数理解析研究所講究録
939 巻 1996 年 49-53

Inverse of the Berge Maximum Theorem

慶應義塾大学商学部　小宮 英敏 (Hidetoshi Komiya)

Summary. We consider an inverse of the Berge maximum theorem. We also give an application of our result to fixed point theory.

1 Introduction

The Berge maximum theorem appears often in the area of general equilibrium theory and it is one of the fundamental principles of mathematical economics.

Theorem 1.1 (Berge) *Let X be a subset of l-dimensional Euclidean space R^l and let Y be a subset of m-dimensional Euclidean space R^m. Let $u : X \times Y \to R$ be continuous and let $S : X \twoheadrightarrow Y$ be continuous and compact-valued. Then, the correspondence $K : X \twoheadrightarrow Y$ defined by*

$$K(x) = \{y \in S(x) : u(x, y) = \max_{z \in S(x)} u(x, z)\}, \quad x \in X \tag{1}$$

is upper hemicontinuous and compact-valued.

The maximum theorem is often used in a specific situation such that the correspondence S is convex-valued and u is quasi-concave in its second variable in addition to the hypotheses of Theorem 1.1 in general equilibrium theory ([1, pages 47 and 72]). In this case, the conclusion of the maximum theorem is strengthened to that the correspondence K is upper hemicontinuous and compact *convex*-valued.

We shall consider an inverse of this special maximum theorem in Section 2. Roughly speaking, we shall consider the following problem:

> Does there exist a continuous function u with some convexity which produces K with the equation (1), provided that a correspondence K is upper hemicontinuous and compact convex-valued?

In section 3, we shall study relationship between the Kakutani fixed point theorem and the Fan-Browder fixed point theorem. We shall derive the Kakutani fixed point theorem from the Fan-Browder fixed point theorem by means of our result obtained in Section 2.

2 Result

Throughout this section, X is a subset of R^l. Let $B(x,\varepsilon)$ and $\overline{B}(x,\varepsilon)$ be open and closed balls with center x and radius ε, respectively. We need several lemmas to prove our result. The first lemma is a special version of [5, Corollary 1] (see also the remarks after [5, Corollary 1]).

Lemma 2.1 *Let $K : X \twoheadrightarrow R^m$ be a nonempty compact convex-valued upper hemicontinuous correspondence. Then there is a sequence $\{A_n\}_{n=1}^{\infty}$ of compact convex-valued continuous correspondences $A_n : X \twoheadrightarrow R^m$ such that, for each $x \in X$,*

$$K(x) \subset A_n(x) \subset A_{n'}(x) \quad \textit{for } n > n'$$

and

$$K(x) = \bigcap_{n=1}^{\infty} A_n(x).$$

Lemma 2.2 *Let $A : X \twoheadrightarrow R^m$ be a nonempty compact-valued lower hemicontinuous correspondence. Then, for any $x \in X$ and $\varepsilon > 0$, there is $\delta > 0$ such that*

$$A(x) \subset A(x') + B(0,\varepsilon), \quad x' \in B(x,\delta) \cap X.$$

Proof Let $x \in X$ and $\varepsilon > 0$. Since A is lower hemicontinuous, for any $y \in A(x)$, there is $\delta(y) > 0$ such that $A(x') \cap B(y, \frac{\varepsilon}{2}) \neq \emptyset$ for $x' \in B(x, \delta(y)) \cap X$. Since $A(x)$ is compact, there are finite number of $y_i \in A(x)$ such that $\bigcup_i B(y_i, \frac{\varepsilon}{2}) \supset A(x)$. Set $\delta = \min_i \delta(y_i)$. Then we claim that $A(x) \subset A(x') + B(0,\varepsilon)$ for $x' \in B(x,\delta)$. Take $x' \in B(x,\delta) \cap X$ and $y \in A(x)$. Then there is y_i such that $B(y_i, \frac{\varepsilon}{2}) \ni y$. Since $x' \in B(x,\delta) \cap X \subset B(x, \delta(y_i)) \cap X$, we have $A(x') \cap B(y_i, \frac{\varepsilon}{2}) \neq \emptyset$, and there is $z \in A(x')$ such that $z \in B(y_i, \frac{\varepsilon}{2})$. Therefore, we have $y \in B(z,\varepsilon)$, and hence $y \in A(x') + B(0,\varepsilon)$, that is, $A(x) \subset A(x') + B(0,\varepsilon)$. □

Lemma 2.3 *Let $A : X \twoheadrightarrow R^m$ be a nonempty compact-valued lower hemicontinuous correspondence. Then, for any $\varepsilon > 0$, $A^\varepsilon : X \twoheadrightarrow R^m$ defined by*

$$A^\varepsilon(x) = A(x) + B(0,\varepsilon), \quad x \in X$$

has open graph in $X \times R^m$.

Proof Take (x,y) in $\mathrm{Gr}(A^\varepsilon)$, where $\mathrm{Gr}(A^\varepsilon)$ denotes the graph of A^ε. Set $\varepsilon' = (\varepsilon - d(y, A(x)))/3 > 0$, where $d(y, A(x))$ denotes the distance between the point y and the set $A(x)$. Then there is $\delta > 0$ such that $Ax \subset Ax' + B(0,\varepsilon')$ for $x' \in B(x,\delta) \cap X$ by Lemma 2.2. If $(x', y') \in (B(x,\delta) \cap X) \times B(y,\varepsilon')$, then we have

$$y' \in B(y,\varepsilon') \subset A(x) + B(0, \varepsilon - 2\varepsilon') + B(0,\varepsilon') \subset$$

$$\subset A(x') + B(0,\varepsilon') + B(0,\varepsilon - 2\varepsilon') + B(0,\varepsilon') \subset A(x') + B(0,\varepsilon) = A^\varepsilon(x').$$

This means that $(x', y') \in \mathrm{Gr}(A^\varepsilon)$, and $\mathrm{Gr}(A^\varepsilon)$ is open in $X \times R^m$. □

The following theorem is our main result.

Theorem 2.1 *Let X be a subset of R^l. Let $K : X \twoheadrightarrow R^m$ be a nonempty compact convex-valued upper hemicontinuous correspondence. Then there exists a continuous function $v : X \times R^m \to [0,1]$ such that*

(i) $K(x) = \{y \in R^m : v(x,y) = \max_{z \in R^m} v(x,z)\}$ *for any* $x \in X$*;*

(ii) $v(x,y)$ *is quasi-concave in* y *for any* $x \in X$.

Proof Let D be the set of all positive dyadic rational numbers, that is, the set of all numbers of the form $n/2^{n'}$ for positive integers n and n'. We construct a family $\{G_t\}_{t \in D}$ of correspondences from X to R^m as follows. For $t \in D$ with $t \geq 1$, set $G_t(x) = R^m$ for all $x \in X$. For $t \in D$ with $0 < t < 1$, take its binary expansion

$$t = \frac{t_1}{2} + \frac{t_2}{2^2} + \cdots + \frac{t_n}{2^n}, \quad \text{where } t_i = 0 \text{ or } 1,$$

and set $l(t) = \min\{i : t_i = 1\}$. Define $G_t : X \twoheadrightarrow R^m$ by

$$G_t(x) = A_{l(t)}(x) + B(0,t), \quad x \in X$$

by means of the sequences $\{A_n\}_{n=1}^{\infty}$ of correspondences obtained in Lemma 2.1. It is easily seen that G_t is convex-valued and $\overline{G_s(x)} \subset G_t(x)$ for $s < t$. Define a correspondence $G_t{}' : X \twoheadrightarrow R^m$ for each $t \in D$ by

$$G_t{}'(x) = \overline{G_t(x)} = A_{l(t)}(x) + \overline{B}(0,t), \quad x \in X.$$

Since $G_t{}'$ is compact convex-valued and continuous, it has closed graph, and hence, we have $\mathrm{Gr}(G_t{}') = \overline{\mathrm{Gr}(G_t)}$. Therefore, we have

$$\overline{\mathrm{Gr}(G_s)} \subset \mathrm{Gr}(G_t) \quad \text{for} \quad s < t.$$

On the other hand, the correspondence G_t has open graph by Lemma 2.3 because A_n is lower hemicontinuous.

Define a function $w : X \times R^m \to [0,1]$ by

$$w(x,y) = \inf\{t \in D : (x,y) \in \mathrm{Gr}(G_t)\}.$$

Then w is continuous and we have

$$\{y \in R^m : w(x,y) \leq s\} = \bigcap_{t>s} G_t(x)$$

for any $s \in R$ and $x \in X$ by [2, Lemma 4.2 and Lemma 4.3]. Therefore, w is quasi-convex in its second variable. We have

$$K(x) = \{y \in R^m : w(x,y) = \min_{z \in R^m} w(x,z)\}$$

for all $x \in X$ because

$$K(x) = \bigcap_{n=1}^{\infty} A_n(x) = \bigcap_{t \in D} G_t(x) = \{y \in R^m : w(x,y) = 0\}.$$

Therefore, $v = -w + 1$ is the desired function. □

3 Application to Fixed Point Theory

We consider two fixed point theorems for correspondences which play crucial roles in general equilibrium theory. One is the Kakutani fixed point theorem and the other is the Fan-Browder fixed point theorem (see, for example, [1] and [6]):

Theorem 3.1 (Kakutani) *Let C be a compact convex subset of R^m. Let $F : C \twoheadrightarrow C$ be a nonempty closed convex-valued upper hemicontinuous correspondence. Then there is a point $x_0 \in C$ such that $x_0 \in F(x_0)$.*

Theorem 3.2 (Fan-Browder) *Let C be a compact convex subset of R^m. Let $\varphi : C \twoheadrightarrow C$ be a nonempty convex-valued correspondence such that $\varphi^{-1}(y)$ is open in C for $y \in C$. Then there is a point $x_0 \in C$ such that $x_0 \in \varphi(x_0)$.*

We show that the Kakutani fixed point theorem can be derived from the Fan-Browder fixed point theorem with the aid of Theorem 2.1. The method of the derivation is inspired by the simple proofs of the K-K-M-S theorem in [3] and [4].

Derivation of Kakutani from Fan-Browder

Let F be a correspondence satisfying the hypothesis of the Kakutani fixed point theorem. Thanks to Theorem 2.1, we can find a continuous function $v : C \times R^m \to [0,1]$ such that $F(x) = \{y \in R^m : v(x,y) = \max_{z \in R^m} v(x,z)\}$ for any $x \in C$ and v is quasi-concave in its second variable. Since $F(x) \subset C$, we have $F(x) = \{y \in C : v(x,y) = \max_{z \in C} v(x,z)\}$. Define $\varphi : C \twoheadrightarrow C$ by

$$\varphi(x) = \{y \in C : v(x,x) < v(x,y)\}, \quad x \in C.$$

Then the correspondence φ satisfies all of the hypotheses of the Fan-Browder fixed point theorem but the nonemptiness of its values. It is obvious that φ has no fixed points. Therefore, the Fan-Browder fixed point theorem assures us the existence of a point x_0 such that $\varphi(x_0) = \emptyset$. This means that $v(x_0, x_0) = \max_{z \in C} v(x_0, z)$, that is, $x_0 \in F(x_0)$. □

4 Concluding Remark

We can interpret Theorem 1.1 in consumption models as follows: the set X is the space of price-wealth pairs, the function u is the utility function of a consumer, and S is the budget constraint of the consumer. Then K is the demand correspondence of the consumer. Thanks to Theorem 2.1, we can represent the demand correspondence K by a function v alone. Thus all the information of the consumers such as his utility and budget constraint is integrated in the function v in Theorem 2.1. The same argument holds in production models and the supply correspondence can be expressed by a single two-variable real-valued function.

References

[1] Debreu, G.: Theory of Value. New Haven, London: Yale University Press 1959

[2] Kelly J.L.: General Topology. Berlin, Heidelberg, New York: Springer 1955

[3] Komiya H.: A Simple Proof of K-K-M-S Theorem. Economic Theory 4, 463–466 (1994)

[4] Krasa S., Yannelis N.C.: An Elementary Proof of the Knaster-Kuratowski-Mazurkiewicz-Shapley Theorem. Economic Theory 4, 467–471 (1994)

[5] Tulcea C.I.: On the Approximation of Upper Semi-continuous Correspondences and the Equilibriums of Generalized Games. Journal of Mathematical Analysis and Applications 136, 267–289 (1988)

[6] Yannelis N.C., Prabhakar N.D.: Existence of Maximal Elements and Equilibria in Linear Topological Spaces. Journal of Mathematical Economics 12, 233–245 (1983)