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Zero-dimensional Schemes and Hilbert functions of Cohen-Macaulay Homogeneous Domains

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Cohen–Macaulay 斉次環の Hilbert 関数や syzygy の振舞いは, Stanley や 日比孝之氏の著作 [19, 11], あるいは本報告集の寺井直樹氏のレポートにもあるように, 組合せ論への応用という面から見ても, 興味深い問題である.

本稿では, Cohen–Macaulay 斉次 "整域" の上記の性質を, [8, 9, 10, 6] 等で展開されている射影曲線の (次数と種数の関連などの) 研究から派生した種々の手法を応用して考察する.

特に重点的に用いたのは, Harris [10] の第3章で "Castelnuovo Theory" と云う名で紹介されている論法である. これは, 基礎体の標数が 0 の場合, 被約かつ既約で超平面に含まれない射影曲線と 一般の超平面の交わりが, (その超平面の中で) "良い" 配置にある事を用いて, この点集合やもとの曲線の Hilbert 関数を考察するものである.

第3節では, 0-次元スキーム (射影空間内の有限個の点集合) の考察を行なうが, ここでは可換代数のホモロジカルな手法を主として用い, Eisenbud–Harris [10, 3, 4] とは全く異なったアプローチをとった. それでも, 本稿の Lemma 2 は 先行する結果よりもむしろ強く, これをを認めれば, [10] の3章の殆んどの結果は, 簡単な曲面論から導ける.

また, 本稿の Theorem 4 で, (A の Krull 次元が 2 で) $s = 3$ の場合は, 曲線の標準埋入に関する Enriques–Petri の古典的な結果に対応し, これの M. Green [6] のアイデアによる証明を なぞった形になっている. Proposition 2 (2) (の証明) も, これの $s = 2$ の場合が、Saint-Donat, Fujita による, 次数 $\geq 2g+2$ の完備線形系で埋め込まれた曲線が 2 次式で定義されると言う結果に対応しており, これの背景を [6] のアイデアに従って見ていく形になる. はしょった解説ではあるが, local duality や syzygy がどのように応用されている様子が分かると思う.

本稿の主要な研究対象である Lemma 16 の同値条件を満たす Cohen–Macaulay 斉次整域の、多項式環上の minimal free resolution は, 一般化された Eagon–Northcot complex (cf., [2, A2.6.1]) を用いて具体的に構成できるのだが, 最後に, この free resolution と今回の研究集会で寺井氏が発表された stacked polytope に付随した Stanley–Reisner 環の minimal free resolution との関連 (と言うほど 大袈裟なものでもないが) について触れる.

1 Introduction

By a *homogeneous algebra* over a field k, we mean here a commutative k-algebra A with identity, together with a vector space direct sum decomposition $A = \bigoplus_{i\geq 0} A_i$, such that: (a) $A_0 = k$, (b) $A_iA_j \subset A_{i+j}$, (c) $\dim_k A_1 < \infty$ and (d) A is generated by A_1 as a k algebra. The *Hilbert function* of A is defined by $H_A(n) := \dim_k A_n$ for $n \geq 0$, while the *Hilbert series* is given by

$$F_A(\lambda) := \sum_{i\geq 0} H_A(i)\lambda^i = \frac{h_0 + h_1\lambda + \cdots + h_s\lambda^s}{(1-\lambda)^d},$$

where d is the Krull dimension of A and $h_0, h_1, \ldots, h_s$ are certain integers satisfying $h_s \neq 0$. We call the vector $(h_0, h_1, \cdots, h_s)$ the *h-vector* of A. The h-vector is nothing other than the d-th difference of the Hilbert function. More precisely, $h_i = \Delta^d H_A(i)$ for all i, where $\Delta H_A(n) = H_A(n) - H_A(n-1)$. And we always have $h_0 = 1$ and $\deg A = \sum_{i=0}^{s} h_i$. If A is Cohen-Macaulay, we have $h_i > 0$ for all $0 \leq i \leq s$. It is clear that the h-vector of A together with its Krull dimension determines the Hilbert function of A and conversely.

A famous theorem of Macaulay–Stanley gives a characterization of a numerical functions which occur as the Hilbert function $H_A(n)$ of a homogeneous k-algebra A. They also gives a numerical characterization of possible h-vectors (that is, numerical functions which occur as the Hilbert function $H_A(n)$ of a Cohen–Macaulay homogeneous algebra A). See [18] for further information.

But very little is known about the Hilbert function of a Cohen-Macaulay homogeneous *domain*, while it is conjectured that the h-vector of a Cohen–Macaulay homogeneous domain is under much stronger restrictions than that of a general Cohen-Macaulay homogeneous k-algebra.

The complete characterizations are obtained in a few special cases :

- when $h_1 \leq 1$ (trivial),
- when $h_1 = 2$ (Gruson and Peskine [8], see also [9]),
- when $h_1 = 3$ and A is Gorenstein (de Negri and Valla [15]).

When $h_1 \geq 4$, the problem become quite difficult. In general case, one of the best known results on the h-vector of a Cohen-Macaulay homogeneous domain is that, if the base field k is algebraically closed field with $\operatorname{char} k = 0$, then $h_i \geq h_1$ for all $1 \leq i \leq s-1$ (cf. [20]).

The following theorem refines the above inequality (in the rest of this note, we assume that the base field k is algebraically closed).

Theorem 1 ([24, Theorem 3.2 (a)]) *Suppose that k is algebraically closed field characteristic 0. Let A be a Cohen–Macaulay homogeneous domain with the h-vector $(h_0, h_1, \ldots, h_s)$. If $h_i = h_1$ for some $2 \leq i \leq s-2$, then $h_1 = h_2 = \cdots = h_{s-1} \geq h_s$. When $h_s \geq 2$, the condition $h_{s-1} = h_1$ also implies the same assertion.*

Remark. (a) For a given sequence $\mathbf{h} = (h_0, h_1, \dots, h_s)$ satisfying $h_0 = 1$, $h_1 = h_2 = \cdots = h_{s-1} \geq h_s$, there exists a Cohen-Macaulay homogeneous domain whose h-vector coincides with $\mathbf{h}$. For example, the projective coordinate ring of an arithmetically Cohen–Macaulay irreducible curve contained in a surface scroll is a two dimensional Cohen–Macaulay homogeneous domain with such a h-vector (see [10] for further information).

(b) When $h_s = 1$, the condition $h_{s-1} = h_1$ does not implies $h_1 = h_2$. For example, a complete intersection of general hypersurfaces of degree ≥ 3 is a Cohen–Macaulay homogeneous domain whose h-vector $(h_0, h_1, \dots, h_s)$ satisfies $s \geq 4$, $h_s = 1$, $h_{s-1} = h_1$ (furthermore $h_{s-i} = h_i$ for all i) but $h_2 > h_1$. Theorem 4 (stated below) concerns what happens when $h_s = 1$ and $h_1 = h_{s-1}$.

To prove Theorem 1, we use technique of Eisenbud and Harris [10, Chapter 3]. More precisely, we will use *uniform position lemma* and generalize a classical result of Castelnuovo which concerns a finite set of points in a projective space.

Lemma 2 ([24, Lemma 2.1]) *Let $X \subset \mathbb{P}^r$ be a not necessarily reduced zero-dimensional subscheme in uniform position. Denote the h-vector of the projective coordinate ring of X by $(h_0, h_1, \dots, h_s)$. If $h_i = h_1$ for some $2 \leq i \leq s-2$, then there is a rational normal curve containing X. If $h_s \geq 2$ and $s \geq 3$, then $h_1 = h_{s-1}$ also implies the same assertion.*

When $i = 2$ and X is reduced, Lemma 2 is a classical result due to Castelnuovo. And, Eisenbud and Harris [3, 4] proved the case $i = 2$ and X is non-reduced.

They use a deformation theory on projective schemes, while we use standard techniques of modern commutative algebra.

Obviously, A is a quotient ring of a polynomial ring $S = k[x_1, \dots, x_v]$ with $v = \dim_k A_1$ and $\deg x_i = 1$ for all $1 \leq i \leq v$. S is used in this meaning in the rest of this section. So $A \simeq S/I$ as a graded k-algebra for some graded ideal $I \subset \bigoplus_{i \geq 2} S_i$.

The h-vector of $A = S/I$ has some information on the degrees of minimal generators of I. The following, in particular part (1), seems more or less well-known. But, we will give a proof in §4 for readers convenience.

Proposition 3 *Suppose that $A = S/I$ is a Cohen–Macaulay homogeneous domain with the h-vector $(h_0, h_1, \dots, h_s)$, $h_1 \geq 2$. Then;*

(1) I is generated by elements of degree $\leq s+1$.

(2) If $h_s < h_1$, then I is generated by elements of degree $\leq s$.

(3) Suppose that $\operatorname{char} k = 0$. *If I is generated by elements of degree $\leq s-1$, then we have $h_i > h_1$ for all $2 \leq i \leq s-2$.*

Remark. Without the assumption that A is an integral domain, Proposition 3 (2) does not hold at all, though (1) of this proposition remains valid for general Cohen–Macaulay homogeneous rings. For example, let $A = S/I$ be as in Proposition 3, and let $\mathrm{Gin}(I)$ be a *generic initial ideal* of I (see [2] for the definition). It is well-known that $S/\mathrm{Gin}(I)$ is a non-reduced Cohen-Macaulay homogeneous ring with the h-vector $(h_0, h_1, \ldots, h_s)$, but $\mathrm{Gin}(I)$ always needs a generator of degre $s+1$.

Theorem 4 *Let $A = S/I$ be a Cohen–Macaulay homogeneous domain with the h-vector $(h_0, h_1, \ldots, h_s)$. Suppose that $s \geq 3$, $h_s = 1$ and $h_1 = h_{s-1} \geq 2$ (note that I is generated by elements of degree $\leq s$ by Proposition 2, in this case). If I actually needs a generator of degree s, the number of minimal generators of I of degree s is $h_1 - 1$ (i.e, $\dim_k[\mathrm{Tor}_1^S(k, S/I)]_s = h_1 - 1$). In this case, $h_1 = h_2 = \cdots = h_{s-1}$ and A is Gorenstein.*

If $s \geq 4$ and $h_1 = h_2 = \cdots = h_{s-1} \geq h_s$, then I needs a generator of degree s or $s+1$, by Proposition 3 (3). When the h-vector of A is $(1, h, h, 1)$, there are two cases. For example, let C be a smooth non-hyperelliptic curve with genus $g \geq 5$, and $A = S/I$ the homogeneous coordinate ring of the canonical embedding $C \subset \mathbb{P}^{g-1}$. A is a 2-dimensional Cohen–Macaulay homogeneous domain with the h-vector $(1, g-2, g-2, 1)$. A well-known theorem of Enriques–Petri says that I needs a generator of degree 3 ($= s$) if and only if C is trigonal or a plane quintic (of course, $\dim_k[\mathrm{Tor}_1^s(k, A)]_3 = g - 3$ in this case). So if C is a general curve, then I is generated by degree 2 ($= s - 1$) elements.

2 Canonical modules

Let $S := k[x_1, \ldots, x_v]$ be a polynomial ring with $\deg x_i = 1$ for all $1 \leq i \leq v$, and let $A \simeq S/I$ be a d-dimensional Cohen–Macaulay homogeneous algebra with the h-vector $(h_0, h_1, \ldots, h_s)$.

For a graded A-module $M = \bigoplus_{i \in \mathbb{Z}} M_i$, we sometime denotes the n-th graded component of M by $[M]_n$ (i.e., $[M]_n = M_n$), and $M(p)$ denotes the graded module with $[M(p)]_i = M_{p+i}$.

The graded minimal free resolution of $A = S/I$ over S is given by

$$0 \to \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{c,j}} \to \cdots \to \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{1,j}} \to S \to A = S/I \to 0,$$

where $c := h_1$ and $\beta_{i,j} := \dim_k[\mathrm{Tor}_i^S(k, A)]_j$ for each i, j. We call $\beta_{i,j}$ the (i,j)-th *Betti number* of A.

To prove Lemma 2, Proposition 3 and Theorem 4, we need the notion of *canonical module.*

Definition 5 *Let the notation be as above. The graded A-module $\omega_A := \mathrm{Ext}_S^c(A, S(-v))$ is called the canonical module of A.*

ω_A is a d-dimensional Cohen–Macaulay A module. The following is well known and a easy consequence of local duality.

Lemma 6 *Let the notation be as above.*
(a) (Stanley [18]) *We have*

$$F(\omega_A, \lambda) := \sum_{i \in \mathbb{Z}} \dim_k [\omega_A]_i \lambda^i = \frac{\lambda^{-s+d}(h_s + h_{s-1}\lambda + \cdots + h_0\lambda^s)}{(1-\lambda)^d}.$$

(b) (c.f., [6]) *Furthermore,* $[\mathrm{Tor}_i^S(k, A)]_j \simeq [\mathrm{Tor}_{c-i}^S(k, \omega_A)]_{v-j}$, *where* $c = h_1 = \mathrm{ht}\, I$.

3 Zero-dimensional schemes

In this section, we work over an algebraically closed field k of arbitrary characteristic unless otherwise specified. By $\mathbb{P}^r$, we denote the projective r space over k. Let $S := k[X_0, \cdots, X_r]$ be the homogeneous coordinate ring of $\mathbb{P}^r$.

Given a subscheme $V \subset \mathbb{P}^r$, we denote by I_V the saturated homogeneous ideal of V. We say that a subscheme $V \subset \mathbb{P}^r$ is *non-degenerate*, if no hyperplane contains V.

Let $X \subset \mathbb{P}^r$ be a zero-dimensional subscheme, and $R := S/I_X$ be the homogeneous coordinate ring of X. Unless otherwise specified, X and R are used in this meaning throughout this note. R is a (not necessarily reduced) 1-dimensional Cohen-Macaulay homogeneous algebra.

The Hilbert function of X is denoted by $H_X : \mathbb{Z} \to \mathbb{N}$ ($n \longmapsto \dim_k R_n$), while the *degree* of X is given by $\deg X = \sup\{H_X(n) | n \geq 0\}$. If X is reduced, then $\deg X$ is equal to the number of points contained in X.

Since R is 1-dimensional, the h-vector $(h_0, h_1, \ldots, h_s)$ of R is given by $h_i = H_X(i) - H_X(i-1)$ and $s = \min\{n | H_X(n) = \deg X\}$. We have that $h_i > 0$ for all $0 \leq i \leq s$, $\deg X = h_0 + h_1 + \cdots + h_s$, and $H_X(n) = h_0 + h_1 + \cdots + h_n$ for all n. h_1 is equal to the dimension of the linear span of X, in particular, if X is non-degenerate then $h_1 = r$.

Let ω_R be the canonical module of R. ω_R is a 1-dimensional Cohen–Macaulay R module. By Lemma 6, we have $[\omega_R]_i = 0$ for all $i < -s+1$, $\dim_k[\omega_R]_{-s+1} = h_s$, and $\dim_k[\omega_R]_{-s+2} = h_s + h_{s-1}$.

We now recall a few well-known geometric conditions on zero-dimensional schemes.

Definition 7 We say that X is *in linearly general position*, if every proper subspace $L \subset \mathbb{P}^r$ satisfies $\deg(L \cap X) \leq 1 + \dim L$, or equivalently, if every subscheme $Y \subset X$ satisfies $H_Y(1) = \min\{\deg Y, r+1\}$.

Definition 8 Let $X \subset \mathbb{P}^r$ be a zero-dimensional subscheme. We say that X is *in uniform position*, if X is in linearly general position and every subscheme $Y \subset X$ satisfies $H_Y(n) = \min\{H_X(n), \deg Y\}$ for all $n \in \mathbb{Z}$.

In the next proposition, we shall say that a map of k-vector spaces $\phi : U \otimes V \to W$ is *1-generic*, if $\phi(u \otimes v) \neq 0$ whenever $u, v \neq 0$, and *non-degenerate*, if $u \in U$ and $\phi(u \otimes V) = 0$ (resp. $v \in V$ and $\phi(U \otimes v) = 0$) imply $u = 0$ (resp. $v = 0$).

Proposition 9 (Kreuzer [13]) *Let $X \subset \mathbb{P}^r$ be a zero-dimensional subscheme in uniform position, and let R be its coordinate ring. Set $s = \min\{n | H_X(n) = \deg X\}$, in other words, s is the "length" of the h-vector of R. If $s \geq 2$, then the multiplication map $S_1 \otimes (\omega_R)_{-s+1} \to (\omega_R)_{-s+2}$ is 1-generic, and the multiplication map $R_n \otimes (\omega_R)_{-s+1} \to (\omega_R)_{-s+n+1}$ is non-degenerate for all $n \geq 0$.*

Remark 10 (a) Let $X \subset \mathbb{P}^r$ be a zero-dimensional subscheme in linearly general position (e.g., uniform position) with the h-vector $(h_0, h_1, \ldots, h_s)$. Then we have $h_i \geq h_1 = r$ for all $1 \leq i \leq s-1$. Hence we have $H_X(n) \geq \min\{1 + nr, \deg X\}$ for all $n \geq 0$ (see for example [13]).

(b) For any non reduced point $x \in X$, the local artinian ring $\mathcal{O}_{X,x}$ has a non-zero socle. Since we have that $\dim_k \mathcal{O}_{X,x}/(a) = \dim_k \mathcal{O}_{X,x} - 1$ where $0 \neq a \in \operatorname{soc}(\mathcal{O}_{X,x})$, there is a subscheme $Y \subset X$ such that $\deg Y = \deg X - 1$ (if X is reduced, $Y = X \setminus \{x\}$ for some $x \in X$). Moreover for each integer $1 \leq n \leq \deg X$, there is a subscheme $Y \subset X$ such that $\deg Y = n$.

On the other hand, if X is in uniform position and has the h-vector $(h_0, h_1, \ldots, h_s)$, then every subscheme $Y \subset X$ is in uniform position again, and its h-vector is given by $(h_0, h_1, h_2, \ldots, h_{i-1}, h_i')$ for some $i \leq s$ and $h_i' \leq h_i$.

We also need the following result from Eisenbud's "1-generic matrix" theory.

Proposition 11 (Eisenbud [1, Theorem 5.1]) *Let $\phi : U \otimes V \to W$ be a linear map of k-vector spaces and M be the matrix with entries in W which corresponds to ϕ (the correspondence between a bilinear map and a linear form matrix is given in the introduction of [1]). If ϕ is 1-generic and $\dim_k V = 2$, then M is equivalent to a unique scroll matrix $M(a_1, \cdots, a_d)$ with $1 \leq a_1 \leq \cdots \leq a_d$, $\sum_1^d a_i = \dim_k U$. That is,*

$$M \simeq M(a_1, \cdots, a_d)$$

$$:= \left(\begin{array}{cccc|ccc|cc} x_{1,0} & x_{1,1} & \cdots & x_{1,a_1-1} & x_{2,0} & \cdots & x_{2,a_2-1} & \cdots & x_{d,a_d-1} \\ x_{1,1} & x_{1,2} & \cdots & x_{1,a_1} & x_{2,1} & \cdots & x_{2,a_2} & \cdots & x_{d,a_d} \end{array} \right)$$

Proof. From a well-known formula on determinantal ideals, the assumption of Theorem 5.1 of [1] is satisfied automatically in this case. □

See [1] for further information on 1-generic matrices.

Proof of Lemma 2. If the assumption of the lemma is satisfied, we have $h_{i+1} \geq 2$ (see Remark 10 (a)). Hence we can find a subscheme $Y \subset X$ whose h-vector is $(h_0, h_1, \ldots, h_i, 2)$ by the argument in Remark 10 (b). Since $[I_Y]_2 = [I_X]_2$ and the defining ideal of a rational normal curve is generated by quadrics, we can replace X by Y. So we may assume that $h_s = 2$, $s \geq 3$ and $h_{s-1} = h_1 = r$. Then we have $\dim_k[\omega_R]_{-s+1} = 2$ by Lemma 6 (a).

Let M be the matrix with entries in $[\omega_R]_{-s+2}$ which corresponds to the multiplication map $S_1 \otimes [\omega_R]_{-s+1} \to [\omega_R]_{-s+2}$. Since this map is 1-generic by Proposition 9, M is equivalent to a scroll matrix $M(a_1, \cdots, a_{d'})$ by Proposition 11, for some $a_1, \ldots, a_{d'}$ such that $1 \leq a_1 \leq \cdots \leq a_{d'}$ and $\sum_1^{d'} a_i = r + 1$.

Easy calculation shows that $\dim_k M \leq \dim_k[\omega_R]_{-s+2} = 2 + h_{s-1} = r + 2$, where $\dim_k M$ means the dimension of the linear span of the entries of M. On the other hand, by the "shape" of the scroll matrix, it is easy to see that $\dim_k M = r + 1 + d$. Hence we have $d = 1$.

So we can find a basis $x_0, \cdots, x_r$ of S_1 and α_0, α_1 of $[\omega_R]_{-s+1}$ respectively such that $x_i \alpha_0 = x_{i-1}\alpha_1$ for all $1 \leq i \leq r$.

Set

$$M' := \begin{pmatrix} x_0 & x_1 & \cdots & x_{r-2} & x_{r-1} \\ x_1 & x_2 & \cdots & x_{r-1} & x_r \end{pmatrix}$$

the scroll matrix of type $M(r)$ with entries in S_1.

An explicit calculation shows that $I_2(M') \cdot [\omega_R]_{-s+1} = 0$. In fact, we have

$$\begin{aligned} & (x_j x_{i-1} - x_{j-1}x_i) \cdot \alpha_0 \\ = \; & x_j x_{i-1}\alpha_0 - x_{j-1}(x_i \alpha_0) \\ = \; & x_j x_{i-1}\alpha_0 - x_{j-1}(x_{i-1}\alpha_1) \\ = \; & x_j x_{i-1}\alpha_0 - x_{i-1}(x_{j-1}\alpha_1) \\ = \; & x_j x_{i-1}\alpha_0 - x_{i-1}(x_j \alpha_0) \\ = \; & 0, \end{aligned}$$

for all $1 \leq i, j \leq r$. Similarly,

$$(x_j x_{i-1} - x_{j-1}x_i) \cdot \alpha_1 = 0.$$

But, the multiplication map $R_2 \otimes [\omega_R]_{-s+1} \to [\omega_R]_{-s+3}$ is non-degenerate by Proposition 9. So we have $I_2(M') \subset I_X$. That is, the rational normal curve defined by $I_2(M')$ contains X. $\square$

The following is the non-reduced version of a classical result due to Castelnuovo.

Corollary 12 (Eisenbud–Harris, [3, Theorem 2.1] and [4, Theorem 2.2]) *Suppose that X is not necessarily reduced zero-dimensional subscheme of $\mathbb{P}^r$ in linearly general position.*

(a) If $\deg X = r+3$, *then there is a unique rational normal curve containing* X.

(b) If $\deg X \geq 2r+3$ *but* X *imposes only* $2r+1$ *conditions on quadrics, then there is a unique rational normal curve containing* X .

Proof. (a) The uniqueness part is easy (see [3]). We can prove the existence of the rational normal curve containing X by the same arguments in our proof of Lemma 2, since X is in uniform position and has the h-vector $(1, r, 2)$.

(b) Let Y be a subscheme of X with $\deg Y = 2r+3$. Then Y is in uniform position and has the h-vector $(1, r, r, 2)$. Since $[I_X]_2 = [I_Y]_2$, the statement follows from Lemma 2 immediately. □

Theorem 13 *Let* $X \subset \mathbb{P}^r$, $r \geq 2$ *be a non-degenerate zero-dimensional subscheme in uniform position and let* $(h_0, h_1, \ldots, h_s)$ *be the h-vector of the projective coordinate ring of* X *(note that* $h_1 = r$*).*

(1) (Green, [6, Corollary (3.c.6)]) *The following are equivalent.*

(a) $[\mathrm{Tor}^S_{r-1}(k, A)]_r \neq 0$.

(a') $\dim_k [\mathrm{Tor}^S_{r-1}(k, A)]_r = r-1$.

(b) X *lies on a (unique) rational normal curve.*

(2) If further $s \geq 4$ *(resp.* $s \geq 3$ *and* $h_s \geq 2$*), then the following conditions are equivalent.*

(c) $h_1 = h_i$ *for some* $2 \leq i \leq s-2$ *(resp.* $2 \leq i \leq s-1$*).*

(c') $h_1 = h_2 = \cdots = h_{s-1} \geq h_s$.

Proof of (2). (b) $\Rightarrow$ (c'): Well-known.

(c') $\Rightarrow$ (c): Obvious.

(c) $\Rightarrow$ (b): From Lemma 2. □

The assumption $h_s \geq 2$ of the "resp. part" of Theorem 13 (2) is necessary. There are many examples of a zero-dimensional subscheme in uniform position whose h-vector satisfies $s \geq 4$, $h_s = 1$, $h_1 = h_{s-1}$ and $h_2 > h_1$. In fact, almost all zero-dimensional complete intersections have such h-vectors.

The following result is a key lemma of the arguments in Eisenbud and Harris [10, §3.b]. They proved this under the additional assumption that $\operatorname{char} k = 0$.

Corollary 14 (Eisenbud–Harris, [10]) *Let* $X \subset \mathbb{P}^r$ *be a zero-dimensional scheme of degree* d *in uniform position. If there is no rational normal curve containing* X *then we have*

$$H_X(n) \geq \begin{cases} \min\{d, n(r+1)\} & \textit{unless } r+1 | d \textit{ and } n = d/(r+1) - 1, \\ d-1 & \textit{if } r+1|d \textit{ and } n = d/(r+1)-1. \end{cases}$$

Proof. The assertion follows from Theorem 13 (2) immediately. □

4 The h-vectors of Cohen-Macaulay homogeneous domains

In this section, we assume that k is an algebraically closed field with $\operatorname{char} k = 0$. The next result plays a key role of this note.

Uniform Position Lemma (Harris, [10, Corollary 3.4]) *Let $C \subset \mathbb{P}_k^r$ be a reduced, irreducible and non-degenerate curve. Then a general hyperplane section $C \cap H$ is a set of points in uniform position in H.*

Remark. When $\operatorname{char} k > 0$, uniform position lemma does not hold! In fact, there are well-known examples of a space curve C such that every secant of C is a multisecant, i.e., every secant of C intersects C at least one more point. It is easy to see that the hyperplane section of C always fails uniform position property.
Rathmann [16] classifies counter examples of uniform position lemma, in the positive characteristic case (the classification is not complete). Like the curve we mensioned above, most of these curves have pathological properties on secant lines or planes.

In virtue of UPL, we can use our results on zero-dimensional schemes to study the Hilbert function of a Cohen–Macaulay homogeneous domain.

Lemma 15 *Let A be a homogeneous Cohen-Macaulay domain over k of dimension d. Then there exists a set of points in uniform position whose projective coordinate ring has the same h-vector as A.*

Proof. By Bertini's theorem and the uniform position lemma, there is a linear regular sequence $\mathbf{x} = x_1, \ldots, x_{d-1} \in A_1$ for which $A/(\mathbf{x})$ is the projective coordinate ring of a set of points in uniform position. It is well-known that A and $A/(\mathbf{x})$ have the same h-vector. □

Proof of Theorem 1. The assertion follows from Lemma 15 and Theorem 13 (2). □

Let A be a d-dimensional Cohen–Macaulay homogeneous domain. Set $v := \dim_k A_1$. And let $S = k[X_1, \ldots, X_v]$ be a polynomial ring over k such that $A \simeq S/I$ as a graded k-algebra, for some homogeneous ideal $I \subset \bigoplus_{i \geq 2} S_i$.
It is easy to see that $X := \operatorname{Proj}(S/I) \subset \mathbb{P}^{v-1}$ is a subvariety (i.e., reduced and irreducible) of dimension $d-1$. Note that h_1 is the *codimension* of $X \subset \mathbb{P}^{v-1}$ (i.e., $h_1 = v - d$).
The following is a domain version of Theorem 13.

Lemma 16 *(1)* (Green, [6, Theorem (3.c.1)]) *Let the notation be as above. If $s \geq 3$ and $h_1 \geq 2$, then the following are equivalent.*

(a) $[\mathrm{Tor}^S_{(h_1-1)}(k,A)]_{h_1} \neq 0$.

(a') $\dim_k[\mathrm{Tor}^S_{(h_1-1)}(k,A)]_{h_1} = h_1 - 1$.

(b) $X \subset \mathbb{P}^{v-1}$ lies on a d-dimensional subvariety of $\mathbb{P}^{v-1}$ with minima degree (i.e., degree $v-d$).

(2) If further $s \geq 4$ (resp. $s \geq 3$ and $h_s \geq 2$), the following statements are also equivalent to the above.

(c) $h_1 = h_2 = \cdots = h_{s-1} \geq h_s$.

(c') $h_i = h_1$ for some $2 \leq i \leq s-2$ (resp. $2 \leq i \leq s-1$)

Proof of Proposition 3. For the convenience, we put $c = h_1$.

(1) Since $[\omega_A]_j = 0$ for all $j < -s+d$, we have $[\mathrm{Tor}^S_i(k,\omega_A)]_j = 0$ for all $j < -s+d+i$. From Lemma 4, it is easy to see that $[\mathrm{Tor}^S_1(k,A)]_j \simeq [\mathrm{Tor}^S_{c-1}(k,\omega_A)]_{c+d-j} = 0$ for all $j > s+1$. So the assertion follows from the fact that $\dim_k[\mathrm{Tor}^S_1(k,A)]_j$ is the number of minimal generators of I of degree j.

(2) Since $[\omega_A]_j = 0$ for all $j < -s+d$ and $\dim_k[\omega_A]_{-s+d} = h_s$, Theorem (3.a.1) of [6] (see also [5]) yields that $[\mathrm{Tor}^S_i(k,\omega_A)]_{-s+d+i} = 0$ for all $i \geq h_s$. By the assumption that $h_s < c$, we have

$$[\mathrm{Tor}^S_1(k,A)]_{s+1} \simeq [\mathrm{Tor}^S_{c-1}(k,\omega_A)]_{-s+d+c-1} = 0.$$

(3) It is easy to see that, if $A = S/I$ satisfies the equivalent conditions of Lemma 16, I is not generated by elements of degree $\leq s-1$. In the next section, we will study the free resolution of A over S, when A satisfies the conditions of Lemma 5 □

Proof of Theorem 4. By Lemma 6, we have

$$[\mathrm{Tor}^S_{c-1}(k,\omega_A)]_{-s+d+c} \simeq [\mathrm{Tor}^S_1(k,A)]_s \neq 0,$$

where $c = h_1$.

Since $h_s = 1$ and $h_{s-1} = h_1$, we see that $[\omega_A]_i = 0$ for all $i < -s+d$, $\dim_k[\omega_A]_{-s+d} = 1$ and $\dim_k[\omega_A]_{-s+d+1} = \dim_k A_1$. Consider the following exact sequence (note that ω_A is a torsion free A-module),

$$0 \to A \to \omega_A(-s+d) \to \mathrm{Coker} \to 0.$$

It is easy to see that $[\mathrm{Coker}]_i = 0$ for all $i \leq 1$, and hence

$$[\mathrm{Tor}^S_{c-1}(k,\mathrm{Coker})]_c = [\mathrm{Tor}^S_c(k,\mathrm{Coker})]_c = 0$$

. So applying the functor $\mathrm{Tor}^S_\cdot(k,-)$ to the above exact sequence, we get

$$[\mathrm{Tor}^S_{c-1}(k,A)]_c \simeq [\mathrm{Tor}^S_{c-1}(k,\omega_A(-s+d))]_c \simeq [\mathrm{Tor}^S_{c-1}(k,\omega_A)]_{-s+d+c} \neq 0.$$

Thus the assertion follows from Lemma 16. □

Another application of Corollary 16 can be found in [25]. We give here the main result of [25] without proof. It improves [21, 12, 14], in some senses.

Theorem 17 ([25]) *Let $C \subset \mathbb{P}^r$, $r \geq 3$ be a reduced, irreducible and non-degenerate curve.*

(1) Suppose that the hyperplane section $Z := C \cap H$ is aithmetically Gorenstein (aGor, for short) for a general hyperplane $H \subset \mathbb{P}^r$, but $C \subset \mathbb{P}^r$ itself is not aGor. Then Z is contained in a rational normal curve of $H \simeq \mathbb{P}^{r-1}$, and $\deg C = \deg Z \equiv 2 \pmod{r-1}$. *If further* $\deg C > r+1$, *then C is contained in a surface with minimal degree.*

(2) Conversely, for a given integer $d \geq r+1$ such that $d \equiv 2 \pmod{r-1}$, there is a smooth, irreducible curve $C \subset \mathbb{P}^r$ with $\deg C = d$ *which is not aGor, but its general hyperplane section is aGor.*

(3) If a general degree d hypersurface section of C is aGor for some d (≥ 2), then C itself is also aGor.

5 Minimal Free Resolution

Let $A = S/I$ be a d-dimensional Cohen–Macaulay homogeneous domain which satisfies the equivalent conditions of Lemma 16.

By several methods, we can compute the Betti numbers of A. For example, according to an idea of Schreyer [21], we can construct the minimal free resolution of A over S as a mapping cone between the complex constructed by Buchsbaum–Eisenbud (c.f., [2, A2.6.1] and [17, (1.5) and (1.6)]) and the Eagon–Northcot complex. Or, we can also compute the Betti numbers of A using an argument similar to [23, Corollary 3.4], since a suitable linear subspace section of $\operatorname{Proj} A$ is a set of points contained in a rational normal curve.

Proposition 18 *Let $A = S/I$ be a Cohen–Macaulay homogeneous domain which satisfies the equivalent conditions of Lemma 16. Put $c := h_1$. Then the Betti numbers of A over S is given by,*

$$\dim_k[\operatorname{Tor}_i^S(k, A)]_j = \begin{cases} 1 & \text{if } i = j = 0, \\ i\binom{c}{i+1} & \text{if } 1 \leq i \leq c-1 \text{ and } j = i+1, \\ (c - i - h_s + 1)\binom{c}{i-1} & \text{if } 1 \leq i \leq c - h_s \text{ and } j = s - 1 + i, \\ (h_s - c + i)\binom{c}{i} & \text{if } c - h_s + 1 \leq i \leq c \text{ and } j = s + i, \\ 0 & \text{otherwise.} \end{cases}$$

When $h_s = 1$, the Betti numbers are relatively simple.

Corollary 19 *Let $A = S/I$ be a d-dimensional Cohen–Macaulay homogeneous domain whose h-vector $(h_0, h_1, \cdots, h_s)$ satisfies $h_s = 1$ and $h_1 = \cdots = h_{s-1}$. If $s \geq 4$, or $s = 3$ and I has a generator of degree 3, then a minimal free resolution of A over S is of the form:*

$$0 \to S(-s-c) \to S(-c)^{b_{c-1}} \oplus S(-s-c+2)^{b_1} \to$$
$$S(-c+1)^{b_{c-2}} \oplus S(-s-c+3)^{b_2} \to \cdots\cdots \to S(-3)^{b_2} \oplus S(-s-1)^{b_{c-2}} \to$$
$$S(-2)^{b_1} \oplus S(-s)^{b_{c-1}} \to S \to A = S/I \to 0,$$

where $c = h_1$ and

$$b_i = i\binom{c}{i+1} \qquad \textit{for all } 1 \leq i \leq c-1.$$

Let $P(v,d)$ be a *stacked d-polytope* with v vertices, $\Delta(P(v,d))$ its boundary complex and $R := k[\Delta(P(v,d))]$ the Stanley–Reisner ring of $\Delta(P(v,d))$ over k (see [22] for the definition). R is a reduced (but non-irreducible) Cohen–Macaulay homogeneous ring of dimension d and embedding dimension v. So R is a quotient ring of a v-dimensional polynomial ring S.

A stacked polytope is a optimal example of *lower bound theorem* (cf. [10]), that is, the h-vector of R is given by $(h_0, h_1, \cdots, h_d)$, where $h_0 = h_d = 1$ and $h_1 = \cdots = h_{d-1} = v - d$. So R has the same h-vectors as the Cohen–Macaulay homogeneous domains which have been studied in this paper.

Recently, Terai and Hibi [22] computed the Betti number of R.

Comparing each Betti number, we get the following.

Proposition 20 *Let R be the Stanley–Reisner ring associated with the boundary complex of a d-dimensional stacked polytope $P(v,d)$. Let A be a Cohen–Macaulay homogeneous domain over k which has the same h-vector as R. If A satisfies the equivalent conditions of Lemma 16 (e.g., $d \geq 4$), then A and R have the same Betti numbers.*

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