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RADON TRANSFORM OF HYPERFUNCTIONS AND PROBLEMS OF SUPPORT

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§1.Introduction

In this talk we define the Radon transform of a class of hyperfunctions and discuss their support. The Radon transform was first introduced by J.Radon in [R], where he proved an inversion formula. Later, the Radon transforms turned out to be useful tools for the study of computerized tomography ([SK]), radio astronomy ([BR]) and so on. There are numerous studies of Radon transform in the shoes of applications. Here, however, we are interested in a purely mathematical problem and study this transform from the viewpoint of micro-local analysis and the theory of hyperfunctions. In section 2, we make some definitions, state the support theorems known already, introduce the sketches of their proofs and mention some remarks around them. In section 3, we define the Radon transform for a class of hyperfunctions, demonstrate some examples. Where growth condition on functions plays an important role. In section 4, we show the support theorem for rapidly deceasing Fourier hyperfunctions, which is what we would like to insist here. For the proof of our main theorem, the fact that our Fourier hyperfunctions decay more rapidly than any polynomials of negative power is essential.

Throughout this talk what we would like to claim with stress is that decay conditions to be assigned on functions are important for the definition of the Radon transfrom and for the proofs of global support theorem. Smoothness of functions affects little.

§2.The Radon transform and support theorems

First we introduce the Radon transform

Definition 2.1. Let f be a function on $\boldsymbol{R}^n$ and ξ be a hyperplane. The Radon transform $Rf(\xi)$ of f is

$$Rf(\xi) \equiv \int_\xi f(x)dx, \tag{2.1}$$

where dx is Euclidean measure on ξ.

When $f \in \mathcal{E}'(\boldsymbol{R}^n)$ we can define $Rf(\xi)$ (cf.[He]) and this idea applies to distributions with some decay condition without modifications. Of course we can define the radon transform on d planes $(1 \leq d \leq n-1)$, but in this talk we mean the ones on the hyperplanes by the word the Radon transform. Support theorem established by S.Helgason ([He]) is as follows;

Theorem 2.2. *$f \in C(\boldsymbol{R}^n)$ satisfies*

$$(i) \quad |x|^k|f(x)| < \infty \quad \text{for } \forall k \in \boldsymbol{N},$$
$$(ii) \quad Rf(\xi) = 0 \quad \text{for } d(0,\xi) > A \quad (A > 0),$$

where d denotes the distance. Then

$$f(x) = 0 \quad \text{for } |x| > A.$$

The concept of his proof is as follows; from the assumption that the Radon transforms vanish, we can claim that any integral over spheres which contain $B(0, A)$ inside vanish, furthermore with the condition of rapid decay it turns out that f and any polinomial are orthogonal on such spheres. Therefore we conclude that $f = 0$ outside $B(0, A)$. Helgason's proof holds for rapidly decreasing distributions with some adjustment. We remark that the decay conditon assigned on f cannot be omitted, to our astonishment there exists a continuous function on $\boldsymbol{R}^n$ any line integral of which vanishes but f is not identically zero (cf. [Z]). In 1991 J.Boman detailed this theorem, in order to state his theorem we need some preparations. Let $Z = \{(x, H)|\ H \subset \boldsymbol{R}^n \text{ isahyperplane } x \in H\}$, $f \in C(\boldsymbol{R}^n)$. For $\rho \in A(Z)$, define

$$R_\rho f(\xi) := \int_\xi f(x)\rho(x,\xi)dx,$$

if possible. Now we introduce Boman's theorem ([B2]).

Theorem 2.3. *Let $f \in C(\boldsymbol{R}^n)$ decays fast enough to be integrable over any hyperplanes and decay faster than any negative power of $|x|$ in Γ, $\rho \in A(Z)$ be able to be extended to a real analytic and positive function on Z, K be a compact subset of $\boldsymbol{R}^n$. Assume $R_\rho f(\xi) = 0$ for all ξ not intersecting K. Then $f = 0$ in $\bigcap_{x \in K}(x + (\Gamma \cup (-\Gamma)))$*

As a corollary of this theorem, Theorem 2.2 follows. In Boman's theory, we take f for the one defined on $\boldsymbol{P}^n$ which is difficult in considering f being a distribution or Fourier hyperfunction with decay. Among main tools to prove Theorem 2.3 are Holmgren's uniqueness theorem (Theorem 8.5.6') and a local vanishing theorem for distributions with analytic parameters (cf. [B1]) which reads as

Theorem 2.4. *Let f be a distribution defined in some neighborhood of the real analytic hypersurface $S \subset \boldsymbol{R}^n$,*

$$N^*(S) \cap WF_A(f) = \emptyset$$

and

$$\partial^\alpha f|_S = 0 \text{ forall } \alpha.$$

Then f vanishes in some neighborhood of S.

This theorem holds for non-quasi-analytic ultradistributions ([TT]), which implies Theorem 2.4 is true for ultradistributions decaying faster than any negative power of $|x|$, and the parameter, by which we mean the regularity near S, can be extended to be quasi-analytic ([B3]). We note that though Holmgren's uniqueness theorem holds for hyperfunctions (Theorem 9.6.6 in [Hö]) Theorem 2.4 does not, which requires another vanishing theorem for hyperfunctions for extension (cf. Section 4).

§3 Radon transform for hyperfinctions

The Radon transform for hyperfuncitons was first discussed in [KT] and studied much in detail in [TK]. In [TK], so as to define Radon transform for hyperfunctions we interpret the definition of the Radon transfrom in three ways;

$$\begin{aligned} Rf(\xi) &= \int_\xi f(x)dx \\ &= \int_{\boldsymbol{R}^n} \delta(x \cdot \omega - t) f(x)dx \\ &= \frac{1}{2\pi}\int_{\boldsymbol{R}} e^{its}ds \int_{\boldsymbol{R}^n} e^{-is\omega \cdot x} f(x)dx, \end{aligned}$$

where $\xi = \{x \cdot \omega = t\}, \omega \in S^{n-1}$.

From this interpretation or by duality argument, we can define the class of hyperfunctions for any element of which the Radon transform is well defined. We call this the class of Radon hyperfunctions, which is not a subclass of Fourier hyperfunctions, unfortunately. Fourier hyperfunctions with some decay conditions, however, belong to this class. Here we impose some examples;

Example 3.1. 1) For $f(x) = J(D)\delta(x-a)$, where $a \in \boldsymbol{R}^n$ is a fixed point and $J(D)$ we have

$$Rf(\omega, t) = J(\omega \frac{d}{dt})\delta(t - a\omega).$$

2) Take $f(x) = 1/(x_1 + ix_2)^{m+1}$ in $\boldsymbol{R}^2$. We have

$$Rf(\omega, t) = 2\pi \frac{(-1)^m}{m!2^m} \frac{(\omega_1 - i\omega_2)^m}{\omega_1 + i\omega_2} \delta^{(m-1)}(t).$$

For $m = 0$ we would have

$$R\frac{1}{x_1 + ix_2} = \pi \frac{1}{\omega_1 + i\omega_2} \mathrm{sgn} t.$$

But this is not covered by our present theory.

3) Consider $f(x) = \prod_{j=1}^n (x_j + i\varepsilon)^{-1}$ with $\varepsilon \geq 0$, we obtain

$$Rf(\omega, t) = -(-2\pi i)^{n-1} \left(\frac{\chi(\omega)}{t + i\varepsilon|\omega_1 + \cdots + \omega_n|} - \frac{\chi(-\omega)}{t - i\varepsilon|\omega_1 + \cdots + \omega_n|} \right),$$

where $\chi(\omega)$ denotes the characteristic function of the part of $\boldsymbol{S}^{n-1}$ lying in the first orthant. This is analytic in t for $\varepsilon > 0$ but discontinuous in ω whatever ε may be. By a similar calculus we obtain the Radon transform of $f(x) = \prod_{j=1}^n (x_j + i\varepsilon)^{-m}$:

$$\begin{aligned} Rf(\omega, t) = & - (-2\pi i)^{n-1} \frac{(2m-2)!}{(m-1)!^2} (\omega_1 \cdots \omega_n)^{m-1} \\ & \times \left(\frac{\chi(\omega)}{t + i\varepsilon|\omega_1 + \cdots + \omega_n|} - \frac{\chi(-\omega)}{t - i\varepsilon|\omega_1 + \cdots + \omega_n|} \right). \end{aligned}$$

This becomes more and more regular in ω as m grows.

§4 Support Theorem for rapidly decreasing Fourier hyperfunctions

In order to state our main theorem, we define Rapidly decreasing Fourier hyperfunctions. For $U \subset \boldsymbol{D}^n + i\boldsymbol{R}^n$ let

$$\mathcal{O}^{(-\infty)}(U) = \{f(z) \in \mathcal{O}(U \cap \boldsymbol{C}^n | \forall K \subset\subset U, \forall m \in \boldsymbol{N}, \sup_{z \in K \cap \boldsymbol{C}^n} |\mathrm{Re} z|^m |f(z)| < \infty\}.$$

Here, $K \subset\subset U$ means that the closure of K in $\boldsymbol{D}^n$ is included in U. We denote the corresponding sheaf on $\boldsymbol{D}^n + i\boldsymbol{R}^n$ by $\mathcal{O}^{(-\infty)}$. The sheaf of rapidly decreasing Fourier hyperfunctions is defined as $\mathcal{H}^n_{\boldsymbol{D}^n}(\mathcal{O}^{(-\infty)})$. Remark that by the term "rapidly decreasing" we mean the decay faster than any negative power of $|x|$ at infinity, which is different use from the usual one for Fourier hyperfunctions.

Support theorem implies the uniqueness of the exterior problem;

Theorem 4.1. *Let $f(x)$ be a rapidly decreasing Fourier hyperfunction. Assume that $Rf(\omega, t)$ vanishes for $|t| \geq A$. Then $f(x)$ vanishes on $|x| \geq A$.*

In [TK], we have extended both proofs by S.Helgason and J.Boman to verify this theorem.

To extend Helgason's proof to rapidly decreasing Fourier hyperfunctions we apply some uniqueness theorem for hyperfunctions containing analytic paremeters (Theorem 4.4.7 in [K]).

Enlarging Boman's certification makes us be in much trouble. First trouble is the interpretation of the Radon transform over hyperplane at infinity when f is regarded the one on $\boldsymbol{P}^n$. In Theorem 2.3, since f is continuous we can extend $f = 0$ at infinity naturally. But here the problem is not so easy, even among exponentially decreasing Fourier hyperfunctions, by which we mean rapidly decraesing ones as usual, there exists the one whose support intersects the sphere at infinity (cf. Example 8.4.6 in [K]). However, the assumption of the Radon transform enable us exclude such critical case when f is of rapidly decay.

The second trouble is, as was mentioned in section 2, a local vanishing theorem (Theorem 2.4) does not apply for hyperfunctions (cf. Note 3.3 in [K]). Fortunately, in this casewe can apply Theorem 4.4.5 in [K] in place of Theorem 2.4.

We would be very happy if we could extend Theorem 2.4 to hyperfunctions, for which we have some problems left to be solved.

First, in this case, f does dot decrease rapidly outside the cone Γ, the interpretation at infinity is not such a simple one.

In this case rapidly decreasing does not seem enough. We know exponontially decay implies this extension if the first problem is cleared.

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