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An application of delay differential equations to market equilibrium

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We study the equilibrium of the market model described by a delay differential equation. We show conditions to ensure the stability of the equilibrium.

Consider the market of a certain commodity. We assume that this market is isolated from all the other markets . We also assume that the market is competitive in that all buyers and sellers take the price of the commodity as given.

Letting p be the price, demand of buyers D is decided by a demand function $D = D(p)$ and supply of sellers S by a supply function $S = S(p)$. These functions are nonnegative, continuous and they satisfy the following:

$D'(p) < 0$ for $0 < p < p_1$ and $D(p) = 0$ for $p \geq p_1$;

$S(p) = 0$ for $0 \leq p \leq p_2$, $S'(p) > 0$ for $p_2 < p < p_3$ and $S(p)$ is a constant for $p \geq p_3$. Here p_i are positive constants.

Suppose $p_2 < p_1$, then there exists a unique equilibrium price p^* such that $D(p^*) = S(p^*)$.

In the traditional market model, the excess demand for the commodity raises the price. Thus, the traditional market equation is given by

$$\dot{p} = D(p(t)) - S(p(t)).$$

In this model demand and supply at time t is decided by the price at t.

In economics, the market is said to be stable if the equilibrium price p^*, that is, the solution $p \equiv p^*$ of the equation is globally asymptotically stable. We follow this economic definition here.

It is easy to prove the market is stable in the traditional model. The stability is independent of the demand and the supply functions. Therefore, this model shows any competitive isolated markets are stable and cannot explain instability of certain markets.

We modify the model and assume a delay between supply and demand. More precisely, we assume supply at time t is decided by the price at $t-h$.

Thus the model is described by the differential equation with a delay $h > 0$

$$\dot{p} = D(p(t)) - S(p(t-h)).$$

The commodity should be obtained, produced, manufactured or transported before supply. So the delay occurs. We show the stability conditions on the delay model and their economic interpretation.

Theorem. *In this model the market is stable if*

$$|D(p) - D(p^*)| > |S(p) - S(p^*)| \text{ for } p \neq p^* \tag{1}$$

is satisfied or if

$$h\left(|D(p) - D(p^*)| + |S(p) - S(p^*)|\right) \leq |p - p^*| \tag{2}$$

is satisfied.

Proof. The thoerem is proved by Liapunov functionals. If condition (1) is satisfied, consider the functional

$$V(p_t) = |p(t) - p^*| + \int_{t-h}^{t} |S(p(\tau)) - S(p^*)| \, d\tau.$$

Then

$$\dot{V}(p_t) \;\leq\; -|D(p(t)) - D(p^*)| + |S(p(t-h)) - S(p^*)|$$

$$
\begin{aligned}
&+|S(p(t)) - S(p^*)| - |S(p(t-h)) - S(p^*)| \\
= \ & -(|D(p(t)) - D(p^*)| - |S(p(t)) - S(p^*)|)
\end{aligned}
$$

is obtained.

Thus, $p \equiv p^*$ is globally asymptotically stable by a standard argument for functional differential equations with finite delay, since the equation is autonomous.

If (2) holds, we consider

$$
\begin{aligned}
V(p_t) &= \left(p(t) - p^* - \int_{t-h}^{t} (S(p(\tau)) - S(p^*))\, d\tau\right)^2 \\
&\quad + \int_{t-h}^{t} (\tau - (t-h))\,(S(p(\tau)) - S(p^*))^2\, d\tau.
\end{aligned}
$$

The derivative along the solution of the equation satisfies

$$
\begin{aligned}
\dot{V}(p_t) &\leq 2(p(t) - p^*)(D(p(t)) - S(p(t))) \\
&\quad + h(D(p(t)) - S(p(t)))^2 + h(S(p(t)) - S(p^*))^2 \\
&\leq -|p(t) - p^*||D(p(t)) - D(p^*)|.
\end{aligned}
$$

Therefore $p \equiv p^*$ is globally asymptotically stable.

This completes the proof.

The result has the following economic meaning.

1. If the amount of the supply cannot be rapidly increased, compare to that of the demand, in some market, the condition (1) is satisfied and the market is stable. This corresponds to the result of the well-known cobweb model(difference equation model).

2. If the commodity can be supplied without spending time, that is, if the delay for supply becomes small enough, condition (2) is satisfied and the market is stable even if the supply curve is steep.

3. Suppose there is a small delay caused by sellers in the market. Buyers can increase demand and demand curve can be steep so that condition (2) is not satisfied. Neither condition (1) hold if the amount of the supply is larger than that of demand. Therefore, stability of the market will not be guaranteed.