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2階$Euler$型方程式の振動問題について

信州大学理学部　　杉江実郎 (Jitsuro Sugie)

大阪府立大学工学部　原　惟行 (Tadayuki Hara)

1. Introduction and statement of results

We consider the oscillation problem for the second order nonlinear differential equation

$$(1) \qquad t^2 x'' + g(x) = 0, \qquad t > 0,$$

where $g(x)$ satisfies locally Lipschitz continuous on $\mathbf{R}$ and

$$xg(x) > 0 \quad \text{if} \quad x \neq 0.$$

A nontrivial solution of (1) is said to be *oscillatory* if it has arbitrarily large zeros. Otherwise, the solution is said to be *nonoscillatory*. In the theory of oscillations, the number $\frac{1}{4}$ very often appears as a critical value. The following result is a good illustration of this fact: all nontrivial solutions of Euler's equation

$$(2) \qquad t^2 x'' + \lambda x = 0$$

are oscillatory if and only if $\lambda > \frac{1}{4}$. Other examples are found in [3, 6] and the references cited therein.

Because of Sturm's separation theorem, the solutions of second order linear differential equations are either all oscillatory or all nonoscillatory, but cannot be both. Thus, we can classify second order linear differential equations into the two types. However, the oscillation problem for (1) is not so easy, because $g(x)$ is nonlinear.

Judging from the oscillation result for Euler's equation (2), we see that all nontrivial solutions of (1) have a tendency to be oscillatory according as $g(x)$ grows larger in some sense; and we must consider the case

$$\frac{g(x)}{x} \to \frac{1}{4} \quad \text{as} \quad |x| \to \infty \tag{3}$$

to solve completely the oscillation problem for (1).

The purpose of this report is to give our answer to this delicate problem. Our main results are stated in the following:

Theorem 1. *Let $\lambda > 0$. Then all nontrivial solutions of* (1) *are oscillatory if*

$$\frac{g(x)}{x} \geq \frac{1}{4} + \frac{\lambda}{\log|x|} \tag{4}$$

for $|x| > R$ with a sufficienlty large $R > 0$.

Theorem 2. *Suppose that there exists a λ with $0 < \lambda < \frac{1}{4}$ such that*

$$\frac{g(x)}{x} \leq \frac{1}{4} + \left(\frac{\lambda}{\log|x|}\right)^2 \tag{5}$$

for $x > R$ or $x < -R$ with a sufficiently large $R > 0$. Then all nontrivial solutions of (1) *are nonoscillatory.*

Remark. We note that condition (3) is satisfied in either case

$$g(x) = \frac{1}{4}x + \frac{\lambda x}{\log|x|} \qquad \text{with} \quad \lambda > 0$$

or

$$g(x) = \frac{1}{4}x + \left(\frac{\lambda}{\log|x|}\right)^2 x \qquad \text{with} \quad 0 < \lambda < \frac{1}{4}$$

for $|x|$ sufficiently large.

2. Some lemmas

The change of variable $t = e^s$ reduces (1) to the equation

$$\ddot{x} - \dot{x} + g(x) = 0, \qquad s \in \mathbf{R},$$

where $\dot{} = \frac{d}{ds}$. This equation is equivalent to the system

$$\begin{aligned} \dot{x} &= y + x \\ \dot{y} &= -g(x) \end{aligned} \tag{6}$$

which is of Liénard type. Note that every solution of (6) exists in the future.

We give some results on the asymptotic behavior of trajectories of (6). We write $\gamma^+(P)$ for the positive semitrajectory of (6) starting at a point $P \in \mathbf{R}^2$.

Lemma 1. *For each point $P = (p, -p)$ with $p > 0$, the positive semitrajectory $\gamma^+(P)$ crosses the negative y-axis.*

Lemma 2. *For each point $P = (-p, p)$ with $p > 0$, the positive semitrajectory $\gamma^+(P)$ crosses the positive y-axis.*

We here introduce a new important concept which is useful in the theory of oscillations. We say that system (6) has *property* (X^+) *in the right half-plane* (resp., *left half-plane*) if, for every point P in the region $\{(x, y) \colon x \geq 0 \text{ and } y > -x\}$ $\left(\text{resp.,} \{(x, y) \colon x \leq 0 \text{ and } y < -x\}\right)$, the positive semitrajectory $\gamma^+(P)$ crosses the curve $y = -x$.

In [1] the authors went into details about property (X^+) and gave some necessary conditions and some sufficient conditions for property (X^+). We state below special cases of those results. Let

$$G(\infty) = \int_0^{\infty} g(\xi)d\xi \qquad \text{and} \qquad G(-\infty) = \int_0^{-\infty} g(\xi)d\xi.$$

Lemma 3 [1, Theorem 4.1]. *Assume $G(\infty) < \infty$ (resp., $G(-\infty) < \infty$). Then system (6) fails to have property (X^+) in the right half-plane (resp., left half-plane).*

Lemma 4 [1, Theorem 5.4]. *Assume $G(\infty) = \infty$ (resp., $G(-\infty) = \infty$). Then system (6) fails to have property (X^+) in the right half-plane (resp., left half-plane) if*

$$|x| \geq 2\sqrt{2G(x)} - h\left(\sqrt{2G(x)}\right) \quad \textit{for } |x| \textit{ sufficiently large,} \tag{7}$$

where $h(r)$ is a continunous function on $[0, \infty)$ such that for r sufficiently large

$$\frac{h(r)}{r} \textit{ is non-increasing and non-negative;} \tag{8}$$

$$r\left(\int_r^{\infty} \frac{h(\xi)}{\xi^2} d\xi\right)^2 \leq \frac{1}{4} h(r). \tag{9}$$

Lemma 5 [1, Theorem 5.2]. *Assume* $G(\infty) = \infty$ *(resp.,* $G(-\infty) = \infty$*). Then system* (6) *has property* (X^+) *in the right half-plane (resp., left half-plane) if*

$$|x| \leq 2\sqrt{2G(x)} - h\left(\sqrt{2G(x)}\right) \quad \text{for } |x| \text{ sufficiently large,} \tag{10}$$

where $h(r)$ *is a continunous function on* $[0, \infty)$ *with*

(11) $\quad \dfrac{h(r)}{r}$ *is non-increasing, non-negative and is not greater than* 2 *for* r *sufficienlty large;*

$$\int^{\infty} \frac{h(r)}{r^2} dr = \infty. \tag{12}$$

3. Proof of the theorems

Proof of Theorem 1.1. Each solution of (1.1) exists in the future. Suppose that system (6) which is equivalent to (1) has property (X^+) in the right and left half-plane. Then it follows from Lemmas 1 and 2 that every solution of (6) keeps on rotating around the origin except the zero solution. Hence, all nontrivial solutions of (1) are oscillatory. Thus, to prove Theorem 1.1, it is enough to show that system (6) has property (X^+) in the right and left half-plane. We will demonstrate this fact by means of Lemma 5. Note that (4) implies $G(\pm\infty) = \infty$.

Let $0 < \nu < \lambda$ and

$$h(r) = \frac{\nu r}{\log r}$$

for r sufficienlty large. Then it is clear that conditions (11) and (12) are satisfied. We next define continuous functions $k(x), K(x)$ and $L(x)$ on $\mathbf{R}$ by

$$k(x) = \frac{\lambda x}{\log |x|}, \quad K(x) = \int_0^x k(\xi) d\xi \quad \text{and} \quad L(x) = \frac{\lambda x^2}{2 \log |x|}$$

for $|x|$ sufficiently large, respectively. Then we have

$$K(x) \geq L(x) - M \qquad \text{for some } M > 0$$

and by (4)

$$G(x) \geq \frac{1}{8}x^2 + K(x) - N \qquad \text{for some } N > 0.$$

Since $xK(x)$ is increasing for $|x|$ sufficiently large, we get

$$\begin{aligned} K\left(2u-\frac{\nu u}{\log|u|}\right)-\frac{\nu u^2}{2\log|u|} &\geq K(u)-\frac{\nu u^2}{2\log|u|} \\ &\geq L(u)-M-\frac{\nu u^2}{2\log|u|} \\ &= \frac{(\lambda-\nu)u^2}{2\log|u|}-M \end{aligned}$$

which tends to ∞ as $|u|\to\infty$. Hence, for $|u|$ sufficiently large

$$\begin{aligned} \frac{1}{2}u^2 &\leq \frac{1}{2}u^2+K\left(2u-\frac{\nu u}{\log|u|}\right)-\frac{\nu u^2}{2\log|u|}-N+\frac{\nu^2u^2}{8(\log|u|)^2} \\ &= \frac{1}{8}\left(2u-\frac{\nu u}{\log|u|}\right)^2+K\left(2u-\frac{\nu u}{\log|u|}\right)-N \\ &\leq G\left(2u-\frac{\nu u}{\log|u|}\right), \end{aligned}$$

namely,

$$\frac{1}{2}u^2 \leq \begin{cases} G(2u-h(u)) & \text{if } \ u>0 \\ G(2u+h(-u)) & \text{if } \ u<0. \end{cases}$$

Letting

$$u=\begin{cases} \sqrt{2G(x)} & \text{if } \ x>0 \\ -\sqrt{2G(x)} & \text{if } \ x<0, \end{cases}$$

we have

$$|x| \leq 2\sqrt{2G(x)}-h\left(\sqrt{2G(x)}\right)$$

for $|x|$ sufficiently large, that is, condition (10) is also satisfied. Thus, by Lemma 5 system (6) has property (X^+) in the right and left half-plane. The proof is complete.

To prove Theorem 2, we need Lemmas 6 and 7 below.

Lemma 6. *Every solution of* (6) *are unbounded except the zero solution.*

Let

$$V(x,y)=\frac{1}{2}y^2+G(x)$$

and consider the curve

$$V(x,y)=V(x_0,y_0),$$

where $x_0 > 0$. Then there exist two points of intersection of the curve with the line $y = -x$. In fact, the equation

$$V(x, -x) = V(x_0, y_0)$$

has exactly two roots because $V(x, -x)$ is increasing for $x > 0$ and decreasing for $x < 0$, and $V(0, 0) = 0$. Let $(-a, a)$ and $(b, -b)$ be the intersecting points, where $a > 0$ and $b > 0$. Define

$$S = \{(x, y) \colon -a \leq x \leq c \ \text{ and } \ V(x, y) \leq V(x_0, y_0)\}$$

in which $c = \max\{b, x_0\}$. Then it is clear that S is a bounded set. Lemma 6 shows that every solution of (6) starting in $S \setminus \{0\}$ does not remain in S. Take note of the vector field of (6) and the fact that

$$\dot{V}_{(6)}(x, y) = xg(x) > 0 \qquad \text{if} \quad x \neq 0.$$

Then we also see that every solution of (6) starting in S^c, the complement of S in $\mathbf{R}^2$, stays in S^c for all future time. Thus, we have

Lemma 7. *Every solution of* (6) *starting in* $S \setminus \{0\}$ *enters* S^c *which is a positive invariant set with respect to* (6).

Proof of Theorem 1.2. We prove only the case that condition (5) is satisfied for $x > R$, because the other case is carried out in the same way.

First, we will show that system (6) fails to have property (X^+) in the right half-plane. If $G(\infty) < \infty$, then this fact is clear because of Lemma 3. Suppose that $G(\infty) = \infty$. To use Lemma 4, we will check that conditions (7)–(9) hold.

Let

$$h(r) = \frac{r}{4(\log r)^2}$$

for r sufficiently large. Then $\frac{h(r)}{r}$ is non-increasing and non-negative; and we have

$$r\left(\int_r^\infty \frac{h(\xi)}{\xi^2} d\xi\right)^2 = \frac{r}{16(\log r)^2} = \frac{1}{4}h(r),$$

that is, conditions (8) and (9) are satisfied. Define continuous functions $k(x)$ and $L(x)$ on $\mathbf{R}$ by

$$k(x) = \left(\frac{\lambda}{\log x}\right)^2 x \quad \text{and} \quad L(x) = \left(\frac{\nu x}{\log x}\right)^2$$

for $x > R$ with $\lambda^2 < 2\nu^2 < \frac{1}{16}$. Then

$$K(x) \equiv \int_0^x k(\xi)d\xi$$

is increasing for $x > R$, and there exist constants $M > 0$ and $N > 0$ such that

$$L(x) + M \geq K(x)$$

and

$$G(x) \leq \frac{1}{8}x^2 + K(x) + N$$

for $x > 0$. Hence, we obtain

$$\begin{aligned}
-\frac{1}{2}uh(u) + \frac{1}{8}(h(u))^2 + K(2u - h(u)) &\leq -\frac{u^2}{8(\log u)^2} + \frac{u^2}{128(\log u)^4} + K(2u) \\
&\leq -\frac{u^2}{8(\log u)^2} + \frac{u^2}{128(\log u)^4} + L(2u) + M \\
&\leq -\frac{(1-32\nu^2)u^2}{8(\log u)^2} + \frac{u^2}{128(\log u)^4} + M \\
&\to -\infty \qquad \text{as} \quad u \to \infty,
\end{aligned}$$

and therefore, for u sufficiently large

$$\begin{aligned}
\frac{1}{2}u^2 &\geq \frac{1}{2}u^2 - \frac{1}{2}uh(u) + \frac{1}{8}(h(u))^2 + K(2u - h(u)) + N \\
&= \frac{1}{8}(2u - h(u))^2 + K(2u - h(u)) + N \\
&\geq G(2u - h(u)).
\end{aligned}$$

Let $u = \sqrt{2G(x)}$. Then we have

$$x \geq 2\sqrt{2G(x)} - h\left(\sqrt{2G(x)}\right)$$

for x sufficiently large. Thus, condition (7) is also satisfied, and so system (6) fails to have property (X^+) in the right half-plane by Lemma 4. Hence, there exists a point $P_0(x_0, y_0)$ with $x_0 \geq 0$ and $y_0 > -x_0$ such that $\gamma^+(P_0)$ runs to infinity without intersecting the curve $y = -x$.

We here suppose that (1) has a oscillatory solution. Let $\gamma^+(Q)$ be the positive semitrajectory which corresponds to the oscillatory solution of (1). By virtue of Lemma 7, we see that $\gamma^+(Q)$ eventually goes around the set S inifinity many times. Hence, it crosses the half-line $\{(x, y): x = x_0 \text{ and } y > y_0\}$ at a point $P_1(x_0, y_1)$ with $y_1 > y_0$. From the uniqueness of solutions for the initial value problem, it turns out that

(i) $\gamma^+(Q)$ coincides with $\gamma^+(P_1)$ except for the arc QP_1.

(ii) $\gamma^+(P_1)$ lies above $\gamma^+(P_0)$.

Hence, $\gamma^+(Q)$ runs to infinity without crossing the curve $y = -x$. This contradicts the fact that $\gamma^+(Q)$ circles the set S. The proof is now complete.

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