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AN EQUILIBRIUM THEOREM FOR SET-VALUED MAPS WITHOUT COMPACTNESS AND ITS APPLICATIONS

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ABSTRACT. The purpose of this paper, we prove two existence theorems which are equilibrium theorem and fixed-point theorem without compactness for set-valued maps.

1. INTRODUCTION AND PRELIMINARIES

Fixed-point theorems of set-valued maps have become an important tool to derive various results in mathematical economics, game theory, and so on. In particular, we note that fixed-point theorem is provided the proof comparatively easy by equilibrium theorem. Also, it is found out that these theorems are equivalent by many mathematicians. However, compactness is necessary to prove them. So, we can extended these theorems without compactness for a reflexive Banach space.

A set-valued map F from a set X into a set Y is a map which associates a subset of Y with each point of X. Equivalently, F can be viewed as a function from the set X into the power set 2^Y. We use the notation for the operation on set-valued maps, which is defined by $F\colon X \rightsquigarrow Y$. The domain of F is the subset of elements $x \in X$ such that $F(x)$ is not empty; $\mathrm{Dom}(F) := \{ x \in X \mid F(x) \neq \phi \}$. The image of F is the union of the images $F(x)$, when x ranges over X; $\mathrm{Im}(F) := \cup_{x\in X} F(x)$.

If X and Y are topological spaces, then the image of a set-valued map F are closed, compact, and so on, we say that F is closed-valued, compact-valued, and so on. Also, for a subset K, $\mathrm{int}K$ will indicate its interior, $\mathrm{cl}K$ will indicate its closure.

If K is a convex subset and if $x \in \mathrm{cl}K$, $T_K(x)$ is called the tangent cone to K at x and is denoted by $T_K(x) := \mathrm{cl}(\cup_{h>0}(K-x)/h)$.

2. DEFINITIONS AND AN EQUILIBRIUM THEOREM OF SET-VALUED MAPS WITHOUT COMPACTNESS

Definition 1. Let $K \subset \mathrm{Dom}(F)$ be a nonempty subset of a Banach space X. Then a subset K is said to be a viability domain of F (or satisfying tangential condition) if and only if

$$\forall x \in K, \qquad F(x) \cap T_K(x) \neq \phi \tag{1}$$

This means that for any point $x \in K$, there exists at least a direction $v \in F(x)$ which is tangent to K at x.

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Definition 2. We shall say that a set-valued map F is upper hemicontinuous at $x_0 \in \mathrm{Dom}(F)$ if and only if for all $y^* \in X^*$, the function

$$x \longmapsto \sigma\big(F(x), y^*\big) := \sup_{y \in F(x)} \langle y, y^* \rangle \tag{2}$$

is upper semi-continuous at x_0.

It is said to be upper hemicontinuous if and only if it is upper hemicontinuous at any point of $\mathrm{Dom}(F)$.

Also, F is said to weak upper hemicontinuous if and only if F is upper hemicontinuous in $\sigma(X, X^*)$.

Lemma 3. *If K_1 and K_2 are convex closed subsets of a Banach space X such that $\theta_X \in \mathrm{int}(K_1 - K_2)$ then*

$$T_{K_1 \cap K_2}(x) = T_{K_1}(x) \cap T_{K_2}(x). \tag{3}$$

Theorem 4. *Let K be a weak compact convex subset of a Banach space X and $\varphi \colon X \times X \longrightarrow \mathbb{R}$ be a function satisfying*

(i) *$\forall y \in K$, $x \to \varphi(x, y)$ is weak lower semicontinuous.*
(ii) *$\forall x \in K$, $y \to \varphi(x, y)$ is concave.*
(iii) *$\forall y \in K$, $\varphi(y, y) \leqq 0$.*

Then, there exists $\bar{x} \in K$ such that $\forall y \in K$, $\varphi(\bar{x}, y) \leqq 0$.

Theorem 5. *Assume that X is a Banach space and that $F \colon X \rightsquigarrow X$ is a weak upper hemicontinuous set-valued map with closed convex values.*

If $K \subset X$ is a convex weak compact viability domain of F then it contains an equilibrium of F.

$$\textit{i.e.}, \qquad \exists \bar{x} \in K \qquad \textit{s.t.} \qquad \theta_X \in F(\bar{x}) \tag{4}$$

Proof. We proceed by contradiction, assuming that the conclusion is false. Hence, for any $x \in K$, $\theta_X \notin F(x)$. Since the images of F are closed and convex, the Hahn-Banach Separation Theorem implies

$$\exists y_x^* \in X^* \backslash \{\theta_{X^*}\} \quad \text{s.t.} \quad \sigma\big(F(x), y_x^*\big) < 0.$$

We set

$$\Gamma_{y^*} := \big\{\, x \in K \;\big|\; \sigma\big(F(x), y^*\big) < 0 \,\big\}.$$

Then K is covered by the subsets Γ_{y^*} when y^* ranges over the dual of X. These subsets are weak open by the very definition of weak upper hemicontinuity of F. So, K can be covered by n such weak open subsets $\Gamma_{y_i^*}$.

Let us consider a continuous partition of unity $(\alpha_i)_{i=1,\dots,n}$ associated with $\Gamma_{y_i^*}$ and introduce the function $\varphi \colon K \times K \longrightarrow \mathbb{R}$ defined by

$$\varphi(x, y) := \sum_{i=1}^{n} \alpha_i(x) \langle y_i^*, x - y \rangle.$$

Being continuous with respect to x and affine with respect to y, the assumptions of Theorem 4 are satisfied. Hence there exists $\bar{x} \in K$ such that for $\bar{y^*} := \sum_{i=1}^n \alpha_i(\bar{x})y_i^*$ we have for any $y \in K$, $\varphi(\bar{x}, y) = \langle \bar{y^*}, \bar{x} - y \rangle \leqq 0$. Hence $-\bar{y^*}$ belongs to the polar cone $T_K(\bar{x})^-$ of the convex subset K at $\bar{x}$.

Since K is a viability domain of F, there exists $v \in F(\bar{x}) \cap T_K(\bar{x})$, and thus

$$\sigma\big(F(\bar{x}), \bar{y^*}\big) \geqq \langle \bar{y^*}, v \rangle \geqq 0.$$

By setting $I(\bar{x}) := \big\{ i \mid \alpha_i(\bar{x}) > 0, i = 1, \ldots, n \big\}$. It is not empty.

Hence,

$$\sigma\big(F(\bar{x}), \bar{y^*}\big) \leqq \sum_{i \in I(\bar{x})} \alpha_i(\bar{x}) \sigma\big(F(\bar{x}), y_i^*\big) < 0.$$

Hence, the latter inequality is then a contradiction of the previous one. □

Definition 6. Let X is a Banach space. We associate with any $x \in X$

$$J(x) := \big\{ x^* \in X^* \mid \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \big\}. \tag{5}$$

Set-valued map $J : X \rightsquigarrow X^*$ is called duality mapping.

Lemma 7. *Assume that X is a reflexive Banach space, $J : X \rightsquigarrow X^*$ is the duality mapping, and x belongs to B satisfying $\|x\| = 1$. Then*

$$T_B(x) = \bigcap_{x^* \in J(x)} \big\{ y \in X \mid \langle y, x^* \rangle \leqq 0 \big\}, \tag{6}$$

where B is the unit ball of X.

Proof. For any $v \in T_B(x)$, there exists a sequence of elements $v_n \in \big(\cup_{h>0}(B-x)/h\big)$ converging to v. Hence, for any n, there exists $h_n > 0$ and $b_n \in B$ such that $v_n = (b_n - x)/h_n$. Since $\langle v_n, x^* \rangle \leqq 0$ for any $x^* \in J(x)$, $\langle v, x^* \rangle \leqq 0$ for any $x^* \in J(x)$. Hence,

$$v \in \bigcap_{x^* \in J(x)} \big\{ y \in X \mid \langle y, x^* \rangle \leqq 0 \big\}.$$

Assume that there exists $y_0 \notin T_B(x)$ such that $\langle y_0, x^* \rangle \leqq 0$ for any $x^* \in J(x)$. Since the sets $T_B(x)$ are closed and convex, the Hahn-Banach Separation Theorem implies

$$\exists \bar{y^*} \in X^* \backslash \{\theta_{X^*}\}, \qquad \exists a \in \mathbb{R} \quad s.t. \quad \langle \bar{y^*}, y_0 \rangle > a > \langle \bar{y^*}, y \rangle \qquad \forall y \in T_B(x)$$

So, we have $\bar{y^*}$ belongs to the normal cone $N_B(x)$ and $a > 0$. We set $y^* := \bar{y^*}/\|\bar{y^*}\|$, then we have $y^* \in J(x)$. Hence,

$$\langle y^*, y_0 \rangle > \frac{a}{\|y^*\|} > 0.$$

So, the latter inequality is then a contradiction of the previous one. □

Theorem 8. *Let $K \subset \mathrm{Dom}(F)$ be a closed convex subset of a reflexive Banach space X and a set-valued map $F\colon X \rightsquigarrow X$ a weak upper hemicontinuous map with nonempty closed convex values satisfying the following assumption :*

$$\limsup_{\substack{\|x\|\to\infty \\ x\in K}} \sup_{x^*\in J(x)} \sigma\big(F(x), x^*\big) < 0. \tag{7}$$

Moreover we posit that K is a viability domain of F. Then there exists an equilibrium $\bar{x} \in K$ of F.

Proof. Assumption (7) implies that there exists $\varepsilon > 0$ and $a > 0$ such that

$$\sup_{\|x\|\geqq a} \sup_{x^*\in J(x)} \sigma\Big(F(x), x^*\Big) \leqq -\varepsilon \quad \text{and} \quad K \cap \mathrm{int}(aB) \neq \phi \tag{8}$$

Also, we know that for any $x \in aB$ with $\|x\| = a$ then, by Lemma 7

$$T_{aB}(x) = \bigcap_{x^*\in J(x)} \{\, y \in X \mid \langle y, x^* \rangle \leqq 0 \,\}. \tag{9}$$

Hence, from (8) and (9), it follows that

$$\forall x \in K \cap aB, \qquad F(x) \subset T_{aB}(x).$$

Next, since θ_X belong to $\mathrm{int}(K + aB)$ from (8), by Lemma 3 we know that

$$\forall x \in K \cap aB, \qquad T_{K\cap aB}(x) = T_K(x) \cap T_{aB}(x).$$

So, the tangential condition implies that

$$\forall x \in K \cap aB, \qquad F(x) \cap T_{K\cap aB}(x) \neq \phi.$$

Hence, $K \cap aB$ becomes the viability domain of F and obviously to prove that convex and weak compact set.

Hence, by Theorem 5 there exists an equilibrium $\bar{x} \in K$ of F. □

Theorem 9. *Let K be a closed convex set of a reflexive Banach space X, and the set-valued map $F\colon X \rightsquigarrow K$ satisfy weak upper hemicontinuous and nonempty closed convex values. We set the set-valued map $G := F - I$ where I denote the identity map from X to X. So, we assume that G is satisfying (7) then F has a fixed point in K.*

Proof. Since G is satisfying (7),

$$\exists a > 0 \quad s.t. \quad \forall x \in K \cap aB, \qquad G(x) \subset T_{aB}(x).$$

Also, since K is convex and $F(K) \subset K$, then $K - x \subset T_K(x)$. So, we deduce that $K \cap aB$ is a viability domain of G because,

$$\forall x \in K, \quad G(x) \subset T_K(x) \cap T_{aB}(x) = T_{K\cap aB}(x).$$

It is also easy to show that $K \cap aB$ is a closed convex set of X.

Hence, by Theorem 5 there exists an equilibrium $\bar{x} \in K$ of G, which is a fixed point of F. □

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