1-q^{h(x)}} = q^{\frac{m(m+1)}{2}} \frac{\prod_{r=2}^{\infty}(1-q^r)^{[\frac{r}{2}]}}{\prod_{r=1}^{\infty}(1-q^r)}. \tag{4.9}$$

If $d = 3$ and $P(x)$ is antisymmetric, we obtain the following theorem. The case of $d = 3$ and $P(x)$ being symmetric essentially reduces to this case.

Theorem 4.2. *Let m be a positive integer. Then*

$$
\begin{aligned}
&\sum_{\lambda=(\alpha+2|\alpha)} (-1)^{\frac{|\lambda|-p(\lambda)}{2}} s_\lambda(x_1,\dots,x_m) \\
&+\sum_{k=1}^{m}\{T_k(a)+(a-1)U_k(a)\} \sum_{\substack{\lambda=(\alpha+2|\alpha)\\ \alpha\not\ni k-1}} (-1)^{\frac{|\lambda|+p(\lambda)}{2}+q(\lambda,k-1)} \\
&\qquad\times\left\{s_{\lambda\uplus(0|k-1)}(x_1,\dots,x_m)-s_{\lambda\uplus(1|k-1)}(x_1,\dots,x_m)\right\} \\
&=\prod_{i=1}^{m}(1+2ax_i+x_i^2)(1-x_i)\prod_{1\le i<j\le m}(1-x_ix_j).
\end{aligned} \tag{4.10}
$$

If $d=4$ and $P(x)$ is antisymmetric, we obtain the following theorem.

Theorem 4.3. *Let m be a positive integer. Then*

$$
\begin{aligned}
&\sum_{\lambda=(\alpha+3|\alpha)} (-1)^{\frac{|\lambda|}{2}+p(\lambda)} s_\lambda(x_1,\dots,x_m) \\
&+\sum_{k=1}^{m}U_{k+1}(a) \sum_{\substack{\lambda=(\alpha+3|\alpha)\\ \alpha\not\ni k-1}} (-1)^{\frac{|\lambda|}{2}+q(\lambda,k-1)} \\
&\qquad\times\left\{s_{\lambda\uplus(0|k-1)}(x_1,\dots,x_m)-s_{\lambda\uplus(2|k-1)}(x_1,\dots,x_m)\right\} \\
&=\prod_{i=1}^{m}(1+2ax_i+x_i^2)\prod_{1\le i\le j\le m}(1-x_ix_j).
\end{aligned} \tag{4.11}
$$

Appendix

Summation Formula for Columns

We now review basic terminology on lattice path method and fix notation. Let $D=(V,E)$ be an acyclic digraph without multiple edges. Further we assume that there are only finitely many paths between any two vertices. Let $\mathcal{P}(u,v)$ denote the set of all directed paths from u to v in D. Fix a positive integer r. An *r-vertex* is an r-tuple $(u_1,u_2,\dots,u_r)$ of vertices of D. Given any pair of r-vertices $\boldsymbol{u}=(u_1,u_2,\dots,u_r)$ and $\boldsymbol{v}=(v_1,v_2,\dots,v_r)$, an *$r$-path* from $\boldsymbol{u}$ to $\boldsymbol{v}$ is an r-tuple $\boldsymbol{P}=(P_1,P_2,\dots,P_r)$ with $P_i\in\mathcal{P}(u_i,v_i)$. Let $\mathcal{P}(\boldsymbol{u},\boldsymbol{v})$ denote the set of all r-paths from $\boldsymbol{u}$ to $\boldsymbol{v}$. Two directed paths P and Q will be said to intersect if they share a common vertex. An r-path $\boldsymbol{P}$ is said to be nonintersecting if P_i and P_j are nonintersecting for any $i\neq j$. Let $\mathcal{P}_0(\boldsymbol{u},\boldsymbol{v})$ denote the subset of $\mathcal{P}(\boldsymbol{u},\boldsymbol{v})$ which consists of all nonintersecting r-paths.

We fix a weight-function w which assigns values in a fixed commutative ring R to each edge of D. Set the weight of a path P to be the product of the weights of its edges and denote it by $w(P)$. If u and v are any pair of vertices in D, define

$$h(u,v)=\sum_{P\in\mathcal{P}(u,v)} w(P).$$

The weight of an r-path is defined to be the product of the weights of its components. The sum of the weights of r-paths in $\mathcal{P}(\boldsymbol{u},\boldsymbol{v})$ (resp. $\mathcal{P}_0(\boldsymbol{u},\boldsymbol{v})$) is denoted by $\mathbf{P}(\boldsymbol{u},\boldsymbol{v})$ (resp. $\mathbf{N}(\boldsymbol{u},\boldsymbol{v})$).

Definition A.1. *If I and J are ordered sets of vertices of D, then I is said to be D-compatible with J if, whenever $u<u'$ in I and $v>v'$ in J, every path $P\in\mathcal{P}(u,v)$ intersects every path $Q\in\mathcal{P}(u',v')$.*

The following lemma is from [GV], but we give a proof here to make this paper self-contained.

Lemma A.1. *(Lindström-Gessel-Viennot) Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two r-vertices in an acyclic digraph D. If $\boldsymbol{u}$ is D-compatible with $\boldsymbol{v}$, then*

$$\mathbf{N}(\boldsymbol{u},\boldsymbol{v})=\det[h(u_i,v_j)]_{1\le i,j\le r}. \tag{A.1}$$

Proof: For $\pi \in \mathfrak{S}_r$, let $\pi(\boldsymbol{v})$ denote the r-vertex $(v_{\pi(1)}, v_{\pi(2)}, \cdots, v_{\pi(r)})$. Then

$$\det[h(u_i, v_j)]_{1\le i,j\le r} = \sum_{\pi\in\mathfrak{S}_r} \operatorname{sgn}(\pi) h(u_1, v_{\pi(1)}) h(u_2, v_{\pi(2)}) \dots h(u_r, v_{\pi(r)}). \tag{A.2}$$

Put

$$\Pi = \{(\pi, \boldsymbol{P}) : \pi \in \mathfrak{S}_r \text{ and } \boldsymbol{P} \in \mathcal{P}(\boldsymbol{u}, \pi(\boldsymbol{v}))\},$$
$$\Pi_0 = \{(\pi, \boldsymbol{P}) : \pi \in \mathfrak{S}_r \text{ and } \boldsymbol{P} \in \mathcal{P}_0(\boldsymbol{u}, \pi(\boldsymbol{v}))\}.$$

Then the right-hand side of (A.2) is a generating function of the set Π of configurations $(\pi, \boldsymbol{P})$ with the weight $w(\pi, \boldsymbol{P}) = \operatorname{sgn}(\pi) w(\boldsymbol{P})$. Now we describe an involution on the set $\Pi \setminus \Pi_0$ which reverse the sign of the associated weight. First fix an arbitrary total order on V. Let $C = (\pi, \boldsymbol{P}) \in \Pi \setminus \Pi_0$. Among all vertices that occurs as intersecting points, let v denote the least vertex with respect to the fixed order. Among paths that pass through v, assume that P_i and P_j are the two whose indices i and j are smallest. Let $P_i(\to v)$ (resp. $P_i(v \to)$) denote the subpath of P_i from u_i to v (resp. from v to $v_{\pi(i)}$). Set $C' = (\pi', \boldsymbol{P}')$ to be the configuration in which $P'_k = P_k$ for $k \neq i, j$,

$$P'_i = P_i(\to v) P_j(v \to), \qquad P'_j = P_j(\to v) P_i(v \to),$$

and $\pi' = \pi \circ (i, j)$. It is easy to see that $C' \in \Pi$ and $w(C') = -w(C)$. Thus $C \mapsto C'$ defines a sign reversing involution and, by this involution, one may cancel all of the terms $\{w(C) : C \in \Pi \setminus \Pi_0\}$ and only the terms $\{w(C) : C \in \Pi_0\}$ remains. Since $\boldsymbol{u}$ is D-compatible with $\boldsymbol{v}$, the configurations $C \in \Pi_0$ occur only when $\pi = \mathrm{id}$, and are counted with the weight 1. This proves the lemma. $\square$

Let I be a finite or countablly infinite totally ordered subset of V. Let I^r be the set of all r-vertices $\boldsymbol{v} = (v_1, v_2, \dots, v_r)$ with $v_i \in I$ for $1 \le i \le r$, and let I_r be the set of all r-vertices $\boldsymbol{v} = (v_1, v_2, \dots, v_r) \in I^r$ such that $v_1 < v_2 < \cdots < v_r$ with respect to the fixed total order on I. Let β_{vw} be an element of the commutative ring R for $(v, w) \in I_2$. We write the assembly of the elements as $B = (\beta_{vw})_{(v,w)\in I_2}$ and regard it as an upper triangular matrix of finite or infinite degree indexed by the totally ordered set I. This upper triangular matrix defines an antisymmetric matrix by the unique way, and we express this antisymmetric matrix by the same symbol B. Suppose r is even. Define the associated generating function of the set of nonintersecting r-paths from $\boldsymbol{u}$ to I weighted by the subpfaffians of B to be

$$Q_I(\boldsymbol{u}; B) = \sum_{\boldsymbol{v}\in I^r} \operatorname{pf}(B_{\boldsymbol{v}}) \mathbf{N}(\boldsymbol{u}, \boldsymbol{v}). \tag{A.3}$$

The difference of our definition from the original one by Stembridge is this whether weighting antisymmetric matrix B is putting on or not. Notice that, since $\boldsymbol{u}$ is D-compatible with I, $\mathbf{N}(\boldsymbol{u}, (v_1, \dots, v_r)) = 0$ unless $v_1 < v_2 < \cdots < v_r$, and this implies that, in the above definition, the sum extends to all r-vertices $\boldsymbol{v} \in I_r$. In particular, if $r = 2$, assuming $\boldsymbol{u} = (u_1, u_2)$ is D-compatible with I, then, we have

$$Q_I(\boldsymbol{u}; B) = \sum_{(v_1,v_2)\in I_2} \beta_{v_1 v_2} \begin{vmatrix} h(u_1, v_1) & h(u_1, v_2) \\ h(u_2, v_1) & h(u_2, v_2) \end{vmatrix}. \tag{A.4}$$

The following theorem is an extension of Theorem 3.1 in [Ste]. Here we give a proof by the lattice path method exploited in it. Indeed, the proof we give here almost follows it except some minor modifications by the addition of B, but one may see that this extension gives us a strong tool.

Theorem A.1. *Let r be an even integer. Let $\boldsymbol{u} = (u_1, u_2, \dots, u_r)$ be an r-vertex and I be a totally ordered set of vertices such that $\boldsymbol{u}$ is D-compatible with I. Let $B = (\beta_{k\ell})_{(k,\ell)\in I^2}$ be an antisymmetric matrix indexed by I whose entries are in R. Then*

$$Q_I(\boldsymbol{u}; B) = \operatorname{pf}\left[Q_I(u_i, u_j; B)\right]_{1\le i<j\le r}. \tag{A.5}$$

Proof: Put $r = 2s$. We may interpret the right-hand side of

$$\begin{aligned}\mathrm{pf}\,[Q_I(u_i, u_j; B)] &= \sum_{\sigma\in\mathfrak{F}_r} \operatorname{sgn}\sigma \prod_{i=1}^{s} Q_I(u_{\sigma(2i-1)}, u_{\sigma(2i)}; B) \\ &= \sum_{\sigma\in\mathfrak{F}_r} \sum_{\substack{\boldsymbol{v}=(v_1,\dots,v_r)\in I^r \\ v_{2i-1}<v_{2i}}} \operatorname{sgn}\sigma \prod_{k=1}^{s} \beta_{v_{2k-1}v_{2k}} \prod_{i=1}^{s} \mathbf{N}((u_{\sigma(2i-1)}, u_{\sigma(2i)}), (v_{2i-1}, v_{2i}))\end{aligned} \tag{A.6}$$

as a generating function of the set

$$\Sigma = \left\{ C = (\sigma, \boldsymbol{v}, \boldsymbol{P}) \,\middle|\, \begin{array}{l} \sigma \in \mathfrak{F}_r,\ \boldsymbol{v} = (v_1, v_2, \dots, v_r) \in I^r, \\ \boldsymbol{P} = (P_1, P_2, \dots, P_r) \text{ with } P_k \in \mathcal{P}(u_k, v_k) \text{ for } 1 \le k \le r, \\ P_{\sigma(2i-1)} \text{ and } P_{\sigma(2i)} \text{ must not intersect for } 1 \le i \le s. \end{array} \right\},$$

where the weight assigned to $C = (\sigma, \boldsymbol{v}, \boldsymbol{P})$ is $\operatorname{sgn}\sigma \prod_{k=1}^{s} \beta_{v_{2k-1}v_{2k}} w(\boldsymbol{P})$. Let

$$\Sigma_0 = \{C = (\sigma, \boldsymbol{v}, \boldsymbol{P}) \in \Sigma : \text{the } r\text{-path } \boldsymbol{P} \text{ is nonintersecting.}\}.$$

We shall show that we can define a sign reversing involution on $\Sigma \setminus \Sigma_0$. Fix an arbitrary total order on V which is consistent with the edges of D. That is to say, if there is an edge directed from u to v, then u precedes v in the total order. Let $C = (\pi, \boldsymbol{P}) \in \Sigma \setminus \Sigma_0$. Among all vertices that occurs as intersecting points, let v denote the the vertex which precedes all other points of intersections with respect to the fixed order. Among paths that pass through v, assume that P_i and P_j are the two whose indices i and j are smallest. Define a new r-path $\boldsymbol{P}' = (P'_1, P'_2, \dots, P'_r)$ with $P'_i = P_i(\to v)P_j(v \to)$, $P'_j = P_j(\to v)P_i(v \to)$ and $P'_k = P_k$ for $k \ne i, j$. Let H be the subgroup of $\mathfrak{S}_r$ generated by the elements $(2k-1, 2k)$ for $1 \le k \le s$ and $(2k-1, 2k+1)(2k, 2k+2)$ for $1 \le k < s$. Then the orbit of $(ij) \circ \sigma$ by H in $\mathfrak{S}_r$ has a unique intersection element with $\mathfrak{F}_r$. Set $\sigma' \in \mathfrak{S}_r(ij) \circ \sigma \cap \mathfrak{F}_r$ to be this unique element. We shall show that $C' = (\sigma', \boldsymbol{v}, \boldsymbol{P}') \in \Sigma$. It suffices to show that the paths $P'_{\sigma'(2k-1)}$ and $P'_{\sigma'(2k)}$ do not intersect for $1 \le k \le s$. The case we need to consider is that either of $\sigma'(2k-1)$ or $\sigma'(2k)$ equals i or j. Without loss of generality, we suppose that $\sigma'(2k-1) = i$ and $\sigma'(2k) \ne j$. Then we have $\sigma(2k-1) = j$ and $\sigma(2k) = \sigma'(2k)$. If the path $P_{\sigma'(2k)}$ had intersected $P'_{\sigma'(2k-1)}$, then, from the minimality of v, there would be no intersection points on the subpath $P_i(\to v)$, and this would imply that $P_{\sigma'(2k)}$ would intersect $P_j(v \to)$. But this is a contradiction to the fact that the paths $P_{\sigma(2k-1)}$ and $P_{\sigma(2k)}$ must not intersect. Thus we have shown that $C' \in \Sigma \setminus \Sigma_0$, and it is easy to see that $C \mapsto C'$ is an involution.

Now we shall show that this involution is sign reversing. Assume that $C' = (\sigma', \boldsymbol{v}, \boldsymbol{P}')$ is the image of $C = (\sigma, \boldsymbol{v}, \boldsymbol{P}) \in \Sigma \setminus \Sigma_0$ by this involution and v, i, j are as the above. We shall show that $\operatorname{sgn}\sigma' = \operatorname{sgn}\sigma$. Let k and l be the integers such that $i = \sigma(2k-1)$ or $\sigma(2k)$ and $j = \sigma(2l-1)$ or $\sigma(2l)$, respectively. Without loss of generality, we may suppose that $\{\sigma(2k-1), \sigma(2k), \sigma(2l-1), \sigma(2l)\} = \{1, 2, 3, 4\}$ and $\sigma(2k-1) = 1$. In the case of $(i, j) = (\sigma(2k-1), \sigma(2l-1))$ or $(i, j) = (\sigma(2k), \sigma(2l))$, if $(\sigma(2k-1), \sigma(2k), \sigma(2l-1), \sigma(2l)) = (1, 3, 2, 4)$ or $(1, 4, 2, 3)$, then it is easy to see that $\operatorname{sgn}\sigma' = \operatorname{sgn}\sigma$. However, we shall show that the condition $(\sigma(2k-1), \sigma(2k), \sigma(2l-1), \sigma(2l)) = (1, 2, 3, 4)$ never happens in this case. There is no loss of generality by supposing that $i = 1$ and $j = 3$. Assume that the vertices u_1, u_2, and u_3 is connected to v_1, v_2, and v_3 in I by the paths P_1, P_2 and P_3, respectively. If $v_1 > v_2$, then P_1 and P_2 must intersect by the D-compatibility, and this violates the condition $C \in \Sigma$. If $v_1 < v_2$, then consider the path $P'_3 = P_3(\to v)P_1(v \to)$ which connects u_3 to v_1. From the D-compatibility, P'_3 must intesect P_2, and further, by the minimality of v, this intersection points must be on $P_1(v \to)$. We have a contradiction as well, and this shows that the condition never happens. In the case of $(i, j) = (\sigma(2k-1), \sigma(2l))$ or $(i, j) = (\sigma(2k), \sigma(2l-1))$, if $(\sigma(2k-1), \sigma(2k), \sigma(2l-1), \sigma(2l)) = (1, 2, 3, 4)$ or $(1, 3, 2, 4)$, then it is easy to see that $\operatorname{sgn}\sigma' = \operatorname{sgn}\sigma$. However, by similar reasoning, one can see that $(\sigma(2k-1), \sigma(2k), \sigma(2l-1), \sigma(2l)) = (1, 4, 2, 3)$ never happens. Thus, we have shown that the above involution is sign reversing, and, in (A.7), one may cancel out all the terms which involve intersecting configulations of paths. Thus, we have

$$\mathrm{pf}\,[Q_I(u_i, u_j; B)] = \sum_{(\sigma, \boldsymbol{v}, \boldsymbol{P})\in\Sigma_0} \operatorname{sgn}\sigma \prod_{k=1}^{s} \beta_{v_{2k-1}v_{2k}} w(\boldsymbol{P}).$$

Suppose $(\sigma, \boldsymbol{v}, \boldsymbol{P}) \in \Sigma_0$. Then, put $\boldsymbol{w} = (w_1, w_2, \ldots, w_r) \in I_r$ such that $\boldsymbol{w}$ has the same support set with $\boldsymbol{v}$, i.e. $\{w_1, w_2, \ldots, w_r\} = \{v_1, v_2, \ldots, v_r\}$. From the D-compatibility, P_i connects u_i with w_i for $1 \leq i \leq r$, and this shows that

$$\mathrm{pf}\,[Q_I(u_i, u_j; B)] = \sum_{\boldsymbol{w} \in I_r} \mathbf{N}(\boldsymbol{u}, \boldsymbol{w}) \sum_{\sigma \in \mathfrak{F}_r} \mathrm{sgn}\, \sigma \prod_{k=1}^{s} \beta_{w_{\sigma(2k-1)} w_{\sigma(2k)}}.$$

This completes the proof. □

References

[DW] A.Dress and W.Wenzel, *A simple proof of an identity concerning pfaffians of skew symmetric matrices*, Adv. Math. **112** (1995), 120–134.

[GV] M.Gessel and G.Viennot, *Determinants, Paths, and Plane Partitions*, preprint.

[Hi] R.Hirota, *Mathematical aspect of the soliton theory from a direct methods point of view*, in Japanese, Iwanami Shoten, 1992.

[Ho] R.Howe, *Dual pairs in physics: harmonic oscillators, photons, electrons, and singletons*, Lect. Appl. Math. (AMS) **21** (1985), 179–207.

[I] M.Ishikawa, *A remark on totally symmetric self-complementary plane partitions*, preprint.

[IOW] M.Ishikawa, S.Okada and M.Wakayama, *Applications of minor summation formulas I, Littlewood's formulas*, J. Alg. (to appear).

[IW1] M.Ishikawa and M.Wakayama, *Minor summation formula of Pfaffians*, in press, Linear and Multilinear Alg. (1995).

[IW2] ———, *Minor summation formula of Pfaffians and Schur functions identities*, Proc.Japan Acad., Ser.A **71** (1995), 54–57.

[IW3] ———, *Applications of minor summation formulas III, Some generating functions of Schur polynomials*, preprint.

[Kn] D. Knuth, *Overlapping pfaffians*, preprint.

[Li] D.E.Littlewood, *The Theory of Group Characters and Matrix Representations of Groups, 2nd. ed.*, Oxford University Press, 1950.

[Ma] I.G.Macdonald, *Symmetric Functions and Hall Polynomials, 2nd Edition*, Oxford University Press, 1995.

[O1] S.Okada, *On the generating functions for certain classes of plane partitions*, J.Combin.Theo.Ser.A **51** (1989), 1–23.

[O2] ———, *Applications of minor-summation formulas to rectangular-shaped representations of classical groups*, preprint.

[Ste] J.Stembridge, *Nonintersecting paths, pfaffians and plane partitions*, Adv.Math. **83** (1990), 96–131.

[Su] T.Sundquist, *Pfaffians, involutions, and Schur functions*, University of Minnesota, PhD thesis.

[Wy] H.Weyl, *The Classical Groups, their Invariants and Representatons, 2nd. Edition.*, Princeton University Press, 1946.

Masao Ishikawa, Department of Mathematics, Faculty of Education, Tottori University, Tottori 680, Japan

E-mail address: `m-ishika@tansei.cc.u-tokyo.ac.jp`

Masato Wakayama, Graduate School of Mathematics, Kyushu Unversity, Hakozaki, Fukuoka 812, Japan

E-mail address: `wakayama@math.kyushu-u.ac.jp`