数理解析研究所講究録
939巻 1996年 81-86

NORMAL STRUCTURE AND FIXED POINT PROPERTY FOR NONEXPANSIVE MAPPINGS

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1. Introduction

Let E be a Banach space and X be a weakly compact convex subset of E. Let $\mathcal{S} = \{T_s;\ s \in S\}$ be a continuous representation of a semitopological semigroup S as non-expansive self-maps on X. In this paper, we shall report, among other things , on some recent results concerning the relation of invariant submean on the space of bounded continuous real-valued functions on S, normal structure of K, and the existence of a common fixed point in X for $\mathcal{S}$. We shall also report on some sufficient or necessary conditions on a locally compact group G such that every weak*-compact convex subset of the Fourier Stieltjes algebra of G has normal structure and hence the fixed point property for nonexpansive mappings.

This paper contains part of our talk given during the 1995 RIMS Symposium on Nonlinear Analysis and Convex Analysis held at Kyoto University, Kyoto. We thank the organizers for their kind invitation to speak and their warm hospitality during the conference.

2. Normal Structure and Submean

Let E be a Banach space, D be a bounded subset of E, $u \in D$. Define

$$r_u(D) = \sup\{\|u - v\|;\ v \in D\}.$$

[1]This research is supported by an NSERC grant.

Then $r_u(D) \leq$ diam$(D) = \sup\{\|v_1 - v_2\|;\ v_1, v_2 \in D\}$. A point u in D is said to be diametral if

$$r_u(D) = \text{diam}(D).$$

Otherwise, u is said to be *non-diametral.*

A convex subset X of E is said to have *normal structure* if each closed convex subset D of X with diam$(D) > 0$ contains a non-diametral point i.e. there exists $u \in D$ such that

$$\sup\{\|u - v\|;\ v \in D\} < \text{diam}(D).$$

As well known compact convex subset of a Banach space E has normal structure. Also, uniformly convex Banach spaces have normal structure (see [4]). However it follows from [1] that weakly compact convex subset of a Banach space need not have normal structure.

Let S be a *semitopological semigroup* i.e. S is a semigroup with a Hausdorff topology such that for each $a \in S$, the mappings $s \mapsto as$ and $s \mapsto sa$ from $S \mapsto S$ are continuous. Let $CB_r(S)$ be the space of bounded real-valued functions on G. A *submean* μ is a real-valued function on $CB_r(S)$ satisfying:

(i) $\mu(f+g) \leq \mu(f) + \mu(g)$, $\quad f, g \in CB_r(S)$;

(ii) $\mu(\alpha f) = \alpha\mu(f)$, $\quad \alpha \geq 0, \quad f \in CB_r(S)$;

(iii) for $f, g \in CB_r(S)$, $\quad f \leq g, \quad \mu(f) \leq \mu(g)$;

(iv) $\mu(c) = c$ for every constant function c.

The notion of submean is due to Mizoguchi-Takahashi [12]. A submean μ is *left invariant* if $\mu(\ell_a f) = \mu(f)$ for all $a \in S$, $f \in CB_r(S)$ where $(\ell_a f)(t) = f(at)$, $t \in S$.

A semitopological semigroup is *left reversible* if $\overline{aS} \cap \overline{bS} \neq \phi$ for $a, b \in S$ (where $\overline{A}$ denotes the closure of A in S). There is a strong relation between left reversibility and submean:

Lemma 1 ([8]). *Let* S *be a semitopological semigroup.*

(a) *If* S *is left reversible, then* $CB_r(S)$ *has a left invariant submean.*

(b) *If* S *is normal, and* $CB_r(S)$ *has a left invariant submean, then* S *is left reversible.*

There is also a relation between normal structure and invariant submean.

Lemma 2 ([9]). *Let* X *be a weakly compact convex subset of a Banach space. If* X *has normal structure, then* X *has the following property:*

(P) *whenever* S *is a semitopological semigroup and* $\mathcal{S} = \{T_s;\ s \in S\}$ *is a continuous representation of* S *as nonexpansive self maps on* X, *if* μ *is a left invariant submean on* $CB_r(S)$, *then the set*

$$A_x = \{y \in X;\ \mu_t(\|T_t x - y\|) = \rho_x\}$$

is a proper subset of X *for some* $x \in X$, *where* $\rho_x = \inf\{\mu_t(\|T_t x - y\|);\ y \in X\}$. *Furthermore, for each* $x \in X$, *the set* A_x *is non-empty, closed, convex and* $\mathcal{S}$*-invariant.*

Lemmas 1 and 2 can be used to obtain the following generalization of Lim's result:

Theorem 3 ([9]). *Let* S *be a semitopological semigroup, and* X *be a non-empty weakly compact convex subset of a Banach space with noraml structure. If* $CB_r(S)$ *has a left invariant submean (e.g. when* S *is left reversible), then whenever* $\mathcal{S} = \{T_s;\ s \in S\}$ *is a continuous representation of* S *as non-expansive self maps on* X, X *contains a common fixed point in* X.

Problem: Does Theorem 3 remain valid when X is a weak*-compact convex subset of a dual Banach space and $\mathcal{S} = \{T_s;\ s \in S\}$ is a weak*-continuous representation of S?

The following is a partial solution to this problem:

Theorem 4 ([8]). *Let* S *be a semitopological semigroup. If* $CB_r(S)$ *has a non-zero left invariant continuous linear functional, then whenever* $\mathcal{S} = \{T_s;\ s \in S\}$ *is a representation of* S *as norm non-expansive mappings on a norm-separable weak*-compact convex subset* X *of a dual Banach space such that the mapping* $S \times X \mapsto X$, $(s, x) \mapsto T_s x$, *is jointly continuous when* X *has the weak*-topology, then* X *has a common fixed point for* $\mathcal{S}$.

3. Fixed Point Property and the Fourier-Stieltjes Algebra

Let G be a locally compact group with a fixed left Haar measure λ. The standard Lebesgue space of integrable functions with respect to λ will be denoted by $L^1(G)$; $CB(G)$ will denote the space of all bounded continuous complex-valued functions on G and $C_{00}(G)$ will denote the space of functions in $CB(G)$ with compact support. Let $P(G) \subseteq CB(G)$ be the set of continuous positive definite functions on G, $B(G)$ its linear span. The space $B(G)$ can be identified with the dual of the group C^*-algebra $C^*(G)$, this latter being the completion of $L^1(G)$ under its largest C^*-norm. Indeed, we have the duality

$$\langle \phi, f \rangle = \int_G \phi f d\lambda, \quad (\phi \in B(G), f \in L^1(G)).$$

With pointwise multiplication and the dual norm, $B(G)$ is a commutative Banach algebra called the *Fourier-Stieltjes algebra* of G. The *Fourier algebra* $A(G)$ of G is the closed linear span of $P(G) \cap C_{00}(G)$ in $B(G)$. When G is abelian, then $A(G) \cong L^1(\widehat{G})$ and $B(G) \cong M(\widehat{G})$ where $\widehat{G}$ is the dual group of G (see [3]).

Let E be a Banach space and X be a weakly compact convex subset of E. We say that X has the fpp (= fixed point property) if every nonexpansive mapping $T : X \to X$ (i.e. $\|Tx - Ty\| \le \|x - y\|$ for every $x,\ y \in X$) has a fixed point. The space X has the fpp if every weakly compact convex subset $X \subseteq E$ has fpp. It is well known that (Browder's Theorem [2]) uniformly convex Banach spaces have fpp. In [5] Kirk, extending

Browder's Theorem, showed that a weakly compact convex subset of a Banach space with normal structure has fpp.

It follows from Alspach's example [1] that if $G = (\mathbb{Z}, +)$, then $A(\mathbb{Z}) \cong L^1(\mathbb{T})$ (hence $B(\mathbb{Z})$) does not have the fpp, where $\mathbb{T} = \{\lambda \in \mathbb{C};\ |\lambda| = 1\}$.

If E is a dual Banach space, E is said to have weak* fpp (= *weak* fixed point property*) if for every weak*-compact convex subset X of E has the fpp. It follows from [11] that if $G = \mathbb{T}$, then each weak*-compact convex subset of $B(\mathbb{T}) = A(\mathbb{T}) \cong \ell_1(\mathbb{Z})$ has normal structure. In particular, $B(\mathbb{T})$ has weak* fpp. More generally:

Theorem 5 ([6]). *Let G be a locally compact abelian group. The following are equivalent:*

(a) *G is compact*

(b) *Each weak*-compact convex subset of $B(G)$ has normal structure.*

(c) *$B(G)$ has weak** fpp.

Problem: Does "$B(G)$ has weak* fpp" imply "G is compact"?

In general, "$B(G)$ has fpp" does not imply "G is compact". Indeed, if G is the Fell group (which is the natural semi-direct product of the p-adic numbers with the compact group of p-adic units for a fixed prime p), then G is non-compact but $B(G)$ has fpp as shown in [7].

A locally compact group G is called an [IN]-group if there is a compact neighborhood U of the identity e such that $x^{-1} \cup x = U$ for all $x \in G$. This includes all groups G such that the left and right uniformities coincide. Examples of [IN]-groups include abelian groups, compact groups and discrete groups.

Theorem 7 ([7]). *If G is a connected* [IN]*-group, then G is compact if and only if $A(G)$ (or $B(G)$) has the* fpp.

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