数理解析研究所講究録
936 巻 1996 年 27-31

Noncommutative Euler Characteristic and its Applications

H. TAKAI (T.M.U.)

In topology, one of the most famous and important invariants of spaces is the so-called Euler (or Euler-Poincaré) characteristic, which is defined as the alternative sum of the Betti numbers of manifolds. Even in noncommutative topology, a generalized notion of Euler characteristic of C*-algebras is well understood in terms of their K-theory. Namely, it is defined as the integer of subtracting torsion-free rank of K_1-theory from that of K_0-theory. It has many nice properties since theory does. There exist many examples of simple C*-algebras whose Euler characteristics are given arbitrary integers, so that one may ask how to classify simple C*-algebras with a given Euler characteristic.

In this report, we answer partially the above problem in the case of separable nuclear simple C*-algebras with semi-finite traces, and we also offer a new example of separable simple non-nuclear C*-algebras with non-commutative Euler characteristic −1. Finally, we exhibit a non-commutative version of the Gauss-Bonnet theorem in closed C^{∞}-manifolds of dimension 2.

First of all, we state the following theorem, in connection with which Rørdam [R] showed that any classifiable separable simple nuclear purely infinite C*-algebra is described as a crossed product of a AT-algebra by a single automorphism up to stable isomorphisms:

Theorem 1. Let A be a separable simple nuclear C*-algebra with a semi-finite lower semi-continuous trace and denote by $\chi(A)$ the Euler characteristic of A. Then $\chi(A) = 0$ if and only if there exists a C*-dynamical system $(B, \mathbb{Z}, \beta)$ such that (1) B is strongly amenable with $\chi(B) \in \mathbb{Z}$, and (2) A is stably isomorphic to $B \times_{\beta} \mathbb{Z}$.

Remark 1. Even if A is purely infinite satisfying U.C.T., it is as a crossed product of an AT-algebra by a single automorphism up to stable isomorphisms, which is done by Rørdam [R]. Especially, the Cuntz algebra $\mathcal{O}_n$ $(n \geq 2)$ is stably isomorphic to the crossed product $(M_{n^\infty} \otimes \mathbb{K}) \times_\beta \mathbb{Z}$ of $M_{n^\infty} \otimes \mathbb{K}$ by the shift automorphism β of the tensor product $M_{n^\infty} \otimes \mathbb{K}$ of the UHF-algebra of type n^∞ and the C*-algebra $\mathbb{K}$ of all compact operators on a countably infinite dimensional Hilbert space, however $\chi(M_{n^\infty} \otimes \mathbb{K}) = +\infty$.

<u>Remark 2.</u> In the case of separable simple nuclear C*-algebras, there may be no example of C*-algebras with negative Euler characteristic. In the case of non-simple nuclear C*-algebras, there are many C*-algebras with negative Euler characteristic.

<u>Remark 3.</u> Several examples of C*-algebras with non-zero Euler characteristic are constructed using basic properties.

<u>Conjecture.</u> Suppose A is a separable simple nuclear C*-algebra, then $\chi(A) \geq 0$.

The proof of Theorem 1 is done by combining the following some key lemmas:

<u>Lemma I.</u> Let $(A, \mathbb{Z}, \alpha)$ be a C*-dynamical system. Suppose $\chi(A)$ is finite, then $\chi(A \times_\alpha \mathbb{Z}) = 0$.

<u>Lemma II (with Matsumoto).</u> Let A be as in Theorem 1. If A has Connes-Jones' Property T in C*-sense, then it is a matrix algebra.

<u>Lemma III.</u> If A is a separable strongly amenable C*-algebra without Property T, then there exist a partial isomery $u \in M(A)$ and a strongly amenable C*-subalgebra B of A such that (1) $uBu^* = B$ and (2) $C^*(B,u)$ is a hereditary C*-subalgebra of A.

In what follows, we study simple C*-algebras with negative Euler characteristics. One of the prototype of such C*-algebras is the reduced C*-algebras of the free groups with n-generators. Their Euler characteristics are $1-n$. We shall generalize this fact for $n = 2$, in other words we seek sufficient conditions for C*-algebras under which their Euler characteristics are -1. Let A be a unital separable simple C*-algebra with unique tracial state τ, and (A, T^2, α) an effective C*-dynamical system with the property that (1) $A'' \cap (A^\alpha)' = \mathbb{C}$ on the Hibert space via τ, and (2) there exist two unitaries $u \in A^\alpha(1,0)$, $v \in A^\alpha(0,1)$. There are many examples satisfying the above conditions. We then have the following theorem:

Theorem 2. Under the above situation with $\chi(A) \in \mathbb{Z}$, it follows that $\chi(A) = -1$.

Remark 4. There exist a C*-dynamical system (A,T^2,α) satisfying the above conditions (1) and (2), but $\chi(A) = +\infty$. There exists an action α of T^2 on $\mathcal{O}_2$ with the condition(1), but $\chi(\mathcal{O}_2) = 0$. Moreover there exist non-effective C*-dynamical system (A,T^n,α) with the conditions (1) and (2), however $\chi(A) < 0$.

Let Γ be a discrete group and π a unitary representation of Γ on a Hilbert space H. Then we can construct a quasi-free action α^π of Γ on the CAR-algebra A(H) via π and denote by $A(\Gamma,\pi)$ the crossed product of A(H) of Γ by α^π.

Corollary 3. Let λ be the left regular representation of F_2 on $\ell^2(F_2)$. Then $\chi(A(F_2,\lambda)) = 0$.

Remark 5. It is no longer true in general that $\chi(A) = 1-n$ for a C*-dynamical system (A,T^n,α) with (1) and (2') unitaries $u_j \in A^\alpha(0,1,0)$ $(1 \leq j \leq n)$ where $(0,1,0)$ is the n-tuple with 1 at j-site and 0 at k-site $(k \neq j)$.

For instance, take the gauge action of T^{2g} on the reduced C*-algebra $C_r^*(\Gamma_g)$ of the fundamental group Γ_g of a closed Riemann surface with genus g ($g \geq 2$). Then $\chi(C_r^*(\Gamma_g)) = 2 - 2g$.

We need the notion of cyclic cohomology to show Theorem 2. Let us take A^∞ the canonical smooth part of A with respect to α, and $H_\lambda^*(A^\infty)$ the cyclic cohomology of A^∞ and $H^*(A^\infty) = H_\lambda^*(A^\infty) \otimes_{H_\lambda^*(\mathbb{C})} \mathbb{C}$ the periodic cyclic cohomology of A^∞. The key lemmas are in what follows, which are of independent interest:

<u>Lemma IV.</u> Under the same situation as Theorem 2, the periodic cyclic cohomology $H^*(A^\infty)$ is described as the following:

$$H^{ev}(A^\infty) = \mathbb{C}[\tau] \quad \text{and} \quad H^{odd}(A^\infty) = \mathbb{C}[\tau_1] \oplus \mathbb{C}[\tau_2]$$

where $\tau_j(a,b) = \tau(a\delta_j(b))$ for a,b in A^∞ and δ_j are the generators of the action α of T^2.

<u>Lemma V.</u>- If there exists a C*-dynamical system (A, G, α) whose smooth part A^∞ is closed under the holomorphic function calculus, then we have that

$$\chi(A) = \dim_{\mathbb{C}} H^{ev}(A^\infty) - \dim_{\mathbb{C}} H^{odd}(A^\infty) .$$

In the last stage of this short note, we briefly remark on how tofind a Gauss-Bonnet formula of certain non-commutative manifolds.

Suppose (A,G,α) is a C*-dynamical system whose smooth part A^∞ is cosed under the holomorphic function calculus. Let $\mathcal{E}$ be a finitely projective A^∞-module. Due to Cnnes [C], there exists a connection ∇ from $\mathcal{E}$ to $\mathcal{E} \otimes_{A^\infty} \Omega^1$ where Ω^1 is the set of all 1-forms of A^∞. Then there exists a $\nabla^\sim$ in $\mathrm{End}_\Omega(\mathcal{E} \otimes_{A^\infty} \Omega)$ such that

$$\nabla^{\sim}(\xi \otimes \omega) = \nabla(\xi)\omega + \xi \otimes d\omega$$

for ξ in $\mathscr{E}$ and ω in Ω where Ω is the Grassman algebra of all p-forms of A^{∞}. Let $2\pi i\theta = (\nabla^{\sim})^2$ be in $End_{\Omega}(\mathscr{E} \otimes_{A^{\infty}} \Omega)$. Suppose there exists a faithful tracial state τ of A^{∞} and $G = T^2$, then we have by Connes [C] that

$$\langle [\mathscr{E}], [S\tau] \rangle = \int \theta$$

where $\int$ is the trace on the graded algebra $End_{\Omega}(\mathscr{E} \otimes_{A^{\infty}} \Omega)$ associated to the graded trace on Ω^n. We can find a finitely projective A^{∞}–module $\varepsilon(A)$ with the property that

$$\langle [\varepsilon(A)], [S\tau] \rangle = \chi(A) .$$

Actually, one may take

$$\varepsilon(A) = \Sigma_{j\geq 0} (-1)^j [\Lambda^j (A^{\infty} \otimes (A^{\infty})^o)]$$

where $(A^{\infty})^o$ is the opposite algebra of A^{∞}.

References

[B] B.Blackadar: K-theory for Operator Algebras, Publ.MSRI.,5 Springer (1986).

[C] A.Connes: Non Commutative Differential Geometry, Publ.Math. IHES., 62 (1986), 257-360.

[R] M.Rørdam: Classification of certain infinite simple C*-algebras, Preprint, (1995).