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Dimensions of Almost Periodic Trajectories for Nonlinear Evolution Equations

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1. Introduction

Let X be a Banach space with its norm denoted by $|\cdot|$. Consider the initial value problem

$$\frac{du}{dt} + A(t)u \ni f(t), \quad s \leq t,$$
$$u(s) = x \tag{1.1}$$

for an X-valued function u, where, for each $t \in R$, $A(t)$ is a nonlinear (possibly multivalued) operator on X. When periodicity or almost periodicity is assumed on the nonlinear operator $A(t)$ and on the forcing term $f(t)$, the existence theorems for the almost periodic solutions have been obtained by many authors (cf. [1], [5], [6], [7]). The purpose of this paper is to estimate the fractal dimensions of the almost periodic trajectories by applying the method in our previous paper [10].

It is well known (cf. [11]) that periodic or almost periodic states occupy the important positions as main gateways in various routes to chaos. When some chaotic or unpredictable behaviors are observed in these actual cases, one cannot expect smoothness for the time- variation of solutions. Here we assume Hölder continuity with its exponent less than one on the periodic nonlinear term and on the forcing term. Then we estimate the dimension of the orbits of the solution by using these exponents. In 2-frequency quasiperiodic case where $A(t+\alpha) = A(t), f(t+1) = f(t)$, α is an irrational real number, by using Diophantine approximation, we can show that the fractal dimension of the orbit of the quasiperiodic solution (exactly, a complete trajectory) is majorized by $1/\delta_1 + 1/\delta_2$ where δ_1, δ_2 are the exponents of Hölder continuity on $A(t), f(t)$, respectively.

Our plan of this paper is as follows: In the present section below we introduce the existence theorem in [5] or [7] for the complete trajectories which are almost periodic and we also introduce our previous result in [10] on fractal dimensions of almost periodic trajectories. In section 2 we remark the assumptions for the existence of the complete trajectory. In section 3 we give the estimate of the dimensions in the 2-frequency case by using Diophantine approximations and in section 4, we investigate a semilinear parabolic equation as an example.

For the moment, assume that problem (1.1) admits a unique solution on $[s, +\infty)$ for every $x \in X$ and $s \in R$. Define the two-parameter family of operators

$$U(s,\tau) : X \to X, \quad s \in R, \quad \tau \in R^+ := [0, +\infty)$$

by $U(s,\tau)x = u(s+\tau)$, then we can obtain the following relations from the uniqueness of solutions.

(u-i) $U(s,0) = I$ (the identity operator), $s \in R$

(u-ii) $U(s, \sigma+\tau) = U(s+\tau, \sigma)U(s,\tau) \quad s \in R, \quad \sigma, \tau \in R^+$.

Under suitable conditions on $A(t)$ and $f(t)$, we can consider the following continuity property on $U(s,\tau)$:

(u-iii) For any fixed $\tau \in R^+$, the one-parameter family of maps $U(s,\tau) : X \to X$ with the parameters $s \in R$ is equicontinuous.

A two-parameter family of the operators on X, which satisfies (u-i), (u-ii) and (u-iii), is called a process or an evolutionary operator (cf. [5] or [4]).

Definitions. The positive trajectory through $(s,x) \in R \times X$ is the map $U(s,\cdot)x : R^+ \to X$. A complete trajectory through (s,x) is a function $u(\cdot) : R \to X$ such that $u(s) = x$ and $u(t+\tau) = U(t,\tau)u(t)$ for all $(t,\tau) \in R \times R^+$.

Define the σ-translate U_σ by $U_\sigma(s,\tau) = U(s+\sigma,\tau)$, then a process U on X is called almost periodic if for any sequence $\{\sigma_n\}$ of R, there exists a subsequence $\{\sigma_n'\}$ of $\{\sigma_n\}$ such that the sequence $\{U_{\sigma_n'}(s,\tau)x\}$ converges to some $V(s,\tau)x$ in X uniformly in $s \in R$ and pointwise $(\tau, x) \in R^+ \times X$. We denote by $\mathcal{H}(U)$ the set of all processes V on X for which there exists a sequence $\{\sigma_n\}$ of R such that

$$U_{\sigma_n}(s,\tau)x \to V(s,\tau)x, \quad \text{uniformly } s \in R \text{ and pointwise in } (\tau,x) \in R^+ \times X.$$

A continuous function $u : R \to X$ is called almost periodic if for any sequence $\{\sigma_n\}$ of R, there exists a subsequence $\{\sigma_n'\}$ of $\{\sigma_n\}$ such that the sequence $\{u(t+\sigma_n')\}$ is uniformly convergent on R. The following equivalent definition is well known (cf. [1]). $u : \mathcal{R} \to X$ is almost periodic if for each $\varepsilon > 0$ there exists $l_\varepsilon > 0$ such that for every $a \in \mathcal{R}$ there exists an point $p \in [a, a+l_\varepsilon]$ with the property

$$|u(t+p) - u(t)| \le \varepsilon \quad \text{for all } t \in \mathcal{R}. \tag{1.2}$$

Here the point p is called an ε-almost period and l_ε is called an inclusion length for ε-almost period.

The existence theorems for the complete trajectories which are almost periodic have been given in [5] or [7]:

Theorem A. *Let U be an almost periodic process, which satisfies*

$$|U(s,\tau)x - U(s,\tau)y| < |x-y|$$

for $s \in R$, $\tau > 0$, $x \neq y$, that is, U is strictly contractive. Assume that for any $V \in \mathcal{H}(U)$ and $(s,x) \in R \times X$, the positive trajectory of V through $(s,x) \in R \times X$, is continuous on R^+ and assume that there exists a $(s_0, x_0) \in R \times X$ such that $\overline{co}\{U(s_0,\tau)x_0 : \tau \in R^+\}$ is compact. Then, for any $V \in \mathcal{H}(U)$, there exists a unique complete trajectory of V which is almost periodic.

Next we introduce our previous results in [10] on fractal dimensions of almost periodic trajectories. Let Σ be a subset of X and let $N_\Sigma(\varepsilon)$, $\varepsilon > 0$, denote the minimum number of balls of X with radius ε which is necessary to cover a subset Σ of X. The fractal dimension of Σ, which is also called the capacity of Σ, is the number

$$\mathcal{D}_F(\Sigma) = \limsup_{\varepsilon \to 0} \frac{\log N_\Sigma(\varepsilon)}{\log 1/\varepsilon}.$$

Theorem B. *Let $u(t) : R \to X$ be almost periodic and Hölder continuous with exponent $\delta : 0 < \delta < 1$, that is,*

$$\sup_{t \neq s} \frac{|u(t) - u(s)|}{|t-s|^\delta} < \infty.$$

If the inclusion length for ε-almost period of $u(t)$ satisfies

$$l_\varepsilon \leq K\varepsilon^{-\vartheta} \tag{1.3}$$

for some $K > 0$ and $\vartheta > 0$, then the fractal dimension of its orbit $\Sigma := \cup_{t \in \mathcal{R}} u(t)$ satisfies the following estimate

$$\mathcal{D}_F(\Sigma) \leq \vartheta + \frac{1}{\delta}. \tag{1.4}$$

Remark 1. It is sufficient to assume locally Hölder continuity in the following sense. For a small $\varepsilon_0 > 0$ there exists $c_0 > 0$ such that

$$|u(t) - u(s)| \leq c_0 |t-s|^\delta$$

for every $t, s \in R : |t-s| \leq \varepsilon_0 << 1$ (cf. [10]).

2. Accretive operators

In this section we specify the assumptions on the nonlinear operator $A(t)$ for the existence of the complete trajectories which are Hölder continuous. To identify the terminology with the several known results (cf. [4] or [8]) we put $\mathbf{A}(t)u := A(t)u - f(t)$. Thus we consider

$$\frac{du}{dt} + \mathbf{A}(t)u \ni 0, \quad u(s) = x. \tag{2.1}$$

We assume that the domain of $\mathbf{A}(t)$ is t-independent:
(A1) $D(\mathbf{A}(t)) = D, \quad t \in R.$
For each $t \in R, \mathbf{A}(t)$ is an m-accretive operator ($-\omega$-type):
(A2) There exists $\omega \in R$ such that

$$< u - v, x - y >_s \geq \omega |x - y|^2, \quad [x, u], [y, v] \in \mathbf{A}(t) \subset X \times X,$$

where for $w, z \in X$

$$< w, z >_s = \sup\{(w, z^*) : z^* \in F(z)\}, \quad F(z) = \{z^* \in X^* : (z, z^*) = |z|^2 = |z^*|_*^2\}.$$

(A3) $\mathcal{R}(I + \lambda \mathbf{A}(t)) = X, \quad t \in R, \quad \lambda > 0.$

Since the Hölder continuity is essential to determine the dimensions of trajectories, we need the following condition on the resolvent $J_\lambda(t) := (I + \lambda \mathbf{A}(t))^{-1}, \ \lambda > 0$:
(A4) There exists a constant $0 < \delta < 1$ and a monotone increasing function $l : R^+ \to R^+$ which satisfies

$$|J_\lambda(t)x - J_\lambda(\tau)x| \leq \lambda l(|x|)|t - \tau|^\delta$$

for $t, \tau \in R, \quad x \in X$.

The conditions above are sufficient to construct the associated two-parameter family of operators $U(s, \tau) : X \to X, \quad (s, \tau) \in R \times R^+$, which also satisfies the following properties (cf. theorem 3.2, proposition 2.1 in [4]):
(u-iv) There exists a monotone increasing function $k : R^+ \to R^+$ which satisfies

$$|U(s, \tau)x - U(s, \sigma)x| \leq k(|x|)|\tau - \sigma|^\delta.$$

(u-v) Given $s \in R, x \in X$, define a Hölder continuous function $u : [s, +\infty) \to X$ by

$$u(s + \sigma) = U(s, \sigma)u(s), \quad u(s) = x$$

and let $y_0 \in \mathbf{A}(s)x_0$ and $y(t) \in \mathbf{A}(t)x_0, t \geq s$, such that $y(s) = y_0$ and $y(t)$ is continuous, then $u(t)$ satisfies

$$\begin{aligned}&\int_0^\sigma < y(s + \tau) - \omega(x_0 - u(s + \tau)), x_0 - u(s + \tau) >_s d\tau \\ &\qquad \geq \frac{1}{2}(|x_0 - u(s + \sigma)|^2 - |x_0 - u(s)|^2) \geq (u(s) - u(s + \sigma), \xi^*)\end{aligned} \tag{2.2}$$

for every $\xi^* \in F(x_0 - u(s))$ (where the second inequality is obvious).

For our purpose we further assume the following conditions.

(U) For each $s \in R, x \in X$, $u(t)$ in (u-v), which is called an integral solution, is weakly differentiable and $|u(t)|$ is differentiable.

(A5) $\omega > 0$ in the condition **(A2)**, that is, $\mathbf{A}(t)$ is uniformly accretive for each $t \in R$.

Under the assumption **(U)**, dividing (2.2) by σ, taking the limit $\sigma \downarrow 0$, we have

$$< y_0 - \omega(x_0 - u(s)), x_0 - u(s) >_s \geq -(u', \xi^*)$$

for every $[x_0, y_0] \in \mathbf{A}(s)$ where u' is the weak derivative of $u(t)$ at $t = s$. Since m-accretivity implies maximal accretivity, we obtain $-u'(s) \in \mathbf{A}(s)u(s)$. If X is a Hilbert space and $u(t) \in D$, the condition **(U)** is satisfied for the weak solutions (cf. [2]).

Since **(A5)** yields relative compactness of each positive trajectory (for instance, see the proof of Lemma 5.3 in [5]) and the process $U(s, \tau)$ is strictly contractive (cf. Theorem 2.1 in [4]), it follows from Theorem A that there exists a unique complete trajectory. In the following sections we treat the complete trajectory $u(t)$.

3. 2-frequency quasiperiodic case

In this section we assume the periodicity

$$A(t + \alpha) = A(t), \quad f(t + 1) = f(t), \quad \text{for every } t \in R$$

where α is an irrational real number: $0 < \alpha < 1$. To estimate the dimension of the orbit Σ of the complete trajectory $u(t)$ we use Diophantine approximations.

Consider the following continued fraction of the number α:

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}} \qquad (a_i \in \mathbf{N}) \tag{3.1}$$

and take the rational approximation as follows. Let $m_0 = 1, n_0 = 0, m_{-1} = 0, n_{-1} = 1$ and define the pair of sequences of natural numbers

$$m_i = a_i m_{i-1} + m_{i-2},$$
$$n_i = a_i n_{i-1} + n_{i-2}, \quad i \geq 1,$$

then the elementary number theory gives the Diophantine approximation

$$|\alpha - \frac{n_i}{m_i}| < \frac{1}{m_i m_{i+1}} < \frac{1}{m_i^2}. \tag{3.2}$$

In view of **(A4)**, we assume Hölder continuity on the operator $A(t)$ and the forcing function $f(t)$ with exponents $\delta_1, \delta_2 : 0 < \delta_1, \delta_2 < 1$, respectively.

(A6-i) There exist a monotone increasing function $L_1 : R^+ \to R^+$ such that, if $t, \tau \in R, x \in D$ and $y \in A(\tau)x$, then there exists $w \in A(t)x$ which satisfies

$$|w - y| \leq L_1(|x|)|t - \tau|^{\delta_1},$$

(A6-ii) There exists a positive constant L_2 such that

$$|f(t) - f(\tau)| \leq L_2|t - \tau|^{\delta_2}, \quad t, \tau \in R.$$

(A7) $A(t)0 \ni 0$ for every $t \in R$.

Remark 2. In **(A6)** it is sufficient to assume locally Hölder conditions as follows (see also Remark 1). The inequalities in **(A6-i,ii)** hold for $|t - \tau| < \varepsilon_0 << 1$ and there exist increasing continuous functions $c_1, c_2 : R^+ \to R^+$:

$$|w - y| \leq L_1(|x|)c_1(|t - \tau|), \quad |f(t) - f(\tau)| \leq c_2(|t - \tau|).$$

for $|t - \tau| \geq \varepsilon_0$. It also suffices for the existence of the complete trajectories (cf. [4]).

Lemma 1. *Assume* **(A1)**, **(A2)**, **(A3)**, **(A6)** *and* **(A7)**. *Then* **(A4)** *holds in the following sense. For* $|t - \tau| < 1$, $x \in X$ *and for the constant* $\gamma_1 = \min\{\delta_1, \delta_2\}$, *the estimate*

$$|J_\lambda(t)x - J_\lambda(\tau)x| \leq \lambda l(|x|)|t - \tau|^{\gamma_1}$$

holds where $J_\lambda(t)x = (I + \lambda \mathbf{A}(t))^{-1}x$, *and* $\mathbf{A}(t)u = A(t)u - f(t), \quad u \in D$.

Proof. Let $z \in X$ and consider the following equalities

$$\begin{cases} u_1(t) + \lambda v_1(t) - f(t) = z, & v_1(t) \in A(t)u_1(t) \\ u_2(\tau) + \lambda v_2(\tau) - f(\tau) = z, & v_2(t) \in A(\tau)u_2(\tau), \quad t, \tau \in R. \end{cases}$$

Forming the difference of these equalities and taking the dual product with $\xi^* \in F(u_1 - u_2)$, we have

$$|u_1(t) - u_2(\tau)|^2 + \lambda(v_1(t) - v_2(\tau), \xi^*) - (f(t) - f(\tau), \xi^*) = 0. \tag{3.3}$$

In view of **(A6-i)**, let $w : R \to X$ satisfy

$$w(t) \in A(t)u_2(\tau), \quad |w(t) - v_2(\tau)| \leq L_1(|u_2(\tau)|)|t - \tau|^{\delta_1}. \tag{3.4}$$

Since $A(t)$ is uniformly m-accretive, it follows from (3.3) that

$$\begin{aligned}(1+\omega\lambda)|u_1(t)-u_2(\tau)|^2 &\leq -\lambda < w(\tau)-v_2(\tau), u_1(t)-u_2(\tau) >_s \\ &\quad + < f(t)-f(\tau), u_1(t)-u_2(\tau) >_s .\end{aligned}$$

Thus (3.4) and **(A6-ii)** give

$$(1+\omega\lambda)|u_1(t)-u_2(\tau)| \leq \lambda L_1(|u_2(\tau)|)|t-\tau|^{\delta_1} + L_2|t-\tau|^{\delta_2}. \tag{3.5}$$

On the other hand, since **(A7)** yields $(I+\lambda A(t))^{-1}0 = 0$,

$$\begin{aligned}|u_2(\tau)| &= |(I+\lambda A(\tau))^{-1}(z+f(\tau)) - (I+\lambda A(\tau))^{-1}0| \\ &\leq |z+f(\tau)| \leq c\end{aligned}$$

for some constant $c > 0$. Thus we have

$$\begin{aligned}|J_\lambda(t)z - J_\lambda(\tau)z| &= |u_1(t)-u_2(\tau)| \\ &\leq (1+\lambda\omega)^{-1}\max\{\lambda L_1(c), L_2\}|t-\tau|^{\gamma_1}. \qquad \square\end{aligned}$$

Theorem 1. *Under the assumtions* **(A1)**, **(A2)**, **(A3)**, **(A5)**, **(A6)**, **(A7)** *and* **(U)**, *assume that there exists a constant* $K_0 > 1$ *such that*

$$m_i > K_0 m_{i-1}, \qquad i = 1, 2, \cdots . \tag{3.6}$$

Then there exists a unique almost periodic complete trajectory $u(t)$ *of system (1.1) which satisfies*

(i) $|u(t)-u(\tau)| \leq c_1|t-\tau|^{\gamma_1}, \quad t,\tau \in R : |t-\tau| < 1$
for some $c_1 > 0$ *and* $\gamma_1 = \min\{\delta_1, \delta_2\}$,
(ii) its inclusion length for ε*-period satisfies*

$$l_\varepsilon \leq c_2 \varepsilon^{-\frac{1}{\gamma_2}} \tag{3.7}$$

for some $c_2 > 0$ *and* $\gamma_2 = \max\{\delta_1, \delta_2\}$.

Consequently, the fractal dimension of its orbit $\Sigma = \bigcup_{t\in R} u(t)$ *satisfies*

$$D_F(\Sigma) \leq \frac{1}{\gamma_1} + \frac{1}{\gamma_2} = \frac{1}{\delta_1} + \frac{1}{\delta_2}.$$

Proof. Since it follows from Lemma 1 that (u-iv) holds, the boundedness of the trajectory $u(t)$ yield (i). Thus it is sufficient to show (ii). Let $(t,\sigma) \in R \times R^+$, then it follows from **(U)** that

$$\left(\frac{du(t+\sigma)}{dt} - \frac{du}{dt}, \xi^*\right) = -(y(t+\sigma)-y(t), \xi^*) + (f(t+\sigma)-f(t), \xi^*),$$

$$y(t+\sigma) \in A(t+\sigma)u(t+\sigma), \quad y(t) \in A(t)u(t)$$

for every $\xi^* \in F(u(t+\sigma)-u(t))$ (see also the remark following the condition **(U)**). From **(A6-i)** there exists $w \in A(t+\sigma)u(t)$ which satisfies

$$|w - y(t)| \leq L_1(|u(t)|)\sigma^{\delta_1}.$$

Hereafter in the proof, to clarify the argument, we use the notation $A(t+\sigma)u(t)$, $A(t)u(t)$ instead of $w, y(t)$, respectively. Applying **(U)** and **(A5)**, we have

$$\frac{1}{2}\frac{d}{dt}|u(t+\sigma)-u(t)|^2 + \omega|u(t+\sigma)-u(t)|^2$$
$$\leq |A(t+\sigma)u(t) - A(t)u(t)||u(t+\sigma)-u(t)| + |f(t+\sigma)-f(t)||u(t+\sigma)-u(t)|.$$

Put $\kappa_1 = \sup_{t\in R}|A(t+\sigma)u(t) - A(t)u(t)|$ and $\kappa_2 = \sup_{t\in R}|f(t+\sigma)-f(t)|$, then it follows from Proposition 1, proved later, that we have

$$\sup_{t\in R}|u(t+\sigma)-u(t)| \leq \frac{1}{\omega}(\kappa_1+\kappa_2). \tag{3.8}$$

Thus, for a small $\varepsilon > 0$, we can admit the constant σ as an ε-almost period of $u(t)$ if the estimate

$$|A(t+\sigma)u(t) - A(t)u(t)| + |f(t+\sigma)-f(t)| \leq \omega\varepsilon$$

holds for every $t \in R$.. Now we can use the method in our previous paper [10].

By proving the following two inequalities; $l_\varepsilon \leq K\varepsilon^{-\frac{1}{\delta_i}}, i = 1, 2$ we will show (ii). We start with the case $i = 2$ and here we put $\delta := \delta_2$.

By taking a sufficiently large number k we define a small constant

$$\varepsilon_k = \frac{L_2}{\omega(1-K_0^{-\delta})} \cdot \left(\frac{1}{m_{k+1}}\right)^{\delta}.$$

Our main subject of the proof is to show that we can take $l_{\varepsilon_k} = m_{k+1}\alpha$. Then (3.7) for $\varepsilon = \varepsilon_k$ holds and, by defining

$$l_\varepsilon = l_{\varepsilon_k} \quad \text{for} \quad \varepsilon_{k+1} < \varepsilon \leq \varepsilon_k,$$

we can obtain

$$l_\varepsilon = l_{\varepsilon_k} = K\varepsilon_k^{-\frac{1}{\delta}} \leq K\varepsilon^{-\frac{1}{\delta}}.$$

Thus, it suffices to find an ε-almost period in every interval $[a - m_{k+1}\alpha, a]$, $a \in \mathcal{R}$.

First we consider the case where $a \geq m_{k+1}\alpha$. Hereafter in the proof, to simplify the terminology, we reset the indices and the notation as follows:

$$m_{k+j} \to m_j, \quad n_{k+j} \to n_j, \quad a_{k+j} \to a_j \quad \text{and} \quad \varepsilon_k \to \varepsilon.$$

Fix the number $i : m_i\alpha \le a < m_{i+1}\alpha$, then, by considering the estimate

$$m_j < (a_j + 1)m_{j-1},$$

we can take a sequence of nonnegative integers $\{k_j\}$, $1 \le j \le i$, which satisfies

$$0 \le k_j \le a_j \ \text{ for } \ 1 \le j \le i-1, \ \ 1 \le k_i \le a_i$$

and

$$\begin{aligned} &k_i m_i\alpha + k_{i-1}m_{i-1}\alpha + \cdots + k_1 m_1\alpha \\ &\quad \le a < k_i m_i\alpha + k_{i-1}m_{i-1}\alpha + \cdots + k_2 m_2\alpha + (k_1+1)m_1\alpha. \end{aligned}$$

Define

$$\begin{aligned} m(k) &:= k_i m_i + k_{i-1}m_{i-1} + \cdots + k_1 m_1, \\ n(k) &:= k_i n_i + k_{i-1}n_{i-1} + \cdots + k_1 n_1 \end{aligned}$$

and note that $m(k)\alpha \in [a - m_1\alpha, a]$, then by Hölder continuity and periodicity we have

$$\begin{aligned} |A(t+m(k)\alpha)u(t) - A(t)u(t)| \ &+ \ |f(t+m(k)\alpha) - f(t)| \\ &= |f(t+m(k)\alpha) - f(t)| \\ &= |f(t+m(k)\alpha) - f(t+n(k))| \\ &\le L_2|m(k)\alpha - n(k)|^\delta. \end{aligned}$$

By Diophantine approximation we can estimate

$$\begin{aligned} |m(k)\alpha - n(k)|^\delta &\le (k_i m_i)^\delta|\alpha - \frac{n_i}{m_i}|^\delta + \cdots + (k_1 m_1)^\delta|\alpha - \frac{n_1}{m_1}|^\delta \\ &\le m_{i+1}^\delta|\alpha - \frac{n_i}{m_i}|^\delta + \cdots + m_2^\delta|\alpha - \frac{n_1}{m_1}|^\delta \\ &\le (\frac{1}{m_i})^\delta + \cdots + (\frac{1}{m_1})^\delta \end{aligned} \tag{3.9}$$

where we use an elementary inequality $(x+y)^\delta \le x^\delta + y^\delta$, $x, y \ge 0$. Thus Hypothesis (3.6) yields

$$\begin{aligned} |A(t+m(k)\alpha)u(t) - A(t)u(t)| \ &+ \ |f(t+m(k)\alpha) - f(t)| \\ &\le L_2(\frac{1}{m_1})^\delta(1 + K_0^{-\delta} + K_0^{-2\delta} + \cdots) \\ &< L_2(\frac{1}{m_1})^\delta\frac{1}{1-K_0^{-\delta}} = \varepsilon\cdot\omega \end{aligned}$$

for every $t \in \mathcal{R}$. Therefore, we can find the ε-almost period $m(k)\alpha$ in any interval $[a, a+l_\varepsilon]$ for $a \geq 0$ such that

$$\sup_{t\in R} |u(t+m(k)\alpha) - u(t)| \leq \varepsilon.$$

For the interval $[a - l_\varepsilon, a]$, $a < 0$, we can take the element $-m(k)\alpha$, since

$$|u(t+m(k)\alpha) - u(t)| \leq \varepsilon \quad \text{for every} \quad t \in \mathcal{R}$$

yields

$$|u(t') - u(t' - m(k)\alpha)| \leq \varepsilon \quad \text{for every} \quad t' = t + m(k)\alpha \in \mathcal{R}$$

and $m(k)\alpha \in [a', a'+l_\varepsilon]$, $a' > 0$, is equivalent to $-m(k)\alpha \in [-a' - l_\varepsilon, -a']$. Finally, for the interval $[a, a+l_\varepsilon]$, $-l_\varepsilon < a < 0$, it contains the null point as an ε-almost period.

Next we treat the case $i = 1$, substituting the role of $m(k)\alpha$ by that of $n(k)$. We denote $\delta := \delta_1$ and reset indices and notations in the same way as the above case for simplicity. Let

$$\varepsilon' := \frac{L_1(c)}{\omega(1 - K_0^{-\delta})} \cdot (\frac{1}{m_1})^\delta$$

where $c = \sup_{t\in R} |u(t)|$, and consider the interval $[a - m_1\alpha - \varepsilon_1, a + \varepsilon_1]$ where

$$\varepsilon_1 := \frac{1}{m_1(1 - K_0^{-1})},$$

then $n(k)$ is in this interval, since $|m(k)\alpha - n(k)| < \varepsilon_1$ (see (3.9) and use (3.6)) and $m(k)\alpha \in [a - m_1\alpha, a]$. By using the argument in the preceding case $i = 2$ we can show that the element $n(k)$ is an ε-almost period in this interval, since we have

$$\begin{aligned}
&|A(t+n(k))u(t) - A(t)u(t)| + |f(t+n(k)) - f(t)| \\
&\quad = |A(t+n(k))u(t) - A(t)u(t)| = |A(t+n(k))u(t) - A(t+m(k)\alpha)u(t)| \\
&\quad \leq L_1(c)|n(k) - m(k)\alpha|^\delta \\
&\quad \leq L_1(c)((\frac{1}{m_i})^\delta + \cdots + (\frac{1}{m_1})^\delta) \leq \varepsilon'\omega.
\end{aligned}$$

It follows that

$$\sup_{t\in R} |u(t+n(k)) - u(t)| \leq \varepsilon'.$$

Thus we have the estimate

$$l_\varepsilon \leq K \min\{\varepsilon^{-\frac{1}{\delta_1}}, \varepsilon^{-\frac{1}{\delta_2}}\} = K\varepsilon^{-\frac{1}{\gamma_2}}.$$

Due to Theorem B we obtain the conclusion. □

In the proof we use the following proposition:

Proposition 1. *Given $\sigma \in R^+$, the estimate*

$$\frac{1}{2}\frac{d}{dt}|u(t+\sigma)-u(t)|^2+\omega|u(t+\sigma)-u(t)|^2 \leq (\kappa_1+\kappa_2)|u(t+\sigma)-u(t)| \tag{3.10}$$

for every $t \in R$ yields

$$|u(t+\sigma)-u(t)| \leq \frac{1}{\omega}(\kappa_1+\kappa_2)$$

for every $t \in R$.

For completeness we give its proof, which is owing to a slight modificaition of Lemmas in [9]. Let $\gamma > 0$ be given arbitrarily.

Lemma 2. *If there exists $t_0 \in R$ such that*

$$|u(t_0+\sigma)-u(t_0)| \leq \frac{1}{\omega}(\kappa_1+\kappa_2)+\gamma,$$

then

$$|u(t+\sigma)-u(t)| \leq \frac{1}{\omega}(\kappa_1+\kappa_2)+\gamma,$$

for every $t \geq t_0$.

Proof. Assume that there exits $\bar{t} \geq t_0$ which satisfies

$$|u(\bar{t}+\sigma)-u(\bar{t})| > \frac{1}{\omega}(\kappa_1+\kappa_2)+\gamma$$

then we shall derive a contradiction.

By the continuity of $u(t)$ there exists an interval $[t_1, t_2]$ such that

$$|u(t_1+\sigma)-u(t_1)| = \frac{1}{\omega}(\kappa_1+\kappa_2)+\gamma$$

and

$$|u(t+\sigma)-u(t)| > \frac{1}{\omega}(\kappa_1+\kappa_2)+\gamma$$

for every $t \in (t_1, t_2]$. By (3.10) it follows that

$$\frac{1}{2}\frac{d}{dt}|u(t+\sigma)-u(t)|^2 \leq 0$$

for every $t \in [t_1, t_2]$. This implies that

$$|u(t+\sigma)-u(t)| \leq \frac{1}{\omega}(\kappa_1+\kappa_2)+\gamma$$

for every $t \in [t_1, t_2]$, which is the contradiction. $\square$

Lemma 3. *There does not exist any point $\tilde{t} \in R$ such that*

$$|u(s+\sigma) - u(s)| > \frac{1}{\omega}(\kappa_1 + \kappa_2) + \gamma \tag{3.11}$$

for every $s \in (-\infty, \tilde{t}]$.

Proof. Assume that there exists $\tilde{t} \in R$ satisfying the above inequality. Let $s \in (-\infty, \tilde{t})$ and integrate inequality (3.10) over $[s, \tilde{t}]$, then we have the inequalities

$$\begin{aligned}
&-\frac{1}{2}|u(\tilde{t}+\sigma) - u(\tilde{t})|^2 + \frac{1}{2}|u(s+\sigma) - u(s)|^2 \\
&\geq \int_s^{\tilde{t}} |u(\zeta+\sigma) - u(\zeta)|\{\omega|u(\zeta+\sigma) - u(\zeta)| - (\kappa_1 + \kappa_2)\}d\zeta \\
&\geq \omega\gamma \int_s^{\tilde{t}} |u(\zeta+\sigma) - u(\zeta)|d\zeta.
\end{aligned}$$

By using (3.11) again, we obtain

$$-\frac{1}{2}|u(\tilde{t}+\sigma) - u(\tilde{t})|^2 + \frac{1}{2}|u(s+\sigma) - u(s)|^2 \geq \omega\gamma(\tilde{t} - s)\{\frac{1}{\omega}(\kappa_1 + \kappa_2) + \gamma\}.$$

Taking the limit for $s \to -\infty$ of the above inequality, the right-hand term increases to $+\infty$, but this contradicts that $u(t)$ is bounded. $\square$

Remark 3. In the rational approximation theory, α is called badly approximable if there exists a positive constant c such that

$$|\alpha - \frac{p}{q}| > (\frac{c}{q})^2 \quad (0 < c < \frac{1}{\sqrt{5}}) \tag{3.12}$$

holds for every rational p/q. It is also known (cf. [12]) that α is badly approximable if and only if the sequence $\{a_n\}$ in the continued fraction of α is bounded. The assumption (3.6) is satisfied if the irrational number α is badly approximable (cf. [10]).

4. Semilinear parabolic equations

In this section we investigate a semilinear parabolic equation, which satisfies the conditions in the preceding sections.

Let Ω be a bounded domain in R^n with a smooth boundary $\partial\Omega$ and let $H^m(\Omega)$, $H_0^m(\Omega)$ be the usual Sobolev spaces and put $X := L^2(\Omega)$. Consider an m-accretive operator (equivalently, maximal monotone) $g(t) \subset X \times X$ which satisfies
(g-i) the domain of g is independent of t, $D(g(t)) = D_g$;
(g-ii) $0 \in g(t)0$;
(g-iii) there exist a monotone increasing function $k : R^+ \to R^+$ and a constant $\delta_1, 0 < \delta_1 < 1$, such that if $t, \tau \in R : |t - \tau| << 1$, $x \in D_g$ and $y \in g(t)x$, then there exists a $w \in g(\tau)x$:

$$|y - w| \leq k(|x|)|t - \tau|^{\delta_1}.$$

Define the operator $A(t)$ on X by

$$A(t)u = -\Delta u + g(t, u), \quad D(A(t)) = D := H^2(\Omega) \cap H_0^1(\Omega) \cap D_g,$$

then, we can easily show that $A(t)$ is uniformly m-accretive, since the smallest eigenvalue of $-\Delta$ is strictly positive. (See also [3] or [4] for the sum of m-accretive operators.)

Consider a forcing function $f(t) : R \to X$, which is also Hölder continuous with its exponent $\delta_2 : 0 < \delta_2 < 1$. Let $\varphi \in X$, then, applying the results in [4], we can show that the initial value problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + g(t)u \ni f(t) & \text{in } \Omega \times (s, +\infty) \\ u(t, x) = 0 \quad x \in \partial\Omega, \ t \geq s \\ u(s, x) = \varphi(x) \quad x \in \Omega \end{cases} \tag{4.1}$$

has a unique weak solution $u : [s, +\infty) \to X$, which is Hölder continuous with its exponent $\delta := \min\{\delta_1, \delta_2\}$. (See [2] for the weak or integral-type solutions.) Furthemore we assume the weak differentiability of the weak solution. (Sufficient conditions for the weak or strong differentiability are also given in [2].)

Assume the periodicity

$$g(t + \alpha) = g(t), \quad f(t + 1) = f(t) \quad \alpha : \text{ irrational}.$$

Then, due to Theorem A, there exists a unique complete trajectory which is almost periodic. Applying Theorem 1, we can estimate the fractal dimension of the complete trajectory $\Sigma = \bigcup_{t \in R} u(t)$

$$D_F(\Sigma) \leq \frac{1}{\delta_1} + \frac{1}{\delta_2}.$$

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