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# On the Faddeev-Newton Equation in the Inverse Scattering Theory

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Let us imagine a quantum mechanical collision process. Given a target, you project another particle in the form of a beam of one particle par unit time from the $\omega$-direction. By observing the scattered particles, you try to understand the target.

What is the mathematical back ground of this experiment? Let us consider the Schrödinger equation in $\mathbf{R}^3$

$$(-\triangle+V)\varphi=E\varphi, \quad E>0. \tag{1}$$

We seek a solution of this equation in the following form

$$\varphi(x)\sim e^{i\sqrt{E}\omega\cdot x}+\frac{e^{i\sqrt{E}r}}{r}f(E,\theta,\omega), \quad r=|x|\to\infty, \tag{2}$$

where $\theta=x/r$. The first term of the right-hand side represents the incident wave and the second term represents the scattered wave. Letting

$$v=\varphi-e^{i\sqrt{E}\omega\cdot x},$$

we have the following equation for $v$

$$(-\triangle+V-E)v=-V(x)e^{i\sqrt{E}\omega\cdot x},$$

and $v$ is written by the resolvent

$$v=-(-\triangle+V-E-i0)^{-1}(V(x)e^{i\sqrt{E}\omega\cdot x}).$$

This $v$ is shown to have the asymptotic expansion given in (2). Up to a multiplicative constant, $f(E,\theta,\omega)$ has the following representation

$$\begin{aligned} f(E,\theta,\omega) &= \int e^{-i\sqrt{E}(\theta-\omega)\cdot x}V(x)dx \\ &\quad -\int e^{-i\sqrt{E}\theta\cdot x}V(x)(-\triangle+V-E-i0)^{-1}(V(\cdot)e^{i\sqrt{E}\omega\cdot})dx. \end{aligned}$$

$f(E,\theta,\omega)$ is called the scattering amplitude and $|f(E,\theta,\omega)|^2$ represents the number of particles reflected to the $\theta$-direction par unit time. The problem is now formulated as follows.

*Problem. Construct the potential $V$ in terms of $f(E,\theta,\omega)$.*

This is the inverse problem of scattering.

In the 50's, this problem was solved for the 1-dimensional case by the celebrated theory of Gel'fand-Levitan-Marchenko. They found an algorithm to construct the potential by means of the scattering amplitude. The core of their theory lies in the following obsevation. Let $\varphi$ be the solution of the Cauchy problem

$$\begin{cases} -\varphi'' + V\varphi = k^2\varphi, & x > 0, \\ \varphi(0) = 0, \quad \varphi'(0) = 1. \end{cases}$$

Then $\varphi$ behaves like

$$\varphi(x,k) = \frac{1}{k}\sin kx(1 + o(1)), \quad |k| \to \infty.$$

This $\varphi$ is an even function of $k$ and entire in $\mathbf{C}$. By the Paley-Wiener theorem, we then have the following representation

$$\varphi(x,k) = \frac{1}{k}\sin kx - \int_0^x K(x,y)\frac{1}{k}\sin kydy.$$

Moreover $K$ satisfies the wave equation

$$(\partial_y^2 - \partial_x^2 + V(x))K(x,y) = 0$$

with a condition on the characteristic

$$-2\frac{d}{dx}K(x,x) = V(x).$$

In this way, the potential $V$ is related with the solution to the Schrödinger equation.

The first rigorous result for the 3-dimensional inverse problem appeared as the Born approximation at high energies. Namely for any non zero vector $\xi \in \mathbf{R}^3$, by taking a series of vectors $\theta, \omega \in S^2$ and energies $E > 0$ tending to infinity in such a way that $\sqrt{E}(\theta - \omega) \to \xi$, one can show that

$$f(E,\theta,\omega) \to \widehat{V}(\xi).$$

Therefore one can recover the potential from the high energy behavior of the scattering amplitude. Although this result clearifies the peculiarity of the 3-dimensional inverse problem, it is not regarded as satisfactory because it is far from the characterization of the scattering amplitude and also the Schrödinger equation is not adapted to high energy physics.

From the middle of 60's, L.D.Faddeev began to publish articles for solving the 3-dimensional inverse problem. His idea consists in introducing a new Green's function different from the classical one. Using this Green's function, Faddeev and, shortly after, R.G.Newton proposed a scheme for investigating the 3-dimensional inverse problem. Let us explain the basic idea.

For a solution $\varphi$ of the Schrödinger equation

$$(-\Delta + V - E)\varphi = 0,$$

let $\psi = e^{-ik\cdot x}\varphi$, $k^2 = E$. Then $\psi$ satisfies

$$(-\Delta - 2ik\cdot\nabla + V)\psi = 0.$$

Letting $\psi = 1 + v$, we have the following equation

$$(-\Delta - 2ik\cdot\nabla + V)v = -V(x)\cdot 1.$$

Now the idea is to complexify the wave vector $k$. Choose an arbitrary direction $\gamma \in S^2$ and a vector $\eta \in \mathbf{R}^3$ such that $\eta\cdot\gamma = 0$. Let

$$\zeta = \eta + z\gamma, \quad z \in \overline{\mathbf{C}_+}.$$

Let us consider the equation

$$(-\Delta - 2i\zeta\cdot\nabla + V)u = f, \tag{3}$$

which we call the Faddeev-Newton equation. If the potential $V$ is absent, this equation has a solution written by the Fourier transformation

$$u(x) = (2\pi)^{-3}\int \frac{e^{ix\cdot\xi}}{\xi^2 + 2\zeta\cdot\xi}\hat{f}(\xi)d\xi \equiv \widetilde{G}(\zeta)f. \tag{4}$$

This defines the Green's function of Faddeev. Then the solution of the equation (3) is given by

$$u = (1 + \widetilde{G}(\zeta)V)^{-1}\widetilde{G}(\zeta)f.$$

Returning to our problem, we obtain a solution of the Schrödinger equation having the following form

$$\varphi = e^{ik\cdot x} + e^{ik\cdot x}(1 + \widetilde{G}(k)V)^{-1}\widetilde{G}(k)f,$$
$$f = -V(x)\cdot 1, \quad k = \eta + s\gamma, \quad s \in \mathbf{R}.$$

Now as a function of $z$, the operator $\widetilde{G}(\eta + z\gamma)$ is analytic in $\mathbf{C}_+$. Therefore, if everything works well, by the Paley-Wiener theorem, we have the following representation

$$\varphi = e^{ik\cdot x} - \int_{\gamma\cdot x}^{\infty} K_\gamma(x,\eta,t)e^{ist}dt.$$

Moreover, $K_\gamma$ satisfies the wave equation

$$(\Delta_x - \frac{\partial^2}{\partial t^2} + \eta^2)K_\gamma = V(x)K_\gamma.$$

Letting $x_1 = \gamma \cdot x, x = (x_1, x')$, we also have the following condition on the characteristic

$$2\frac{\partial}{\partial x_1} K_\gamma(x, \eta, x_1) = V(x) e^{i\eta \cdot x'}.$$

This corresponds to the heart of the 1-dimensional inverse scattering theory of Gel'fand-Levitan-Marchenko. On the basis of these observations, Faddeev-Newton proposed a scheme of the construction of the potential from the given scattering amplitude and they succeeded in establishing an astonishing analogy between the 1-dimensional and the 3-dimensional cases.

The chracteristic feature of the 3-dimensional inverse scattering theory is its overdeterminancy. In the 1-dimensional case, both of the potential and the scattering amplitude are the functions of one variable. However, in the 3-dimensional case, the scatteing amplitude consists of five variables, while the potential is a function of three variables. In order that a function $f(E, \theta, \omega)$ be the sacttering amplitude associated with a potential $V(x)$, it must satisfy some compatibility conditions, to find which is the main difficulty of the multi-dimensional inverse problem. A shortcoming of the Faddeev-Newton theory is that their characterization of the scattering amplitude contains some misterious condition which is too big as an assumption.

An important progress was then brought by Nachman-Ablowitz and Khenkin-Novikov by using the $\overline{\partial}$-equation, which enables us to characterize completely the scattering amplitude introduced by Faddeev. Note that this is not the physical scattering amplitude obtained by solving the Lippman-Schwinger equation but is written by the Green's function of Faddeev. Khenkin-Novikov also derived a remarkable result of reconstructing the small potential from the scattering amplitude at fixed energy. Nachman also showed a beautiful application to the inverse eigenvalue problem.

Because of its importance in the inverse scattering theory, it remains a worthwhile pursuit to study the detailed properties of the Faddeev-Newton equation. In the works of Faddeev and Newton, an explicit representation of the Green's function plays an important role. In our approach, we characterize the Green's function of Faddeev by pseudo-differential operators. We introduce a radiation condition and the uniqueness theorem for the Green's function of Faddeev formulated in terms of Ps.D.Op.'s. This will make possible to extend the theory to other operators such as higher order elliptic operators or 1st order systems. Our next aim is the reconstruction of the potential from finite energies and its integral representation. When the potential decays sufficiently rapidly, this problem has already been discussed by Newton and Khenkin-Novikov. Since this is an important subject, we extend it to slowly decreasing potentials in this work.

More precisely, we consider the following equation in $\mathbf{R}^n, n \geq 2$:

$$(-\triangle - 2iz\gamma \cdot \nabla - \lambda^2)u = f, \tag{5}$$

where $\gamma \in S^{n-1}, z \in \overline{\mathbf{C}_+}, \lambda > 0$. This equation is related with the Green's function of Faddeev in the following way:

$$e^{-ix\cdot\eta} \int \frac{e^{ix\cdot\xi}}{\xi^2 + 2z\gamma \cdot \xi - \lambda^2} d\xi = \int \frac{e^{ix\cdot\xi}}{\xi^2 + 2(\eta + z\gamma) \cdot \xi} d\xi,$$

where $\eta \in \mathbf{R}^n$ and satisfies $\eta^2 = \lambda^2, \eta \cdot \gamma = 0$.

We introduce an arbitrary direction $\gamma \in S^{n-1}$. For $z \in \overline{\mathbf{C}_+}$, let

$$\zeta = z\gamma = \zeta_R + i\zeta_I.$$

For $x \in \mathbf{R}^n$, let

$$x_\gamma = \gamma \cdot x, \quad x_\perp = x - (\gamma \cdot x)\gamma,$$
$$X = < x >, \quad X_\perp = < x_\perp > .$$

We need the following function spaces. For $s \in \mathbf{R}$, we define

$$u \in L^{2,s} \Longleftrightarrow \|u\|_s = \|X^s u\|_{L^2(\mathbf{R}^n)} < \infty,$$

$$u \in L^{2,s}_\perp \Longleftrightarrow \|u\|_{\perp,s} = \|X^s_\perp u\|_{L^2(\mathbf{R}^n)} < \infty.$$

Since our localization procedure for the Faddeev-Newton equation is rather involved, it will be helpful to explain the basic strategy. For a constant $\lambda > 0$, we devide the $\xi$-space into three regions.

$$I = \begin{cases} (\xi + \zeta_R)^2 \leq \frac{3}{4}\lambda^2 + (\mathrm{Re}\ z)^2, \\ \text{or} \\ (\xi + \zeta_R)^2 \geq \frac{5}{4}\lambda^2 + (\mathrm{Re}\ z)^2, \end{cases}$$

$$II = \begin{cases} \frac{1}{2}\lambda^2 + (\mathrm{Re}\ z)^2 \leq (\xi + \zeta_R)^2 \leq \frac{3}{2}\lambda^2 + (\mathrm{Re}\ z)^2, \\ \text{and} \\ (\xi_\gamma + \mathrm{Re}\ z)^2 \geq \frac{1}{8}\lambda^2 + (\mathrm{Re}\ z)^2, \end{cases}$$

$$III = \begin{cases} \frac{1}{2}\lambda^2 + (\mathrm{Re}\ z)^2 \leq (\xi + \zeta_R)^2 \leq \frac{3}{2}\lambda^2 + (\mathrm{Re}\ z)^2, \\ \text{and} \\ (\xi_\gamma + \mathrm{Re}\ z)^2 \leq \frac{1}{4}\lambda^2 + (\mathrm{Re}\ z)^2. \end{cases}$$

For the case $I$, we have

$$|\xi^2 + 2\zeta_R \cdot \xi - \lambda^2| \geq \frac{1}{4}\lambda^2.$$

Therefore $(h_0(\xi, \zeta) - \lambda^2)^{-1}$ makes sense.

On the complement of the region $I$, the sign of $\xi_\gamma$ plays an important role. For the case $II$, we have

$$|\xi_\gamma| \geq \delta_1 > 0$$

for a constant $\delta_1 > 0$, if $\lambda$ and Re$z$ vary over compact sets in $\{\lambda > 0\}$ and $\mathbf{R}$ respectively. Therefore one can adopt a smooth localization with respect to $\xi_\gamma$. For the case $III$, one must adopt a sharp cut-off function with respect to $\xi_\gamma$. However, since

$$|\xi_\perp|^2 \geq \frac{1}{4}\lambda^2,$$

one can employ the localization with respect to the angle between $x_\perp$ and $\xi_\perp$. Let $M_\pm$ be the operator defined by

$$M_\pm u = \mathcal{F}^{-1}(F(\pm\gamma \cdot \xi \geq 0)\hat{u}(\xi)),$$

where $\mathcal{F}$ denotes the Fourier transformation and $F(\pm\gamma \cdot \xi \geq 0)$ is the characteristic function of the set $\{\xi; \pm\gamma \cdot \xi \geq 0\}$.

**DEFINITION** $\mathcal{L}_1^{(\pm)}$ *is the set of operators of the form* $L_1^{(\pm)} = \rho_1(\gamma \cdot D_x)M_\pm$, *where*

$$\rho_1(\xi_\gamma) = 1 \quad \text{if} \quad (\xi_\gamma + \mathrm{Re}\ z)^2 > \frac{1}{4}\lambda^2 + (\mathrm{Re}\ z)^2,$$

$$\rho_1(\xi_\gamma) = 0 \quad \text{if} \quad (\xi_\gamma + \mathrm{Re}\ z)^2 < \frac{1}{8}\lambda^2 + (\mathrm{Re}\ z)^2.$$

$\mathcal{L}_0^{(\pm)}$ *is the set of operators of the form* $L_0^{(\pm)} = \rho_0(\gamma \cdot D_x)M_\pm$, *where*

$$\rho_0(\xi_\gamma) = 1 \quad \text{if} \quad (\xi_\gamma + \mathrm{Re}\ z)^2 < \frac{1}{8}\lambda^2 + (\mathrm{Re}\ z)^2,$$

$$\rho_0(\xi_\gamma) = 0 \quad \text{if} \quad (\xi_\gamma + \mathrm{Re}\ z)^2 > \frac{1}{4}\lambda^2 + (\mathrm{Re}\ z)^2.$$

We introduce two classes of Ps.D.Op.'s associated with the cases $II$ and $III$. Let $\sigma$ be a constant such that $0 < \sigma < 1$.

**DEFINITION** $S^{(\pm)}(\zeta, \sigma)$ *be the set of functions* $p^{(\pm)}(x, \xi)$ *having the following properties.*

*(1)* $p^{(\pm)}(x, \xi) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$.

*(2)* $|\partial_x^\alpha \partial_\xi^\beta\, p^{(\pm)}(x, \xi)| \le C_{\alpha\beta} < x >^{-|\alpha|}, \quad \forall \alpha, \beta.$

*(3)* $supp_\xi\, p^{(\pm)}(x, \xi) \subset \{\xi; \frac{1}{2}\lambda^2 + (\mathrm{Re}\ z)^2 \le (\xi + \zeta_R)^2 \le \frac{3}{2}\lambda^2 + (\mathrm{Re}\ z)^2\}$.

*(4)* $-\sigma < \inf_{x,\xi} \pm \theta(x, \xi + \zeta_R)$ *on supp* $p^{(\pm)}(x, \xi)$, *where*

$$\theta(x, \xi) = \frac{x}{< x >} \cdot \frac{\xi}{|\xi|}.$$

**DEFINITION** $S_\perp^{(\pm)}(\zeta, \sigma)$ *be the set of functions* $p_\perp^{(\pm)}(x_\perp, \xi_\perp)$ *having the following properties.*

*(1)* $p_\perp^{(\pm)}(x_\perp, \xi_\perp) \in C^\infty(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1})$.

*(2)* $|\partial_{x_\perp}^\alpha \partial_{\xi_\perp}^\beta\, p_\perp^{(\pm)}(x_\perp, \xi_\perp)| \le C_{\alpha\beta} < x_\perp >^{-|\alpha|}, \quad \forall \alpha, \beta.$

*(3)* $supp_{\xi_\perp}\, p_\perp^{(\pm)}(x_\perp, \xi_\perp) \subset \{\xi_\perp; \frac{1}{4}\lambda^2 \le |\xi_\perp|^2 \le \frac{3}{2}\lambda^2 + (\mathrm{Re}\ z)^2\}$.

*(4)* $-\sigma < \inf_{x_\perp, \xi_\perp} \pm \theta(x_\perp, \xi_\perp)$ *on supp* $p_\perp^{(\pm)}(x_\perp, \xi_\perp)$.

We now introduce a class of functions satisfying the modified radiation condition of the pseudo-differential form. Suppose that $V$ satisfies

$$|V(x)| \le C(1 + |x|)^{-1-\delta_0}, \quad \delta_0 > 0.$$

One can also allow certain mild local sigularities for $V$. Let

$$s_0 = \min(\frac{3}{4}, \frac{1+\delta_0}{2}).$$

**DEFINITION** *Let* $1/2 < s \leq s_0$. $\mathcal{R}_\gamma^s$ *is the set of functions* $u \in H^2_{loc} \cap \mathcal{S}'$ *such that*
*(1)* $L_1^{(\pm)} u \in L^{2,-s}, \quad \forall L_1^{(\pm)} \in \mathcal{L}_1^{(\pm)}$,
*(2)* $L_0^{(\pm)} u \in L_\perp^{2,-s}, \quad \forall L_0^{(\pm)} \in \mathcal{L}_0^{(\pm)}$,
*and there exists a constant* $0 < \sigma < 1$ *such that*
*(3)* $P^{(\pm)} L_1^{(\pm)} u \in L^{2,s-1}, \quad \forall P^{(\pm)} \in S^{(\pm)}(\zeta, \sigma)$,
*(2)* $P_\perp^{(\pm)} L_0^{(\pm)} u \in L_\perp^{2,s-1}, \quad \forall P_\perp^{(\pm)} \in S_\perp^{(\pm)}(\zeta, \sigma)$.

Note that $u \in \mathcal{R}_\gamma^s$ implies $u \in L^{2,-s}$.

One can then show that if $u$ is a solution to the Faddeev-Newton equation

$$(-\Delta - 2iz\gamma\cdot\nabla - \lambda^2)u = f,$$

$u \in \mathcal{R}_\gamma^s$ if and only if

$$u(x) = (2\pi)^{-n} \int \frac{e^{ix\cdot\xi}}{\xi^2 + 2z\gamma\cdot\xi - \lambda^2} \hat{f}(\xi) d\xi.$$

This theorem can be used to derive an equation between the Faddeev resolvent $(-\Delta - 2iz\gamma\cdot\nabla - \lambda^2)^{-1}$ and the usual ones $(-\Delta - E \pm i0)^{-1}$, which is the key to the Faddeev-Newton approach to the inverse scattering theory. This characterization theorem also holds for the operators with short-range potentials. Let us define the set of exceptional points by

$$\mathcal{E}(\lambda, \gamma) = \{z \in \overline{\mathbf{C}}_+; -1 \in \sigma_p(G_{\gamma,0}(\lambda, z)V)\}.$$

Then $\mathcal{E}(\lambda,\gamma) \cap \mathbf{C}_+$ is discrete and $\mathcal{E}(\lambda,\gamma) \cap \mathbf{R}$ is a closed null set. One can then show that for $\lambda > 0$ and $z \in \overline{\mathbf{C}}_+$, $z \in \mathcal{E}(\lambda,\gamma)$ if and only if there exists $u \in \mathcal{R}_\gamma^s$ such that $u \neq 0$ and

$$(-\Delta - 2iz\gamma\cdot\nabla + V - \lambda^2)u = 0.$$

We next assume that the potential $V(x)$ satisfies the following assumption:

$$|\partial^\alpha V(x)| \leq C(1+|x|)^{-3/2-\epsilon-|\alpha|}, \quad |\alpha| \leq n-1$$

for some $\epsilon > 0$. Suppose we are given a set of positive measure $S$ on $\mathbf{R}$ such that $a = \inf S > 0$ and that we are given the scattering amplitude $A(E)$ for $E \in S$. One can then reconstruct $\widehat{V}(\xi)$ for $|\xi| < 2\sqrt{a}$. Moreover if $S$ is a half line : $S = [E_0, \infty)$, for sufficiently large $\lambda > 0$ and $\omega, \omega' \in S^{n-1}$ such that $\omega \neq \omega', \omega\cdot\gamma = \omega'\cdot\gamma = 0$, one can construct a function $\mathcal{C}_\gamma(\lambda, t; \omega, \omega')$ such that

$$\widehat{V}(\lambda(\omega - \omega')) = \mathcal{C}_\gamma(\lambda, t_0; \omega, \omega') + \text{p.v.}\frac{i}{\pi}\int_{-\infty}^{\infty} \frac{\mathcal{C}_\gamma(\lambda, t; \omega, \omega')}{t - t_0} dt$$

holds for any $t_0 \in \mathbf{R}$.

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