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Algorithmic Aspects of D-modules *

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Abstract

We consider an algebraic D-module, i.e. a system of linear partial differential equations with *polynomial* coefficients. Our main purpose is to present algorithms for computing the characteristic variety and the multiplicity of such a system.

1 Involutive bases

We denote by $A_n := \mathbf{C}[x]\langle\partial\rangle = \mathbf{C}[x_1, \ldots, x_n]\ \langle\partial_1, \ldots, \partial_n\rangle$ the Weyl algebra or the ring of differential operators with polynomial coefficients. We consider a system of linear partial differential equations

$$\mathcal{M}\ :\ \sum_{j=1}^{r} P_{ij}u_j = 0 \qquad (i = 1, \ldots, s)$$

for unknown functions $u_1, \ldots, u_r$, where P_{ij} are elements of A_n.

The characteristic variety $\mathrm{Char}(\mathcal{M})$ of $\mathcal{M}$ is by definition an analytic subset of the complex cotangent bundle $T^*\mathbf{C}^n$ and it represents the analytic nature of the system $\mathcal{M}$. The characteristic variety is defined analytically, i.e., through the sheaf of rings $\mathcal{D}$ of linear partial differential operators with analytic coefficients. Hence even for a system of equations with polynomial coefficients as above, it does not seem obvious, at least to the present author, that the characteristic variety can be computed purely algebraically in finitely many steps.

The key point of our argument consists in proving that for a system with polynomial coefficients, a Gröbner basis in the Weyl algebra with an appropriate monomial order gives a so-called involutive basis in the analytic sense.

We denote by $\mathcal{E}$ the sheaf on $T^*\mathbf{C}^n$ of rings of microdifferential operators of finite order. For a vector $\vec{P} = (P_1, \ldots, P_r) \in \mathcal{E}^r$, we denote by $m := \mathrm{ord}(\vec{P})$ the maximum of the order of each component P_i as microdifferential operator, and by $\sigma(\vec{P})$ the vector

$$\sigma(\vec{P}) = (\sigma_m(P_1), \ldots, \sigma_m(P_r)),$$

where $\sigma_m(P_i)$ stands for the principal symbol of order m, which is an analytic function of $(x,\xi) = (x_1, \ldots, x_n, \xi_1, \ldots, \xi_n) \in T^*\mathbf{C}^n$. We denote by $\mathcal{O}_{T^*\mathbf{C}^n}$ the sheaf on $T^*\mathbf{C}^n$ of holomorphic functions and by $\mathcal{O}_{T^*\mathbf{C}^n,x^*}$ its stalk at $x^* \in T^*\mathbf{C}^n$.

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Theorem 1 *Let $\vec{P}_1, \dots, \vec{P}_s$ be elements of $(A_n)^r$ and assume that they are A_n-involutive; i.e., for any $\vec{P} \in A_n\vec{P}_1 + \dots + A_n\vec{P}_s$, we have*

$$\sigma(\vec{P}) \in \mathbf{C}[x,\xi]\sigma(\vec{P}_1) + \dots + \mathbf{C}[x,\xi]\sigma(\vec{P}_s).$$

Then, for any point x^* of $T^*\mathbf{C}^n$, $\vec{P}_1, \dots, \vec{P}_s$ are $\mathcal{E}$-involutive at x^*; i.e., for any $\vec{P} \in \mathcal{E}_{x^*}\vec{P}_1 + \dots + \mathcal{E}_{x^*}\vec{P}_s$, we have

$$\sigma(\vec{P}) \in \mathcal{O}_{T^*\mathbf{C}^n,x^*}\sigma(\vec{P}_1) + \dots + \mathcal{O}_{T^*\mathbf{C}^n,x^*}\sigma(\vec{P}_s).$$

This theorem is a special case of Proposition 2.0.9 of [K] (p. 15) stated without proof. We can use the Gröbner basis theory for the polynomial ring and for the Weyl algebra in two respects: first, in order to prove this theorem concretely (cf. [O1]), and second, in order to compute an A_n-involutive basis. In [O1], Theorem 1 is proved for $\mathcal{D}$ instead of $\mathcal{E}$; however, the same argument applies to this case by replacing $\mathcal{O}_{\mathbf{C}^n}[\xi]$ by $\mathcal{O}_{T^*\mathbf{C}^n}$.

2 Gröbner basis for the Weyl algebra

Let us review the Gröbner basis theory for modules over the Weyl algebra. We fix a total order $\prec$ of $\mathbf{N}^{2n}$ with $\mathbf{N} := \{0, 1, 2, \dots\}$ that satisfies the following conditions:

(1) $(0,0) \preceq (\alpha, \beta)$ for any $\alpha, \beta \in \mathbf{N}^n$;

(2) if $(\alpha, \beta) \prec (\alpha', \beta')$, then $(\alpha + \alpha'', \beta + \beta'') \prec (\alpha' + \alpha'', \beta' + \beta'')$ for any $\alpha'', \beta'' \in \mathbf{N}^n$;

(3) if $|\beta| < |\beta'|$, then $(\alpha, \beta) \prec (\alpha', \beta')$ for any $\alpha, \alpha' \in \mathbf{N}^n$.

Here we use the notation $|\beta| = \beta_1 + \dots + \beta_n$ for $\beta = (\beta_1, \dots, \beta_n) \in \mathbf{N}^n$.

An element P of A_n is written as a finite sum

$$P = \sum_{\alpha,\beta} a_{\alpha\beta} x^\alpha \partial^\beta$$

with $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$, $\partial^\beta = \partial_1{}^{\beta_1} \dots \partial_n{}^{\beta_n}$, $a_{\alpha\beta} \in \mathbf{C}$ for $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$. Then we define the leading exponent $\mathrm{lexp}(P)$, the order $\mathrm{ord}(P)$, and the leading coefficient $\mathrm{lcoef}(P)$ of P by

$$\begin{aligned} \mathrm{lexp}(P) &= \max{}_\prec\{(\alpha, \beta) \in \mathbf{N}^{2n} \mid a_{\alpha\beta} \neq 0\}, \\ \mathrm{ord}(P) &= \max\{|\beta| \mid a_{\alpha\beta} \neq 0\}, \\ \mathrm{lcoef}(P) &= a_{\alpha\beta} \quad \text{with} \quad (\alpha, \beta) = \mathrm{lexp}(P), \end{aligned}$$

where $\max_\prec$ denotes the maximum element with respect to the order $\prec$. When $\mathrm{ord}(P) \leq m$ we write

$$\sigma_m(P) = \sum_{\alpha,|\beta|=m} a_{\alpha\beta} x^\alpha \xi^\beta$$

with $\xi = (\xi_1, \dots, \xi_n)$. If $\mathrm{ord}(P) = m$, we write simply $\sigma(P) = \sigma_m(P)$ and call it the principal symbol of P.

Moreover, for an r-vector $\vec{P} = (P_1, \dots, P_r) \in (A_n)^r$, we define its order, the leading point $\mathrm{lp}(\vec{P})$, the leading exponent and the leading coefficient by

$$\begin{aligned} \mathrm{ord}(\vec{P}) &= \max\{\mathrm{ord}(P_\nu) \mid \nu = 1, \dots, r\}, \\ \mathrm{lp}(\vec{P}) &= \max\{\nu \in \{1, \dots, r\} \mid \mathrm{ord}(P_\nu) = \mathrm{ord}(\vec{P})\}, \\ \mathrm{lexp}(\vec{P}) &= (\mathrm{lexp}(P_\nu), \mathrm{lp}(\vec{P})) \quad \text{with} \quad \nu = \mathrm{lp}(\vec{P}), \\ \mathrm{lcoef}(\vec{P}) &= \mathrm{lcoef}(P_\nu) \quad \text{with} \quad \nu = \mathrm{lp}(\vec{P}). \end{aligned}$$

Let N be a left A_n-submodule of $(A_n)^r$. Then the set $E(N)$ of leading exponents of N is defined by

$$E(N) = \{\mathrm{lexp}(\vec{P}) \mid \vec{P} \in N, \ \vec{P} \neq 0\} \subset \mathbf{N}^{2n} \times \{1, \dots, r\}.$$

We introduce a total order $\prec$ in the set $\mathbf{N}^{2n} \times \{1, \dots, r\}$ by

$$\begin{aligned} (\alpha, \beta, \nu) \prec (\alpha', \beta', \nu') \iff & (|\beta| < |\beta'|) \\ \text{or} \quad & (|\beta| = |\beta'| \quad \text{and} \quad \nu < \nu') \\ \text{or} \quad & (|\beta| = |\beta'| \quad \text{and} \quad \nu = \nu' \quad \text{and} \quad (\alpha, \beta) \prec (\alpha', \beta')) \end{aligned}$$

for $\alpha, \beta, \alpha', \beta' \in \mathbf{N}^n$ and $\nu, \nu' \in \{1, \dots, r\}$.

Definition 1 A finite subset $\mathbf{G}$ of a left A_n-submodule N of $(A_n)^r$ is called a *Gröbner basis* of N if

$$E(N) = \bigcup_{\vec{P} \in \mathbf{G}} (\mathrm{lexp}(\vec{P}) + \mathbf{N}^{2n})$$

holds, where we put

$$(\alpha, \beta, \nu) + \mathbf{N}^{2n} = \{(\alpha + \alpha', \beta + \beta', \nu) \mid \alpha', \beta' \in \mathbf{N}^n\}.$$

The algorithm of constructing a Gröbner basis from a given set of generators of N is similar to the Buchberger algorithm ([B]) for ideals of polynomial rings as was first pointed out by Galligo ([G]), which was generalized by Takayama ([T]) to a more general setting.

The following lemma is an immediate consequence of the definitions above:

Lemma 1 *Put $\mathbf{G} = \{\vec{P}_1, \dots, \vec{P}_s\}$ with $\vec{P}_1, \dots, \vec{P}_s \in (A_n)^r$. Assume that $\mathbf{G}$ is a Gröbner basis of the left A_n-submodule N of $(A_n)^r$ generated by $\mathbf{G}$. Then $\mathbf{G}$ is A_n-involutive.*

3 Characteristic variety

Now consider the system $\mathcal{M}$ defined in the first section and put $\vec{P}_i = (P_{i1}, \dots, P_{ir})$ for $i = 1, \dots, s$.

Theorem 2 *Let* $\mathbf{G}$ *be a Gröbner basis of the left* A_n*-submodule* $N := A_n\vec{P}_1 + \ldots + A_n\vec{P}_s$ *of* $(A_n)^r$. *Put*

$$\mathbf{G}_\nu = \{\vec{P} \in \mathbf{G} \mid \mathrm{lp}(\vec{P}) = \nu\}$$

for each $\nu \in \{1, \ldots, r\}$. *Then the characteristic variety of* $\mathcal{M}$ *is given by* $\mathrm{Char}(\mathcal{M}) = \bigcup_{\nu=1}^r V_\nu$ *with*

$$V_\nu = \{(x, \xi) \in T^*\mathbf{C}^n \mid \sigma(\vec{P})_\nu(x, \xi) = 0 \quad \text{for any} \quad \vec{P} \in \mathbf{G}_\nu\},$$

where $\sigma(\vec{P})_\nu$ *denotes the* ν*-th component of the vector* $\sigma(\vec{P})$.

4 Multiplicity

We regard $\mathcal{M}$ as a left coherent $\mathcal{E}$-module on $T^*\mathbf{C}^n$ by

$$\mathcal{M} := \mathcal{E}^r / \mathcal{E}\vec{P}_1 + \ldots + \mathcal{E}\vec{P}_s.$$

Let $u_1, \ldots, u_r$ be the residue classes of the unit vectors $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1) \in \mathcal{E}^r$. For each integer j, let us denote by $\mathcal{E}(j)$ the subsheaf of $\mathcal{E}$ consisting of operators of order at most j and put

$$\mathcal{M}(j) := \mathcal{E}(j)u_1 + \ldots + \mathcal{E}(j)u_r \subset \mathcal{M}.$$

Let x^* be a non-singular point of the characteristic variety $V = \mathrm{Char}(\mathcal{M})$. Then the *multiplicity* of $\mathcal{M}$ at x^* is defined as the multiplicity of the coherent $\mathcal{O}_{T^*\mathbf{C}^n}$-module

$$\overline{\mathcal{M}} := \mathcal{O}_{T^*\mathbf{C}^n} \otimes_{\mathcal{O}_{T^*\mathbf{C}^n}(0)} (\mathcal{M}(0)/\mathcal{M}(-1))$$

along V, where $\mathcal{O}_{T^*\mathbf{C}^n}(0)$ denotes the subsheaf of $\mathcal{O}_{T^*\mathbf{C}^n}$ consisting of functions homogeneous of degree 0 with respect to ξ (cf. [K]). We denote by $\mathrm{mult}_V(\overline{\mathcal{M}}, x^*)$ the multiplicity of $\overline{\mathcal{M}}$ along V at x^*. Let $\overline{u}_i$ be the residue class of u_i in $\overline{\mathcal{M}}$ and put

$$\overline{\mathcal{M}}_\nu := \mathcal{O}_{T^*\mathbf{C}^n}\overline{u}_1 + \ldots + \mathcal{O}_{T^*\mathbf{C}^n}\overline{u}_\nu \subset \overline{\mathcal{M}}.$$

Then we have

$$\mathrm{mult}_V(\overline{\mathcal{M}}, x^*) = \sum_{\nu=1}^r \mathrm{mult}_V(\overline{\mathcal{M}}_\nu/\overline{\mathcal{M}}_{\nu-1}, x^*).$$

Using the same notation as in Theorem 2, let $\mathcal{I}_\nu$ be the ideal of $\mathcal{O}_{T^*\mathbf{C}^n}$ generated by $\{\sigma(\vec{P})_\nu \mid \vec{P} \in \mathbf{G}_\nu\}$. Then we can show that $\overline{\mathcal{M}}_\nu/\overline{\mathcal{M}}_{\nu-1} = \mathcal{O}_{T^*\mathbf{C}^n}/\mathcal{I}_\nu$. Let J be the maximal ideal of $\mathcal{O}_{T^*\mathbf{C}^n, x^*}$ and put

$$h(k) := \sum_{j=1}^r \dim_{\mathbf{C}} \mathcal{O}_{T^*\mathbf{C}^n, x^*}/((\mathcal{I}_\nu)_{x^*} + J^k)$$

for $k \in \mathbf{N}$. Then $h(k)$ is a polynomial of k for k sufficiently large (Hilbert polynomial). Denote the leading term of $h(k)$ by $ck^d/d!$. Then d is the dimension of V and we have $c = \mathrm{mult}_V(\mathcal{O}_{T^*\mathbf{C}^n}/\mathcal{I}_\nu, x^*)$. This d and c can be obtained by computing a so-called standard basis of $\mathcal{I}_\nu$ introduced by Hironaka. In general, to compute a standard basis is difficult;

however, in our situation, since the ideal is generated by polynomials, we can compute a standard basis by using the Buchberger algorithm and the technique of homogenization as was shown by Lazard.

Once we get the multiplicity of each irreducible component of the characteristic variety of a holonomic system, we can compute, at least in principle, the local index

$$\chi_p(\mathcal{M}) := \sum_i (-1)^i \dim_{\mathbf{C}} \mathcal{E}xt^i_{\mathcal{D}}(\mathcal{M}, \mathcal{O}_{\mathbf{C}^n})_p$$

at a point $p \in \mathbf{C}^n$ by virtue of the index theorem of Kashiwara ([K]), where we regard $\mathcal{M}$ as a left $\mathcal{D}$-module $\mathcal{M} = \mathcal{D}^r/\mathcal{D}\vec{P}_1 + \ldots + \mathcal{D}\vec{P}_s$.

5 Effective computation in $\mathcal{D}$

As is shown so far, in order to compute the characteristic cycle of an algebraic D-module, we do not need to carry out computation in the ring $\mathcal{D}_n$ of differential operators with convergent power series coefficients. However, for some other computations of $\mathcal{D}$-modules such as verifying an isomorphism between two $\mathcal{D}$-modules, computing an induced system of a $\mathcal{D}$-module along a non-characteristic submanifold, the notion of Gröbner basis (or standard basis) for left ideals of the ring $\mathcal{D}_n$ as was formulated by Castro ([C]) seems indispensable. This notion is rather abstract and not sufficiently suited for actual computation. However, if the ideal in question is geretated by elements of the Weyl algebra, then we have an algorithm of computing a Gröbner basis in $\mathcal{D}_n$ via a kind of homogenization ([O2]).

References

[B] Buchberger, B.: Ein algorithmisches Kriterium für die Lösbarkeit eines algebraischen Gleichungssystems. Aequationes Math. **4** (1970), 374–383.

[C] Castro, F.: Calculs effectifs pour les idéaux d'opérateurs différentiels. Travaux en Cours **24** (1987), Hermann, Paris, pp. 1–19.

[G] Galligo, A.: Some algorithmic questions on ideals of differential operators. Lect. Notes Comput. Sci., **204** (1985), 413–421.

[K] Kashiwara, M.: Systems of Microdifferential Equations. Birkhäuser, Boston, 1983.

[O1] Oaku, T.: Computation of the characteristic variety and the singular locus of a system of differential equations with polynomial coefficients. Japan J. Indust. Appl. Math. **11** (1994), 485–497.

[O2] Oaku, T.: Tangent cone algorithm for differential operators. Winter Workshop on Computer Algebra, Tokyo, January 1994, Japan Society for Symbolic and Algebraic Computation, pp. 37–40.

[T] Takayama, N.: Gröbner basis and the problem of contiguous relations. Japan J. Appl. Math. **6** (1989), 147–160.