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Second-order directional derivatives of sup-type functions

Sup-型関数の２次の方向微分について

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Abstract

In this paper, we deal with the following sup-type function:

$$S(x) := \sup_{t\in T} G(x(t), t) \quad x \in X, \tag{1}$$

where T is a compact metric space, X is a subspace of the set of all n-dimensional vector-valued continuous functions $C(T)^n$ equipped with the uniform norm. We denote by G_x and G_{xx} the gradient (row) vector and the Hesse matrix of f w.r.t. x, respectively, and assume them to be continuous on $R^n \times T$. This sup-type function is induced from a phase constraint

$$G(x(t), t) \leq \quad {}^{\forall}t \in T$$

which appears in variational problems and optimal control problems [15].

On the other hand, another sup-type function has been deeply studied:

$$S_0(x) := \sup_{t\in T} G(x, t) \quad x \in R^n, \tag{2}$$

Clarke[1], Correa and Seeger[2], Danskin [3], Dem'yanov and Malozemov[4] Demyanov and Zabrodin[5], Hettich and Jongen[6], Ioffe[7], Kawasaki[8][9] [10][11][13], Shiraishi[17], Seeger[16], Wetterling[18]. We encounter this sup-type function in Tchebycheff approximation. When T depends on x, the minimization problem of $S_0(x)$ becomes a parametric optimization problem. To tell the truth, $S_0(x)$ is a special case of $S(x)$. Indeed, if we take as X $\{x \in C(T)^n \mid x(t) \equiv \text{constant vector} \in R^n\}$, then $S(x)$ reduces to $S_0(x)$. So $S(x)$ inherits a lot of properties from $S_0(x)$.

論文の概要

次の Sup-型関数の１次と２次の方向微分について考察する。

$$S(x) := \sup_{t \in T} G(x(t), t) \quad x \in X, \tag{3}$$

ただし T はコンパクト距離空間, X は n 次元ベクトル値連続関数全体 $C(T)^n$ の部分空間とする。

この Sup-型関数は変分問題や最適制御問題の相条件

$$G(x(t), t) \leq \quad {}^{\forall} t \in T$$

を考察するとき出会う。本論文では、この相条件から包絡線がが生成されるかどうかを調べるために、sup-型関数 $S(x)$ の２次の方向微分を表す公式を与える。

一方、従来よく研究されてきた Sup-型関数は次の関数である。

$$S_0(x) := \sup_{t \in T} G(x, t) \quad x \in R^n, \tag{4}$$

この関数はチェビシェフ近似問題と密接に関係する。さらに、集合 T が x に依存してよいとすれば、$S_0(x)$ の最小化問題はパラメトリック最適化問題になる。$S(x)$ が $S_0(x)$ と本質的に異なる点は、後者は x と t が独立に動けるのに対し、前者は x が t に依存することである。しかしながら、$S_0(x)$ は $S(x)$ のスペシャルケースと見なすこともできる。つまり、X として n 次元ベクトル値定数関数全体 $\{x(t) \equiv a \mid a \in R^n\}$ をとればよい。従って、$S(x)$ は $S_0(x)$ の多くの性質を受け継ぐことになる。結論を先に述べると、相条件からも包絡線が生成される。

In the following, we denote by $T(x)$ the set of all extreme points $G(x(\cdot), \cdot)$, that is,

$$T(x) := \{t \in T \; ; \; G(x(t), t) = S(x)\}, \quad x \in C(T)^n.$$

THEOREM 1 *The function $S(x)$ is continuous.*

THEOREM 2 *The function $S(x)$ is directionally differentiable in any direction $y \in X$, and its directional derivative is given by*

$$S'(x; y) = \max\{G_x(x(t), t) y(t); \; t \in T(x)\}. \tag{5}$$

Applying Theorem 2 to the sup-type function induced from the one-sided phase constraint:

$$s(t) \leq x(t) \ ^{\forall}t, \tag{6}$$

where $s(t)$ is a given continuous function, we get the following result:

COROLLARY 1 *Let $s \in C(T)$. Take $G(x,t) := s(t) - x$ for any $x \in \mathrm{R}$ and $t \in T$. Then*

$$S'(x;y) = -\min_{t\in T(x)} y(t).$$

Taking constant functions as $x(t)$ and $y(t)$ in Theorem 2 , we get Danskin's formula.

COROLLARY 2 *(Danskin[3]) The function $S_0(x)$ is directionally differentiable in any direction $y \in R^n$ and its directional derivative is given by*

$$S_0'(x;y) = \max\{G_x(x,t)y;\ t \in T(x)\}. \tag{7}$$

Next, we consider a second-order directional derivative of $S(x)$.

DEFINITION 1 *The upper second-order directional derivative of $S(x)$ at x in the direction y is defined by*

$$\overline{S}''(x;y) = \mathop{limsup}_{\varepsilon\to+0} \frac{S(x+\varepsilon y) - S(x) - \varepsilon S'(x;y)}{\varepsilon^2} \tag{8}$$

DEFINITION 2 *([9]) For any functions $u,\ v \in C(T)$ satisfying*

$$\begin{cases} u(t) \geq 0 \ ^{\forall}t \in T, \\ v(t) \geq 0 \ \ if \ \ u(t) = 0, \end{cases} \tag{9}$$

we define a function $E : T \to [-\infty, +\infty]$ by

$$E(t) := \begin{cases} sup\left\{limsup\frac{v(t_n)^2}{4u(t_n)};\ \{t_n\}\ satisfies\ (11)\right\}, & if\, t \in T_0, \\ 0 & if\, u(t) = v(t) = 0\ and\ t \notin T_0, \\ -\infty & otherwise, \end{cases} \tag{10}$$

$$T_0 := \left\{t \in T;\ ^{\exists}t_n \to t\ s.t.\ u(t_n) > 0,\ -\frac{v(t_n)}{u(t_n)} \to +\infty\right\}. \tag{11}$$

THEOREM 3 *Let x and y be arbitrary functions in $C(T)^n$. Then it holds that*

$$\overline{S}''(x;y) = \max\left\{\frac{y(t)^T G_{xx}(x(t),t)y(t)}{2} + E(t) \ ; \ t \in T(x;y)\right\}, \tag{12}$$

where $T(x;y) := \{t \in T(x) \ ; \ S'(x;y) = G_x(x(t),t)y(t)\}$ and $E(t)$ is defined via Definition 2 by taking

$$u(t) = S(x) - G(x(t),t), \quad v(t) = S'(x;y) - G_x(x(t),t)y(t). \tag{13}$$

Taking constant functions as $x(t)$ and $y(t)$ in Theorem 3 , we get the following formula due to [9].

COROLLARY 3 *Let x and y be arbitrary points in R^n. Then it holds that*

$$\overline{S}''(x;y) = \max\left\{\frac{y^T G_{xx}(x,t)y}{2} + E(t) \ ; \ t \in T(x;y)\right\}, \tag{14}$$

where $E(t)$ is defined via Definition 2 by taking

$$u(t) = S(x) - G(x,t), \quad v(t) = S'(x;y) - G_x(x,t)y. \tag{15}$$

We proved in [9] and [10] that an envelope is formed from $G(x,t)$ when $E(t) > 0$ at some point t. Similarly, an envelope is formed from $G(x(t),t)$ when $E(t) > 0$ at some t.

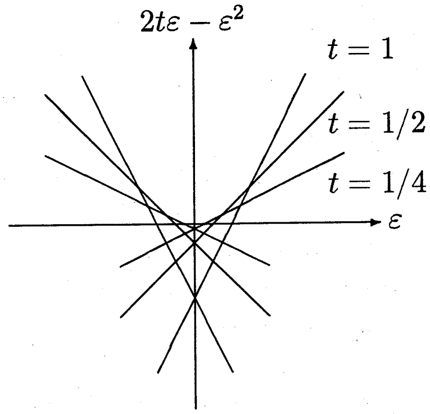
Example We can find an envelope even in the simplest one-sided phase constraint:

$$x(t) \geq 0 \ \ ^\forall t,$$

that is, $G(x,t) = -x$. Let $x(t) := t^2$, $T := [-1,1]$ and $y(t) := -2t$. Then

$$\begin{aligned}
\phi(\varepsilon) &:= S(x+\varepsilon y) \\
&= \max_{|t|\leq 1}\{-x(t) - \varepsilon y(t)\} \\
&= \max_{|t|\leq 1}\{2t\varepsilon - t^2\} \\
&= \begin{cases} \varepsilon^2 & |\varepsilon| \leq 1 \\ |2\varepsilon| - 1 & |\varepsilon| \geq 1 \end{cases}
\end{aligned}$$

For each $t \in [-1,1]$, the function $2t\varepsilon - t^2$ is affine w.r.t. ε. However, these affine functions form the envelope $\phi(\varepsilon) = \varepsilon^2$ near $\varepsilon = 0$.



It is clear from the definition of the upper second-order directional derivative that

$$\overline{S}''(x;y) = \limsup_{\varepsilon \to +0} \frac{\phi(\varepsilon) - \phi(0) - \varepsilon\phi'(0)}{\varepsilon^2} = 1.$$

On the other hand, the functions $u(t)$ and $v(t)$ defined by (13) become

$$u(t) = S(x) - G(x(t), t) = 0 - (-x(t)) = t^2,$$

$$v(t) = S'(x;y) - G_x(x(t), t)y(t) = -y(0) - (-y(t)) = -2t,$$

respectively. Hence

$$E(t) = \begin{cases} 1, & t = 0, \\ -\infty, & t \neq 0. \end{cases}$$

Since $G(x,t)$ is affine w.r.t. x, its second partial derivative vanishes. So the right hand side of (12) equals 1.

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