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SOME NOTIONS OF CONVEXITY FOR SET-VALUED MAPS AND THEIR RELATIONS

新潟大学大学院自然科学研究科　黒岩　大史　(DAISHI KUROIWA)

ABSTRACT. We establish some notions of convexity of set-valued maps. These notions are generalizations of notions of convexity of vector-valued maps. Also we investigate some relations among the generalized concepts of convexity.

1. INTRODUCTION AND PRELIMINARIES

The notions of convexity of sets and functions play an important role in various fields of mathematics. For example, separation theorem, fixed point theorem, and minimax theorem are closely connected with convexity. The notion of convex of real-valued functions has been extended in several ways in order to generalize results above, see [2-5].

Also, set-valued analysis has been very widely developed and produced many applications in recent years. For example, fixed point theorem for set-valued maps has been generalized by many authors and has been applied to various problems, for instance, which are game theory, economic theory, and so on, see [1].

The aim of this paper is to give various concepts concerned with convexity of set-valued maps, and to investigate some relations among them. These notions are generalization of notions of convexity of vector-valued maps. For a notion of convexity of vector-valued maps, there are some generalized notions for set-valued maps. But it is difficult to investigate all notions of convexity of set-valued maps. Hence, we define six notions of convexity of set-valued maps and investigate some relations among them.

First, we give the preliminary terminology used throughout this paper. Let X be a real vector space, C a nonempty convex subset of X, Y a real topological vector space, P a convex cone in Y, and Y^* the continuous dual space of Y. For any vector space V, let θ_V be the null vector in V. For a nonempty subset A of Y, we set $A^+ \equiv \{y^* \in Y^* | \langle y^*, a\rangle \geq 0, \text{ for all } a \in A\}$. The set A^+ is called the *positive polar cone* of A. We define a relation $\leq_P$ in Y by the convex cone P : for $y_1, y_2 \in Y$, $y_1 \leq_P y_2 \iff y_2 - y_1 \in P$. If P is pointed, that is, $P \cap (-P) = \{\theta_Y\}$, the relation $\leq_P$ is an order relation in Y. In this paper, however, we do not assume that P is pointed. We mention about set-valued map. F is called a *set-valued map* from C to Y if F is a map from C to 2^Y, which is the power set of Y, and then we write $F : C \rightsquigarrow Y$. For a set-valued map $F : C \rightsquigarrow Y$, we define $\mathrm{Graph}(F) \equiv \{(x,y) | x \in C,\ y \in F(x)\}$ and $\mathrm{Dom}(F) \equiv \{x \in C | F(x) \neq \emptyset\}$. The set $\mathrm{Graph}(F)$ is called the *graph* of F and $\mathrm{Dom}(F)$ is called the *domain* of F. In this paper,

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we consider the case $C = \mathrm{Dom}(F)$. There are two manners to define the inverse image by a set-valued map $F : C \rightsquigarrow Y$ of a subset M of Y, $F^{-1}(M) \equiv \{x \in C | F(x) \cap M \neq \emptyset\}$ and $F^{+1}(M) \equiv \{x \in C | F(x) \subset M\}$. The subset $F^{-1}(M)$ is called the *inverse image* of M by F and $F^{+1}(M)$ is called the *core* of M by F. A set-valued map F is said to be *convex-valued* (resp. *closed-valued*, *compact-valued*, and so on), if for any $x \in C$ the set $F(x)$ is a convex set (resp. a closed set, a compact set, and so on). A set-valued map F is said to be *P-convex-valued* if for any $x \in C$ the set $F(x) + P$ is a convex set. Details of set-valued maps can be found in [1].

2. Definitions of Convexity of Set-Valued Maps and their Relations

Next, we mention various types of convexity of set-valued maps. Such notions of convexity of set-valued maps are generalized notions of convexity of vector-valued maps. We note that there are many ways with respect to such generalization. In detail, see [4].

Definition 1. A set-valued map $F : C \rightsquigarrow Y$ is said to be

(i) *convex* if for every x_1, $x_2 \in C$, $y_1 \in F(x_1)$, $y_2 \in F(x_2)$, and $\lambda \in (0,1)$, there exists $y \in F(\lambda x_1 + (1-\lambda)x_2)$ such that $y \leq_P \lambda y_1 + (1-\lambda)y_2$;

(ii) *convexlike* if for every x_1, $x_2 \in C$, $y_1 \in F(x_1)$, $y_2 \in F(x_2)$, and $\lambda \in (0,1)$, there exists $(x, y) \in \mathrm{Graph}(F)$ such that $y \leq_P \lambda y_1 + (1-\lambda)y_2$;

(iii) *properly quasiconvex* if for every x_1, $x_2 \in C$, $y_1 \in F(x_1)$, $y_2 \in F(x_2)$, and $\lambda \in (0,1)$, there exists $y \in F(\lambda x_1 + (1-\lambda)x_2)$ such that either $y \leq_P y_1$ or $y \leq_P y_2$;

(iv) *quasiconvex* if for every x_1, $x_2 \in C$, $y_1 \in F(x_1)$, $y_2 \in F(x_2)$, and $\lambda \in (0,1)$, if $y \in Y$ satisfies $y_1 \leq_P y$ and $y_2 \leq_P y$, then there exists $y' \in F(\lambda x_1 + (1-\lambda)x_2)$ such that $y' \leq_P y$;

(v) *naturally quasiconvex* if for every x_1, $x_2 \in C$, $y_1 \in F(x_1)$, $y_2 \in F(x_2)$, and $\lambda \in (0,1)$, there exists $y \in F(\lambda x_1 + (1-\lambda)x_2)$ and $\eta \in [0,1]$ such that $y \leq_P \eta y_1 + (1-\eta)y_2$;

(vi) *$*$-quasiconvex* if for each $y^* \in P^+$, function $x \longmapsto \inf_{y \in F(x)} \langle y^*, y \rangle$ is quasiconvex on C.

Then we have the following:

Proposition 1. *The following statements hold:*

(i) *F is convex if and only if $\mathrm{Graph}(F) + \{\theta_X\} \times P$ is a convex set;*

(ii) *F is convexlike if and only if $F(C) + P$ is a convex set;*

(iii) *F is quasiconvex if and only if for all $y \in Y$, the set $F^{-1}(y - P)$ is a convex set.*

Proof. First we prove (i). Let F be convex. We show that $\mathrm{Graph}(F) + \{\theta_X\} \times P$ is a convex set. For any z_1, $z_2 \in \mathrm{Graph}(F) + \{\theta_X\} \times P$, there exist (x_1, y_1), $(x_2, y_2) \in \mathrm{Graph}(F)$, p_1, $p_2 \in P$ such that $z_1 = (x_1, y_1) + (\theta_X, p_1)$ and $z_2 = (x_2, y_2) + (\theta_X, p_2)$. Since F is convex, for any $\lambda \in (0,1)$, there is $y \in F(\lambda x_1 + (1-\lambda)x_2)$ such that such that $y \leq_P \lambda y_1 + (1-\lambda)y_2$. Hence, $\lambda z_1 + (1-\lambda)z_2 = \lambda\{(x_1, y_1) + (\theta_X, p_1)\} + (1-\lambda)\{(x_2, y_2) + (\theta_X, p_2)\} = (\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) + (\theta_X, \lambda p_1 + (1-\lambda)p_2) \in \mathrm{Graph}(F) + \{\theta_X\} \times P + \{\theta_X\} \times P = \mathrm{Graph}(F) + \{\theta_X\} \times P$. Therefore, $\mathrm{Graph}(F) + \{\theta_X\} \times P$ is convex.

We show F is convex when Graph$(F) + \{\theta_X\} \times P$ is convex. For any x_1, $x_2 \in C$, $y_1 \in F(x_1)$, $y_2 \in F(x_2)$, and $\lambda \in (0,1)$, since Graph$(F) + \{\theta_X\} \times P$ is a convex set and $(x_1, y_1), (x_2, y_2) \in$ Graph$(F)+\{\theta_X\}\times P$, $\lambda(x_1,y_1)+(1-\lambda)(x_2,y_2) \in$ Graph$(F)+\{\theta_X\}\times P$. Then there exists $p \in P$ such that $(\lambda x_1 + (1-\lambda)x_2,\ \lambda, y_1 + (1-\lambda)y_2 - p) \in$ Graph(F). Hence there is $y \in F(\lambda x_1 + (1-\lambda)x_2)$ such that $y = y_1 + (1-\lambda)y_2 - p$. Thus, F is convex.

Proof of (ii) is similar to (i).

Finally, we show (iii). Let F be quasiconvex and let $y \in Y$. We show that $F^{-1}(y-P)$ is a convex set. We can assume that $F^{-1}(y-P) \neq \emptyset$. For any x_1, $x_2 \in F^{-1}(y-P)$, and $\lambda \in (0,1)$, we have $F(x_1) \cap (y-P) \neq \emptyset$, and $F(x_2) \cap (y-P) \neq \emptyset$. Then, there exist $y_1 \in F(x_1)$, and $y_2 \in F(x_2)$ such that $y_1, y_2 \in y - P$. Since F be quasiconvex, there exists $y' \in F(\lambda x_1 + (1-\lambda)x_2)$ such that $y' \in y - P$. Then $\lambda x_1 + (1-\lambda)x_2 \in F^{-1}(y-P)$ and it is shown that $F^{-1}(y-P)$ is a convex set.

Next, we prove that F is quasiconvex when $F^{-1}(y-P)$ is convex for each $y \in Y$. For any x_1, $x_2 \in C$, $y_1 \in F(x_1)$, $x_2 \in F(x_2)$, *lambda* $\in (0,1)$, and $y \in Y$ with $y_1 \leq_P y$ and $y_2 \leq_P y$, we have $x_1, x_2 \in F^{-1}(y-P)$. Since $F^{-1}(y-P)$ is convex, $\lambda x_1 + (1-\lambda)x_2 \in F^{-1}(y-P)$. Then, there exists $y' \in F(\lambda x_1 + (1-\lambda)x_2)$ such that $y' \in y - P$. Hence F is quasiconvex. □

Proposition 2. *The following statements hold:*

(i) *every convex map is also convexlike;*
(ii) *every convex map is also naturally quasiconvex;*
(iii) *properly quasiconvex map is also naturally quasiconvex;*
(iv) *naturally quasiconvex map is also quasiconvex;*
(v) *naturally quasiconvex map is also $*$-quasiconvex.*

Proof. It is easy to show that (i), (ii), (iii), and (iv). We prove that naturally quasiconvex map is also $*$-quasiconvex. Let F be naturally quasiconvex. If for some $y^* \in P^*$, $x \longmapsto \inf_{y\in F(x)} \langle y^*, y\rangle$ is not quasiconvex on C, there exist x_1, $x_2 \in C$, and $\lambda \in (0,1)$ such that, $\inf_{y\in F(\lambda x_1+(1-\lambda)x_2)} \langle y^*, y\rangle > \max\{\inf_{y\in F(x_1)} \langle y^*, y\rangle, \inf_{y\in F(x_2)} \langle y^*, y\rangle\}$. Then there exist $y_1 \in F(x_1)$, $y_2 \in F(x_2)$ such that $\inf_{y\in F(\lambda x_1+(1-\lambda)x_2)} \langle y^*, y\rangle > \max\{\langle y^*, y_1\rangle, \langle y^*, y_2\rangle\}$ Since F is naturally quasiconvex, there exists $y \in F(\lambda x_1 + (1-\lambda)x_2)$ and $\eta \in [0,1]$ such that $y \leq_P \eta y_1 + (1-\eta)y_2$. Hence $\langle y^*, y\rangle \leq \eta\langle y^*, y_1\rangle + (1-\eta)\langle y^*, y_2\rangle \leq \max\{\langle y^*, y_1\rangle, \langle y^*, y_2\rangle\}$. This is a contradiction. Therefore F is $*$-quasiconvex. □

We give a relation between $*$-quasiconvex and naturally quasiconvex the following:

Theorem 1. *Assume that Y is a locally convex space and F is a P-convex-valued map. If F is $*$-quasiconvex, then F is also naturally quasiconvex.*

Proof. The proof is by contradiction. Assume that F is not naturally quasiconvex. Then, there exist x_1, $x_2 \in C$, $y_1 \in F(x_1)$, $y_2 \in F(x_2)$, $\lambda \in (0,1)$ such that

$$F(\lambda x_1 + (1-\lambda)x_2) \cap \{[y_1, y_2] - P\} = \emptyset,$$

where $[y_1, y_2] \equiv \{\lambda y_1 + (1-\lambda)y_2 | \lambda \in [0,1]\}$. This condition is equivalent to the following:

$$F(\lambda x_1 + (1-\lambda)x_2) - [y_1, y_2] + P \not\ni \theta_Y.$$

Since $F(\lambda x_1 + (1-\lambda)x_2) + P$ is closed convex and $[y_1, y_2]$ is compact convex, we have $F(\lambda x_1 + (1-\lambda)x_2) - [y_1, y_2] + P$ is closed convex. Hence, by separation theorem, there exist $\overline{y^*} \in Y^* \setminus \{\theta_{Y^*}\}$ and $\alpha \in \boldsymbol{R}$ such that

$$\langle \overline{y^*}, y\rangle - \langle \overline{y^*}, \eta y_1 + (1-\eta y_2)\rangle + \langle \overline{y^*}, p\rangle > \alpha > 0$$

for all $\eta \in [0,1]$, $p \in P$, and $y \in F(\lambda x_1 + (1-\lambda)x_2)$. Since $\langle \overline{y^*}, p\rangle$ is bounded below with respect to $p \in P$, we deduce that $\overline{y^*} \in P^+$. Then, we have

$$\inf_{y \in F(\lambda x_1 + (1-\lambda)x_2)} \langle \overline{y^*}, y\rangle > \max\left\{ \inf_{y \in F(x_1)} \langle \overline{y^*}, y\rangle, \inf_{y \in F(x_2)} \langle \overline{y^*}, y\rangle \right\}.$$

This contradicts that the function $x \mapsto \inf_{y \in F(x)} \langle \overline{y^*}, y\rangle$ is quasiconvex. □

Finally, we give a relation between naturally quasiconvex and convexlike. For this, we recall two concepts of continuity of set-valued maps, lower semicontinuous and upper semicontinuous. A set-valued map $F : C \rightsquigarrow Y$ is said to be

(i) *lower semicontinuous* (l.s.c.) at $x \in C$ if for any open set U with $F(x) \cap U \neq \emptyset$, there exists a neighborhood V of x such that $V \subset F^{-1}(U)$.

(ii) *upper semicontinuous* (u.s.c.) at $x \in C$ if for any open set U with $F(x) \subset U$, there exists a neighborhood V of x such that $V \subset F^{+1}(U)$.

F is said to be lower semicontinuous (resp. u.s.c.) if and only if it is lower semicontinuous at every point of C (resp. u.s.c.). When F is a single-valued map, (i) and (ii) are equivalent to the continuity of vector-valued maps. Then we have the following:

Theorem 2. *We assume that P is closed and F is upper semicontinuous and P-convex-valued. If F is naturally quasiconvex then it is convexlike.*

Proof. This theorem proves by contradiction. We assume that F is not convexlike. Then, there exist x_1, $x_2 \in C$, $y_1 \in F(x_1)$, $y_2 \in F(x_2)$, $\overline{\lambda} \in (0,1)$ such that

$$F(C) \cap \{\overline{\lambda} y_1 + (1-\overline{\lambda})y_2 - P\} = \emptyset,$$

hence,

$$F([x_1, x_2]) \cap \{\overline{\lambda} y_1 + (1-\overline{\lambda})y_2 - P\} = \emptyset, \tag{2.1}$$

where $[x_1, x_2] \equiv \{\lambda x_1 + (1-\lambda)x_2 | \lambda \in [0,1]\}$. Also, we define the following vectors and sets: for all $\lambda \in [0,1]$,

$$x(\lambda) \equiv \lambda x_1 + (1-\lambda)x_2 \quad \text{and} \quad y(\lambda) \equiv \lambda y_1 + (1-\lambda)y_2;$$

$$B_1 \equiv \bigcup_{\lambda \in [\overline{\lambda}, 1]} (y(\lambda) - P) \setminus (y(\overline{\lambda}) - P) \quad \text{and} \quad B_2 \equiv \bigcup_{\lambda \in [0, \overline{\lambda}]} (y(\lambda) - P) \setminus (y(\overline{\lambda}) - P).$$

From naturally quasiconvexity of F, for each $\lambda \in [0,1]$,

$$F(x(\lambda)) \cap \bigcup_{\eta \in [0,1]} (y(\lambda) - P) \neq \emptyset.$$

Here, we define the number $a \equiv \sup\{\lambda \in [0,1] | F(x(\eta)) \cap B_2 \neq \emptyset,\ \forall \eta \in [0,\lambda]\}$. Clearly $a \in [0,1]$ because $F(x(0)) \cap B_2 = F(x_2) \cap B_2 \neq \emptyset$. From (2.1) and convexity of $F(x(a))$, one of the following condition (2.2) and (2.3) holds:

$$F(x(a)) \cap B_1 \neq \emptyset \text{ and } F(x(a)) \cap B_2 = \emptyset; \tag{2.2}$$

$$F(x(a)) \cap B_1 = \emptyset \text{ and } F(x(a)) \cap B_2 \neq \emptyset. \tag{2.3}$$

First, we consider the case when (2.2) is true. Clearly, $a \neq 0$. From (2.1) and (2.2), $F(x(a)) \cap \{B_2 \cup (y(\overline{\lambda}) - P)\} = \emptyset$. Then

$$F(x(a)) \subset \{B_2 \cup (y(\overline{\lambda}) - P)\}^c.$$

Now, we show that $B_2 \cup (y(\overline{\lambda}) - P)$ is a closed set in Y. In fact, let a net $\{z_\alpha\} \subset B_2 \cup (y(\overline{\lambda}) - P)$ converges to $z \in Y$. Since $B_2 \cup (y(\overline{\lambda}) - P) = \bigcup_{\lambda \in [0,\overline{\lambda}]} (y(\lambda) - P)$, there exists $\{\lambda_\alpha\} \subset [0,\overline{\lambda}]$ and $\{p_\alpha\} \subset P$ such that $z_\alpha = y(\lambda_\alpha) - p_\alpha$. Since the set $[y(0), y(\overline{\lambda})]$ is compact, there exist a subnet $\{\beta\} \subset \{\alpha\}$ and $\hat{\lambda} \in [0,\overline{\lambda}]$ such that $\lim_\beta y(\lambda_\beta) = y(\hat{\lambda})$ and $-\lim_\beta p_\beta = \lim_\beta (z_\beta - y(\lambda_\beta)) = z - y(\hat{\lambda})$. Since P is closed, $z - y(\hat{\lambda}) \in -P$. Hence $z \in y(\hat{\lambda}) - P \subset \bigcup_{\lambda \in [0,\overline{\lambda}]} (y(\lambda) - P)$. Thus, $B_2 \cup (y(\overline{\lambda}) - P)$ is a closed set in Y.

Since $\{B_2 \cup (y(\overline{\lambda}) - P)\}^c$ is an open set and F is upper semicontinuous, there exists a neighbourhood V of $x(a)$ such that

$$F(V) \cap \{B_2 \cup (y(\overline{\lambda}) - P)\} = \emptyset.$$

From this, there exists $c \in (0,a)$ such that $[x(c), x(a)] \subset V$. Hence, for each $\lambda \in [c,a]$, $F(x(\lambda)) \cap B_2 = \emptyset$. This contradicts the definition of a.

In the similar way, it is shown that (2.3) is not true. This is a contradiction.

Consequently, we have completed the proof. □

Remark 2.1. Theorem 2 is a generalization of Lemma 2.1(iv) in [5].

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Department of Mathematical Science, Graduate School of Science and Technology, Niigata University, Niigata 950-21, Japan