数理解析研究所講究録
935 巻 1996 年 87-102

ウェーブレット変換と擬微分作用素

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0. Introduction -Definitions and Theorems-

We define a class of wavelet transforms as a continuous and micro-local version of the Littlewood-Paley decompositions. Hörmander's wave front sets as well as Besov and Triebel-Lizorkin spaces may be characterized in terms of our wavelet transforms. We remark that our decompositions can be regarded linearly independent.
This paper consists of two parts. The former part is the comparison between the wave front sets defined by our wavelet transforms and Hörmander's wave front sets. The latter part is the characterization of Besov, Triebel-Lizorkin spaces by using our wavelet transforms. First, we define our wavelet transforms as follows;

Definition 1. Suppose that the function $\psi(x)$ (called wavelet) has the following properties;
$\psi(x) \in \mathcal{S}(\mathbb{R}^n)$, $\hat{\psi}(\xi) \in C_0^\infty(\mathbb{R}^n)$ and $\hat{\psi}(\xi) \geqq 0$. Let Ω=supp$\hat{\psi}(\xi)$, $(0,\cdot\cdot,0,1)$ is the central axis of Ω, and r_ξ is any rotation which sends $\xi/|\xi|$ to $(0,\cdot\cdot,0,1)$. When $n=1$, $\Omega \subset (0,\infty)$ and when $n \geqq 2$, Ω is connected, does not contain the origin 0 and $\psi(x)=\psi(rx)$ for any $r \in SO(n)$ satisfying $r(0,\cdot\cdot,0,1)=(0,\cdot\cdot,0,1)$. Then our wavelet transform is defined as follows;
for $f(t) \in \mathcal{S}'(\mathbb{R}^n)$, $(x,\xi) \in \mathbb{R}^{2n}$,

$$W_\psi f(x,\xi) = \begin{cases} \int_{\mathbb{R}} f(t)|\xi|^{1/2}\overline{\psi(\xi(t-x))}dt, & \text{if } n=1, \\ \int_{\mathbb{R}^n} f(t)|\xi|^{n/2}\overline{\psi(|\xi|r_\xi(t-x))}dt, & \text{if } n \geqq 2. \end{cases}$$

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Remark 1. $W_\psi f(x,\xi)$ is rewritten as follows;

$$\int_{\mathbb{R}^n} \hat{f}(\tau)\cdot|\xi|^{-\frac{n}{2}}\hat{\psi}(\frac{r_\xi}{|\xi|}\tau)\cdot e^{i\tau x}d\tau.$$

From this, the meaning of our wavelet transforms is clear.

Remark 2. Our wavelet transforms in $\mathbb{R}^n$ are the reduced versions of those defined by R.Murenzi(See,[6]). Our purpose is to carry out the analogy of the microlocal anaiysis L.Hörmander succeeded in [3].

Remark 3. The domain of a wavelet transformation is usually the L_2-space(See,[1]), but can be extended to $\mathcal{S}'$,that is, the dual space of $\mathcal{S}$. It is easy to see that the image of $\mathcal{S}$ by this transformation is also $\mathcal{S}$.

Now, we define our wave front set $WF_\psi(f)(\subset \mathbf{R}^n_x\times\mathbf{R}^n_\xi)$ of $f\in\mathcal{S}'(\mathbb{R}^n)$ as follows.

Definition 2. $(x_0,\xi^0)\notin WF_\psi(f)$ is defined as follows:
there exists a neighbourhood $U(x_0)$ of x_0 and a conic neighbourhood $\Gamma(\xi^0)$ of ξ^0 such that $|W_\psi f(x,\xi)|= O(|\xi|^{-N})$ as $|\xi|$ tends to ∞ for any $N\in\mathbb{N}$ in $U(x_0)\times\Gamma(\xi^0)$.

Moreover, we define the refinement $WF^{(s)}_\psi(f)$ as follows.

Definition 3.

$$(x_0,\xi^0)\notin WF^{(s)}_\psi(f)\Leftrightarrow \iint_{U(x_0)\times\Gamma(\xi^0)} |W_\psi f(x,\xi)|^2(1+|\xi|^2)^s dx d\xi<\infty.$$

It is clear that if $f\in L_2(\mathbb{R}^n)$,

$$WF_\psi(f) = \text{the closure of} \bigcup_{s\geqq 0} WF^{(s)}_\psi(f).$$

We need the following definition to state Theorem 1.

Definition 4. Let

$$\text{cone}\Omega = \{t\xi|\xi\in\Omega, t>0\}.$$

$(x_0,\xi^0)\notin\overline{WF}^\psi$ is defined as follows:
$x_0\notin \text{proj}_x WF$ and $\xi^0\in\mathbb{R}^n$,
or $x_0\in\text{proj}_x WF$ and r(coneΩ) does not intersect $\{\xi\in\mathbb{R}^n;(x_0,\xi)\in WF\}$ for any $r\in SO(n)$ with r(coneΩ) including ξ^0.
Here, $\text{proj}_x WF$ denotes the projection of WF onto x-space.

Theorem 1. *Let $f \in L_2(\mathbb{R}^n)$, and $s \geqq 0$. When $n = 1$, $WF_{\psi}^{(s)}(f) = WF^{(s)}(f)$. When $n \geqq 2$, $WF_{\psi}^{(s)}(f) \subseteq \overline{WF^{(s)}(f)}^{\psi}$ and $WF^{(s)}(f) \subseteq \overline{WF_{\psi}^{(s)}(f)}^{\psi}$. We have the same inclusions between $WF_{\psi}(f)$ and $WF(f)$.*

The latter part of this paper is the characterization of Besov, Triebel-Lizorkin spaces by using our wavelet transforms. We use continuous decompositions not only of the radial direction but also of the unit sphere of the frequency space. (See,J.Peetre[4],H.Triebel[5])

Definition 5.
Let $\phi(x)$ be a rapidly decreasing function whose Fourier transform is compactly supported in $\frac{1}{2} \leqq |\xi| \leqq 2$. Moreover, suppose that any half line starting from the origin intersects supp $\hat{\phi}(\xi)$.
Let $\phi_r(x)$ be $r^n\phi(rx)$. Then, $\hat{\phi}_r(\xi)$ is equal to $\hat{\phi}(\frac{\xi}{r})$.

Definition of Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$. $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ $(s > 0, 1 \leqq p, q \leqq \infty)$ is defined by the following:

$$\left(\int (r^s \|\phi_r * f(x)\|_{L_p(dx)})^q \frac{dr}{r}\right)^{\frac{1}{q}} < \infty.$$

Definition of Triebel-Lizorkin spaces $\dot{F}^s_{p,q}(\mathbb{R}^n)$. $f \in \dot{F}^s_{p,q}(\mathbb{R}^n)$ $(s > 0, 1 \leqq p < \infty, 1 \leqq q \leqq \infty)$ is defined by the following:

$$\left\| \left(\int (r^s \cdot \phi_r * f(x))^q \frac{dr}{r}\right)^{\frac{1}{q}} \right\|_{L_p(dx)} < \infty.$$

Theorem 2. *$f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ $(s > 0,\ 1 \leqq p, q \leqq \infty)$ can be characterized by the following:*

$$\left\| |\xi|^s \left\| |\xi|^{\frac{n}{2}} |W_{\psi} f(x,\xi)| \right\|_{L_p(dx)} \right\|_{L_q(\frac{d\xi}{|\xi|^n})} < \infty.$$

Theorem 3. *$f \in \dot{F}^s_{p,q}(\mathbb{R}^n)$ $(s > 0,\ 1 \leqq p < \infty,\ 1 \leqq q \leqq \infty)$ can be characterized by the following:*

$$\left\| \left\| |\xi|^{s+\frac{n}{2}} |W_{\psi} f(x,\xi)| \right\|_{L_q(\frac{d\xi}{|\xi|^n})} \right\|_{L_p(dx)} < \infty.$$

1. Wave front sets defined by *our* wavelet transforms and Hörmander's wave front sets

As we have already defined, the wavelet $\psi(x)$ is of essentially two parameters that is rotationally invariant around Θ when $n \geqq 2$. For the purpose of proving **Theorem 1**, we prepare three propositions.
(Here, $\Theta = (0, \cdots, 0, 1) \in \mathbb{R}^n$.)

Proposition 1(Parseval formula and inversion formula).
For $f, g \in L_2(\mathbb{R}^n)$,

$$\iint W_\psi f(x,\xi)\overline{W_\psi g(x,\xi)}dx d\xi = C_\psi \int f(t)\overline{g(t)}dt.$$

Here,

$$C_\psi = (2\pi)^n \int \frac{|\hat{\psi}(\xi)|^2}{|\xi|^n} d\xi.$$

From this, we also have:

$$f(t) = C_\psi^{-1} \iint W_\psi f(x,\xi) \cdot |\xi|^{\frac{n}{2}} \psi(|\xi| r_\xi (t-x)) dx d\xi,$$

when $n \geqq 2$. When $n = 1$, $|\xi| r_\xi (t-x)$ is replaced by $\xi(t-x)$.
(For $f \in \mathcal{S}'(\mathbb{R}^n)$, this inversion formula must be regarded in the distribution sense.)

Proposition 2(Locality).
If $x_0 \notin supp f$, then there exists a neighbourhood $U(x_0)$ of x_0 such that $W_\psi f(x,\xi)$ is rapidly decreasing in ξ with respect to $x \in U(x_0)$ uniformly.

Proposition 3(Global Sobolev property).

$$f \in H^s(\mathbb{R}^n) \Leftrightarrow \iint |W_\psi f(x,\xi)|^2 (1+|\xi|^2)^s < \infty.$$

これらの証明は容易故省略する (See, [1], [9])

Proof of **Theorem 1.** It suffices to show when $n \geqq 2$. Moreover, by the fact that $WF_\psi(f)$=the closure of $\bigcup_{s \geqq 0} WF_\psi^{(s)}(f)$, it suffices to prove the statement for any $s \geqq 0$ fixed.

Step.1

Let $(0, \xi^0) \notin \overline{WF^{(s)}(f)}^{\psi}$. If we take a conic neighbourhood $\Gamma(\xi^0)$ of ξ^0 as the union of all r(coneΩ), where r is any rotation with ξ^0 included in r(coneΩ), then there exists a function $\phi(x) \in C_0^\infty(\mathbb{R}^n)$ which is always equal to 1 near $x = 0$ and satisfies $\int_{\Gamma(\xi^0)} |(\hat{\phi f})(\xi)|^2(1+|\xi|^2)^s d\xi < \infty$. This follows from the definition of $\overline{W}^{\psi}$, the definition of Hörmander's wave front set and Heine-Borel's lemma.

What we want to say is that there exist a conic neighbourhood $\tilde{\Gamma}(\xi^0)$ of ξ^0 and a neighbourhood $U(0)$ of 0, satisfying:

$$\iint_{U(0)\times\tilde{\Gamma}(\xi^0)} |W_\psi f(x,\xi)|^2(1+|\xi|^2)^s < \infty$$

Here,using the inversion formula, we divide $W_\psi f(x,\xi)$ into two parts:

$$W_\psi f(x,\xi) = |\xi|^{\frac{n}{2}} \int (\phi f)(t) \cdot \overline{\psi(|\xi| r_\xi (t-x))} dt \qquad (1)$$

$$+ |\xi|^{\frac{n}{2}} \int ((1-\phi)f)(t) \cdot \overline{\psi(|\xi| r_\xi (t-x))} dt \qquad (2)$$

If $U(0) \Subset \{\phi(x) \equiv 1\}$,then, by the argument of propotion 2, (2) is rapidly decresing in $|\xi|$ with respect to $x \in U(0)$ uniformly. Therefore, it is clear that $(0, \xi_0) \notin WF_\psi^{(s)}((1-\phi)f)$. On the other hand, if we take $\tilde{\Gamma}(\xi^0)$ sufficiently small, then we get the following:

$$\iint_{U(0)\times\tilde{\Gamma}(\xi^0)} |W_\psi(\phi f)(x,\xi)|^2(1+|\xi|^2)^s dx d\xi$$

$$\leqq \int_{\tilde{\Gamma}(\xi^0)} d\xi \int_{\mathbf{R}_x^n} |W_\psi(\phi f)(x,\xi)|^2(1+|\xi|^2)^s dx d\xi$$

$$= (2\pi)^n \int_{\mathbf{R}_\tau^n} d\tau |(\hat{\phi f})(\tau)|^2 \int_{\tilde{\Gamma}(\xi^0)} \frac{d\xi}{|\xi|^n}(1+|\xi|^2)^s \hat{\psi}(\frac{r_\xi}{|\xi|}\tau)^2$$

If we change variables from τ to $\omega = \frac{r_\xi}{|\xi|}\tau$ as before , ω must be in Ω. Therefore, we can see that τ stays in $\Gamma(\xi^0)$ because we took $\tilde{\Gamma}(\xi^0)$

very small. The inequality above is followed by:

$$\leqq (2\pi)^n \int\limits_{\Gamma(\xi^0)} d\tau |(\hat{\phi f})(\tau)|^2 \int\limits_{\Omega} \frac{d\omega}{|\omega|^n}(1+\frac{|\tau|^2}{|\omega|^2})^s \hat{\psi}(\omega)^2$$

$$\leqq C \int\limits_{\Gamma(\xi^0)} |(\hat{\phi f})(\tau)|^2(1+|\tau|^2)^s d\tau < \infty \text{(Here, C is a constant.)}$$

Therefore, $(0,\xi^0) \notin WF_{\psi}^{(s)}(\phi f)$.

Step.2

Let $(0,\xi^0) \notin \overline{WF_{\psi}^{(s)}(f)}^{\psi}$. If we take a conic neighbourhood $\Gamma(\xi^0)$ of ξ^0 as the union of all r(coneΩ), where r is any rotation with ξ^0 included in r(coneΩ), then there exists a neighbourhood $U(0)$ of $x=0$ and satisfies $\iint\limits_{U(0)\times\Gamma(\xi^0)} |W_\psi f(x,\xi)|^2(1+|\xi|^2)^s dx d\xi < \infty$, as in **Step 1.**
Here, using the inversion formula, we divide f into two parts:
$f = f_\Gamma + f_{\Gamma^c}$, where

$$f_\Gamma(t) = C_\psi^{-1} \iint\limits_{\Gamma(\xi^0)\times\mathbf{R}_x^n} W_\psi f(x,\xi)\cdot|\xi|^{\frac{n}{2}}\psi(|\xi|r_\xi(t-x))dx d\xi$$

$$f_{\Gamma^c}(t) = C_\psi^{-1} \iint\limits_{\Gamma(\xi^0)^c\times\mathbf{R}_x^n} W_\psi f(x,\xi)\cdot|\xi|^{\frac{n}{2}}\psi(|\xi|r_\xi(t-x))dx d\xi.$$

Then,

$$\widehat{f_{\Gamma^c}}(\tau) = C_\psi^{-1}\int_{\Gamma(\xi^0)^c}\int_{\mathbf{R}_x^n} W_\psi f(x,\xi)\cdot|\xi|^{-\frac{n}{2}}\hat{\psi}(\frac{r_\xi}{|\xi|}\tau)e^{-i\tau\cdot x}dx d\xi$$

If we take a sufficiently small conic neighbourhood $\tilde{\Gamma}(\xi^0)$ of ξ^0, then we obtain

$$\hat{\psi}(\frac{r_\xi}{|\xi|}\tau) \equiv 0 \text{ for any} \tau \in \tilde{\Gamma}(\xi^0) \text{ and for any } \xi \in \Gamma(\xi^0)^c.$$

Therefore, it follows $(0,\xi^0) \notin WF^{(s)}(f_{\Gamma^c})$.

Next, we choose $\phi(x) \in C_0^\infty(\mathbf{R}^n)$ satisfying that $\mathrm{supp}\phi(x) \subset U(0)$ and that $\phi(x) \equiv 1$ in some neighbourhood $U_1(0)$ of 0. Then, we further divide $f_\Gamma(t)$ into two parts:

$$f_\Gamma = f_{\Gamma,\phi} + f_{\Gamma,1-\phi}, \text{where}$$

$$f_{\Gamma,\phi}(t) = C_\psi^{-1} \iint_{\Gamma(\xi^0)\times\mathbf{R}_x^n} \phi(x) \cdot W_\psi f(x,\xi) \cdot |\xi|^{\frac{n}{2}} \psi(|\xi| r_\xi (t-x)) dx d\xi$$

$$f_{\Gamma,1-\phi}(t) = C_\psi^{-1} \iint_{\Gamma(\xi^0)\times\mathbf{R}_x^n} (1-\phi(x)) W_\psi f(x,\xi) \cdot |\xi|^{\frac{n}{2}} \psi(|\xi| r_\xi (t-x)) dx d\xi$$

Let $U_2(0) \Subset \{\phi(x) \equiv 1\}$, then we can easily see that $f_{\Gamma,1-\phi}(t)$ is C^∞ with respect to $t \in U_2(0)$, by **Proposition 2,** and 'the exchange of order of differentiation and integration'. Therefore, it follows $(0,\xi^0) \notin WF^{(s)}(f_{\Gamma,1-\phi})$.

Lastly, we want to show $(0,\xi^0) \notin WF^{(s)}(f_{\Gamma,\phi})$. This is the heart of matter in proving **Theorem 1.** In fact, more strongly, we can show the global Sobolev property of $f_{\Gamma,\phi}$.

$$\widehat{f_{\Gamma,\phi}}(\tau) = C_\psi^{-1} \iint_{\Gamma(\xi^0)\times\mathbf{R}_x^n} \phi(x) \cdot W_\psi f(x,\xi) \cdot |\xi|^{-\frac{n}{2}} \hat{\psi}(\frac{r_\xi}{|\xi|}\tau) e^{-i\tau\cdot x} dx d\xi$$

Here, if we put $g(x,\xi) = \phi(x) W_\psi f(x,\xi) \cdot (1+|\xi|^2)^{\frac{s}{2}}$, then we can see

$$\iint_{\Gamma(\xi^0)\times\mathbf{R}_x^n} |g(x,\xi)|^2 dx d\xi < \infty.$$

(This follows from the hypothesis and from the fact that $\mathrm{supp}\phi(x)$ is included in $U(0)$.)

If we denote the Fourier partial transform of $g(x,\xi)$ from x to τ by $\hat{g}(\tau,\xi)$,

$$\begin{aligned}
&\widehat{f_{\Gamma,\phi}}(\tau)(1+|\tau|^2)^{\frac{s}{2}}\\
=&C_\psi^{-1}\iint\limits_{\Gamma(\xi^0)\times\mathbf{R}_x^n} g(x,\xi)e^{-i\tau\cdot x}\cdot|\xi|^{-\frac{n}{2}}\hat{\psi}(\frac{r_\xi}{|\xi|}\tau)(\frac{1+|\tau|^2}{1+|\xi|^2})^{\frac{s}{2}}dxd\xi\\
=&C_\psi^{-1}(2\pi)^{\frac{n}{2}}\int_{\Gamma(\xi^0)}\hat{g}(\tau,\xi)\cdot K(\tau,\xi)d\xi
\end{aligned}$$

Here, $K(\tau,\xi)$ is defined by $|\xi|^{-\frac{n}{2}}\hat{\psi}(\frac{r_\xi}{|\xi|}\tau)(\frac{1+|\tau|^2}{1+|\xi|^2})^{\frac{s}{2}}$.
Because supp$\hat{\psi}$ is a compact set not including the origin 0 (by the defintion of ψ), there exists a constant C such that

$$|K(\tau,\xi)|\leqq C|\xi|^{-\frac{n}{2}}\hat{\psi}(\frac{r_\xi}{|\xi|}\tau).$$

Therefore, by using the result in the proof of Proposition 1(i.e. the continuous decomposition of the unity), the integral $\int|K(\tau,\xi)|^2d\xi$ is bounded from above. (the bound is $(2\pi)^{-n}C_\psi C^2$.)
After all, we obtain the following inequality:

$$\begin{aligned}
\int|\widehat{f_{\Gamma,\phi}}(\tau)|^2(1+|\tau|^2)^s d\tau \leqq& C_\psi^{-1}C^2\int(\int_{\Gamma(\xi^0)}|\hat{g}(\tau,\xi)|^2d\xi)d\tau\\
=&C'\int_{\Gamma(\xi^0)}d\xi\int_{\mathbf{R}_\tau^n}|\hat{g}(\tau,\xi)|^2d\tau\\
=&C'\iint\limits_{\Gamma(\xi^0)\times\mathbf{R}_x^n}|g(x,\xi)|^2dxd\xi<\infty.
\end{aligned}$$

(Theorem 1) q.e.d.

2. Characterization of Besov, Triebel-Lizorkin Spaces via *our* continuous wavelet transforms

Now we prove **Theorem 2** and **Theorem 3.**

Theorem 2. *$f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ $(s > 0,\ 1 \leqq p, q \leqq \infty)$ can be characterized by the following:*

$$\left\| |\xi|^s \, \left\| |\xi|^{\frac{n}{2}} |W_\psi f(x,\xi)| \right\|_{L_p(dx)} \right\|_{L_q(\frac{d\xi}{|\xi|^n})} < \infty.$$

Proof. **Sufficiency:**
For simplicity, let

$$\psi_{|\xi|, r_\xi} * f(x) = |\xi|^{\frac{n}{2}} W_\psi f(x,\xi),$$

$$\phi_{|\xi|} = \int \psi_{|\xi|, r_\xi} d\theta_\xi, \text{ where } d\theta_\xi \text{ is the Haar measure on } S^{n-1}.$$

Then, $\hat{\phi}_r(\xi)$ is compactly supported in $C_1 r \leqq |\xi| \leqq C_2 r$ (because $\hat{\psi}(\xi)$ is compactly supported.) and any half line starting from the origin intersects supp $\hat{\phi}_r(\xi)$.

$$\begin{aligned}(\int | \int \psi_{|\xi|, r_\xi} * f(x) d\theta_\xi |^p dx)^{\frac{q}{p}} &\leqq (\int (\int |\psi_{|\xi|, r_\xi} * f(x)|^p dx)^{\frac{1}{p}} d\theta_\xi)^q \\ &\leqq C \int (\int |\psi_{|\xi|, r_\xi} * f(x)|^p dx)^{\frac{q}{p}} d\theta_\xi\end{aligned}$$

The first inequality is due to the continuous version of the Minkowskii inequality and the second one is due to the Hölder inequality. After integrating both hand sides of this inequality with respect to $|\xi|^{sq-1} d|\xi|$, we can see that the usual Besov norm can be bounded from above by the Besov norm via the wavelet transform.

Necessity: Let,

$$\hat{\sigma_r}(\tau)^2 = (2\pi)^n C_\psi^{-1} \int \hat{\psi}(\frac{r_\xi}{r}\tau)^2 d\theta_\xi.$$

(See the proof of **Proposition 1.**) Then, supp$\hat{\sigma_r}(\tau)$ is located in $C_1 r \leqq |\tau| \leqq C_2 r$ and

$$\int \hat{\sigma_r}(\tau)^2 \frac{dr}{r} = 1,$$

that is,

$$\int \sigma_r * \sigma_r(x) \frac{dr}{r} = \delta(x).$$

By using this continuous decomposition of the unity,

$$\|\psi_{|\xi|,r_\xi} * f(x)\|_{L_p(dx)} \leqq \int \|\psi_{|\xi|,r_\xi} * \sigma_r\|_{L_1(dx)} \cdot \|f * \sigma_r\|_{L_p(dx)} \frac{dr}{r} \quad (1)$$

Because the Fourier transform of $\psi_{|\xi|,r_\xi} * \sigma_r$ is not equal to 0 only when $C_3|\xi| \leqq r \leqq C_4|\xi|$, and because the L_1 norm of $\psi_{|\xi|,r_\xi}$ and σ_r is bounded,

$$\begin{aligned}(1) \leqq & C \int_{C_3|\xi|}^{C_4|\xi|} \|f * \sigma_r\|_{L_p(dx)} \frac{dr}{r} \\ = & C \int_{C_3}^{C_4} \|f * \sigma_{t|\xi|}(x)\|_{L_p(dx)} \frac{dt}{t}\end{aligned}$$

The last term above is independent of the rotation $d\theta_\xi$, and moreover,

$$\left\| |\xi|^s \, \|f * \sigma_{t|\xi|}\|_{L_p(dx)} \right\|_{L_q(\frac{d\xi}{|\xi|^n})} = t^{-s} \left\| |\xi|^s \, \|f * \sigma_{|\xi|}\|_{L_p(dx)} \right\|_{L_q(\frac{d\xi}{|\xi|^n})},$$

we can conclude that the Besov norm via the wavelet transform is bounded from above by the usual Besov norm.

(**Theorem 2**) q.e.d.

Theorem 3. *$f \in \dot{F}^s_{p,q}(\mathbb{R}^n)$ $(s > 0,\ 1 \leqq p < \infty,\ 1 \leqq q \leqq \infty.)$ can be characterized by the following:*

$$\left\| \left\| |\xi|^{s+\frac{n}{2}} |W_\psi f(x,\xi)| \right\|_{L_q(\frac{d\xi}{|\xi|^n})} \right\|_{L_p(dx)} < \infty.$$

Proof. **Sufficiency:** As in Theorem.2, let

$$\phi_{|\xi|} = \int \psi_{|\xi|,r_\xi} d\theta_\xi.$$

Then,

$$\begin{aligned}||\xi|^s(\phi_{|\xi|} * f(x))|^q = & \left| \int (|\psi_{|\xi|,r_\xi} * f(x)||\xi|^s) d\theta_\xi \right|^q \\ \leqq & C \int |(\psi_{|\xi|,r_\xi} * f(x))|\xi|^s|^q d\theta_\xi.\end{aligned}$$

Hence, we can easily see that the usual Triebel-Lizorkin norm is bounded from above by the norm via the wavelet transform.

Necessity: This part needs very deep results which are continuous versions of the work of C.Fefferman-E.M.Stein[2] and H.Triebel[5].

First, we state the results without proof.(The proof is carried out in the same way as in the discrete case. See[2][5].)

Claim 1.(Continuous version of [2]) Let $f(x,y)$ be a function of $(x,y) \in \mathbb{R}^n_x \times \mathbb{R}^n_y$, and $Mf(x,y)$ be a maximal function of $f(x,y)$ with respect to x. Then,

$$\left\| \left(\int |Mf(x,y)|^q \frac{dy}{|y|^n} \right)^{\frac{1}{q}} \right\|_{L_p(dx)} \leqq C_{p,q} \left\| \left(\int |f(x,y)|^q \frac{dy}{|y|^n} \right)^{\frac{1}{q}} \right\|_{L_p(dx)},$$

where $1 < p < \infty, 1 < q \leqq \infty$.

Claim 2.(Continuous version of maximal inequalities in [5]) Let p,q,r be

$$0 < p < \infty, 0 < q \leqq \infty, \text{and } 0 < r < min(p,q).$$

Let $\hat{f}(\xi,y)$ be the Fourier partial transform of $f(x,y)$ with respect to x, and $\Omega_{|y|}$ be a set including the support of $\hat{f}(\xi,y)$ with respect to ξ. Let the diameter $d_{|y|}$ of $\Omega_{|y|}$ be a continuous function of $|y|$, and $d_{|y|} > 0$. Then the following inequality holds:

$$\begin{aligned} &\left\| \left(\int \left(\sup_{z \in \mathbb{R}^n} \frac{|f(x-z,y)|}{1+|d_{|y|}z|^{\frac{n}{r}}} \right)^q \frac{dy}{|y|^n} \right)^{\frac{1}{q}} \right\|_{L_p(dx)} \\ \leqq & C \left\| \left(\int |f(x,y)|^q \frac{dy}{|y|^n} \right)^{\frac{1}{q}} \right\|_{L_p(dx)}. \end{aligned}$$

Claim 3.(Continuous version of multiplier theorem in [5]) Let

$$0 < p < \infty, 0 < q \leqq \infty, \text{and } \kappa > n\left(\frac{1}{2} + \frac{1}{min(p,q)}\right).$$

Let $\Omega_{|y|}$, $d_{|y|}$ be as in Claim.2. Then, the following inequality holds:

$$\begin{aligned} &\left\| \left(\int ((M(\cdot,y) * f(\cdot,y))(x))^q \frac{dy}{|y|^n} \right)^{\frac{1}{q}} \right\|_{L_p(dx)} \\ \leqq & C \sup_{y \in \mathbb{R}^n} \left\| \hat{M}(d_{|y|}\cdot,y)|H_2^\kappa \right\| \cdot \left\| \left(\int |f(x,y)|^q \frac{dy}{|y|^n} \right)^{\frac{1}{q}} \right\|_{L_p(dx)}. \end{aligned}$$

Claim 1 is essential in proving **Claim 2.**
In proving **Claim 3**, we need **Claim 2** and the following inequality:

$$\sup_{z\in\mathbb{R}^n} \frac{|(M(\cdot,y) * f(\cdot,y))(x-z)|}{1+|d_{|y|}z|^{\frac{n}{r}}}$$

$$\leqq C \sup_{z\in\mathbb{R}^n} \frac{|f(x-z,y)|}{1+|d_{|y|}z|^{\frac{n}{r}}} \cdot \left\| \hat{M}(d_{|y|}\cdot, y)|H_2^\kappa \right\|.$$

(Here, $0 < r < min(p,q), \kappa > \frac{n}{2} + \frac{n}{r}$.)

As in the proof of the necessary condition of Theorem 2, we use the continuous decomposition of the unity:

$$\int \sigma_r * \sigma_r(x) \frac{dr}{r} = \delta(x).$$

$$\left\| \left\| |\xi|^s (\psi_{|\xi|,r_\xi} * f(x)) \right\|_{L_q(\frac{d\xi}{|\xi|^n})} \right\|_{L_p(dx)}$$

$$= \left\| \left\| \int |\xi|^s (\psi_{|\xi|,r_\xi} * \sigma_r) * (f * \sigma_r)(x) \frac{dr}{r} \right\|_{L_q(\frac{d\xi}{|\xi|^n})} \right\|_{L_p(dx)}$$

$$\leqq \int_{C_1}^{C_2} \frac{dt}{t} \left\| \left\| |\xi|^s (\psi_{|\xi|,r_\xi} * \sigma_{t|\xi|}) * (f * \sigma_{t|\xi|}) \right\|_{L_q(\frac{d\xi}{|\xi|^n})} \right\|_{L_p(dx)}$$

We apply **Claim 3** to the integrand of the last term above;

$$d_{|\xi|} = C|\xi|, \text{ and}$$

$$\widehat{\psi_{|\xi|,r_\xi} * \sigma_{t|\xi|}}(\tau) = \hat{\psi}(\frac{r_\xi}{|\xi|}\tau) \cdot \hat{\sigma}(\frac{\tau}{t|\xi|}).$$

Thus,

$$\sup_\xi \left\| \hat{\psi}(\frac{r_\xi}{|\xi|} C|\xi| \cdot \tau) \cdot \hat{\sigma}(\frac{C|\xi| \cdot \tau}{t|\xi|}) |H_2^\kappa \right\|$$

is bounded from above. Therefore, the Triebel-Lizorkin norm via the wavelet transform is bounded from above by the usual norm.

(Theorem 3)q.e.d.

Remark 4. Theorem 3. can be extended to the case when $0 < p < \infty$. The case when $0 < q \leqq \infty$ remains to be proved. Also in Theorem 2, the case when $0 < p \leqq \infty$, $0 < q \leqq \infty$ remains to be proved. Such troubles occur because we used the Hölder inequality and the Minkowskii inequality in the proof of Theorem 2,3.

3. 擬微分作用素 $S^{0}_{1,\delta}$ ($0\leqq\delta<1$) と我々のウェーブレット変換との関係

Theorem 4. $a(s,\eta)\in S^{0}_{1,\delta}$ ($0\leqq\delta<1$) に対して、定数 C が存在して、

$$\| af(s)\|_{L_2(ds)} \leqq C\| f(t)\|_{L_2(dt)}$$

が成り立つ。

証明の方針　[7]の方針に従って、Proposition 1 より、相空間上の積分作用素の積分核が、Schurの補題を満たすことを示す。すなわち、

$$K(z,\zeta;x,\xi)=\iiint e^{-i(t-s)\cdot\eta}\,a(s,\eta)\,\psi_{x,\xi}(t)\,\overline{\psi_{z,\zeta}(s)}\,d\eta\,dt\,ds$$

に対して、定数 M が存在して、

$$\iint |K(z,\zeta;x,\xi)|\,dx\,d\xi < M$$

$$\iint |K(z,\zeta;x,\xi)|\,dz\,d\zeta < M$$

が成り立つことを示す。ここで、

$$\psi_{x,\xi}(t)=|\xi|^{\frac{n}{2}}\psi(|\xi|V_{\xi}(t-x))$$

のことである。

<u>Definition 6.</u> 方向別Triebel空間 $F^s_{p,q}(\Xi)$ $(-\infty<s<\infty,$ $1\leqq p<\infty,\ 1\leqq q\leqq\infty)$ の定義（ただし、$|\Xi|=1$）。

$f\in F^s_{p,q}(\Xi_\varepsilon)$ は次で定義される；

$$\|\phi_0*f(x)\|_{L_p(dx)}+\Big\|\,|\xi|^s\|\,|\xi|^{\frac{n}{2}}W_\psi f(x,\xi)\|_{L_p(dx)}\Big\|_{L_q\left(\frac{d\xi}{|\xi|^n};\ |\xi|\geqq 1,\ \left|\frac{\xi}{|\xi|}-\Xi\right|<\varepsilon\right)}<+\infty,$$

ただし、$\phi_0(x)\in\mathcal{S}(\mathbb{R}^n)$, $\hat{\phi}_0(\xi)$ は compact support $\{\xi\in\mathbb{R}^n\mid |\xi|\leqq 2\}$ を持つ。
また、上式の最後の部分は，ξ の積分範囲を示したものである。

<u>Remark 5.</u> Definition 6. は Theorem 3. により示唆された定義である。Besov空間についても全く同様に出来る。

<u>Theorem 5.</u> $a(x,\xi)\in S^m_{1,\delta}$ $(m>0)$ が,

$$|\partial^\alpha_\xi\partial^\beta_x a(x,\xi)|\leqq C_{\alpha,\beta}(1+|\xi|)^{-|\alpha|+\delta|\beta|},\ \left|\frac{\xi}{|\xi|}-\Xi\right|<\varepsilon$$

を満たすとする。

$\varepsilon_1<\varepsilon_0<\varepsilon$ ならば、$f\in F^{s'}_{p,q}\ (s'<s)\cap F^s_{p,q}(\Xi_{\varepsilon_0})$ に対して、定数 C が存在して、

$$\|af\|_{F^s_{p,q}(\Xi_{\varepsilon_1})}+\|af\|_{F^{s'-m}_{p,q}}\leqq C\left(\|f\|_{F^s_{p,q}(\Xi_{\varepsilon_0})}+\|f\|_{F^{s'}_{p,q}}\right)$$

が成り立つ。

証明の方針

我々が定義したウェーブレット変換を用いて、
$\|af\|_{F^s_{p,q}(\Xi_{\varepsilon_1})}$ を積分表示すること、及び、
[8] を用いれば、

$$\|af\|_{F^s_{p,q}(\Xi_{\varepsilon_1})} \leqq C_1(\|f\|_{F^s_{p,q}(\Xi_{\varepsilon_0})}+\|f\|_{F^{s'}_{p,q}})$$

が導かれる。

$$\|af\|_{F^{s'-m}_{p,q}} \leqq C_2\|f\|_{F^{s'}_{p,q}}$$

は、[8] の結果そのものである。

Remark 6. Theorem 5. は Besov 空間についても全く同様に出来る。

Remark 7. 周波数空間(ξ-space)のみならず、実空間(x-space)をも局所化した定理を述べることも出来る。

$$\begin{cases} 0,\ 1,\ 2\ 章の内容は\ [9]\ と同じ。 \\ \qquad 3\ 章の詳細は\ [10]\ に書く予定。 \end{cases}$$

謝辞. [8] に関連して御教示して頂いた山崎昌男先生に、深く感謝致します。

Acknowledgements. The auther would like to express his sincere gratitude to Prof. H. Komatsu, Prof. K. Kataoka, Prof. K. Asada and Dr. S. Tanabe for many valuable suggestions and encouragement. He would like to express his sincere gratitude also to Mr. S. Yamazaki for helpful discussions.

また、今回の「超函数と微分方程式」で話す機会を与えて下さったことに、深く感謝致します。

REFERENCES

1. Daubechies I., *Ten Lectures on Wavelets*, SIAM CBMS-61, 1992.
2. Fefferman C ;Stein E M., *Some maximal inequalities*, Amer.J.Math. 93 (1971), 107–115.
3. Hörmander L., *Fourier integral operators.I*, Acta.Math. 127 (1971), 79–183.
4. Peetre J., *New Thoughts on Besov Spaces*, Duke Univ.,Durham, 1976.
5. Triebel H., *Theory of Function Spaces*, Birkhäuser, 1983.
6. Murenzi R., *Wavelet transforms associated to the n-dimensional Euclidean group with dilations: signals in more than one dimension*, in Wavelets, J.M. Combes, A. Grossmann, and Ph. Tchamitchian, eds., Springer-Verlag, Berlin, (1989), 239–246.
7. Asada K., 波束変換と振動積分のL_2有界性, 京大数理解析講究録. 795 (1992), 111–121
8. Päivärinta L., *Pseudo differential operators in Hardy-Triebel spaces*, Zeitschrift für Analysis und ihre Anwendungen. 2 (1983), 235–242
9. S. Moritoh, *Wavelet transforms in $\mathbb{R}^n$ — its relation with wave front sets and Besov, Triebel-Lizorkin spaces —*, Preprint
10. S. Moritoh, *Wavelet transforms and pseudo differential operators*, (in preparation)