数理解析研究所講究録
940 巻 1996 年 47-61

Quadratic optimal control problems for linear damped second order systems in Hilbert spaces

Jun-hong HA（河準洪）
Division of System Science, The Graduate School of Science and Technology, Kobe University
Shin-ichi NAKAGIRI（中桐信一）
Department of Applied Mathematics, Faculty of Engineering, Kobe University

1 Introduction

In the memorial work of Lions[6] the optimal control theory for distributed parameter systems has been developed in full extent. The optimal control theory of [6] covers a wide variety of distributed parameter systems, e.g., elliptic, parabolic, hyperbolic and other types of systems, and a great number of results including optimality conditions in terms of adjoint state equations for quadratic cost problems are established. Especially in [6] Lions studied the quadratic cost control problems for hyperbolic controlled systems of the form

$$\left.\begin{array}{l} \ddot{y}+A(t)y=f+Bv, \quad v\in\mathcal{U}_{ad}\subset\mathcal{U}, \\ y(0)=y_0\in V, \quad \dot{y}(0)=y_1\in H, \end{array}\right\} \tag{1.1}$$

where H, V are Hilbert spaces, $V\hookrightarrow H\hookrightarrow V'$ is a Gelfand triple, f is a forcing function, $A(t)$ is the differential operator defined by some bilinear form on V, B is a controller, v is a control and $y=y(v)$ denotes the solution state for given $v\in\mathcal{U}_{ad}\subset\mathcal{U}$, $\mathcal{U}_{ad}$ is an admissible subset of the Hilbert space $\mathcal{U}$ of control variables. The attached quadratic cost functional to (1.1) is given by

$$J(v)=\|Cy(v)-z_d\|_M^2+(Rv,v)_U, \quad v\in\mathcal{U}, \tag{1.2}$$

where M is a Hilbert space of observation variables, z_d is a desired obvervation state in M and C is an observation operator, and R is a positive definite, symmetric operator on $\mathcal{U}$. The quadratic optimal control problem is to find and characterize an element $u\in\mathcal{U}_{ad}$, called optimal control, such that

$$\inf_{v\in\mathcal{U}_{ad}} J(v)=J(u). \tag{1.3}$$

In this paper, we study the quadratic cost optimal control problem for linear nonautonomous systems governed by damped second order equations of the form

$$\left.\begin{aligned} &\ddot{y}(v)+A_2(t)\dot{y}(v)+A_1(t)y(v)=f(t)+Bv, \quad v\in\mathcal{U}_{ad}\subset\mathcal{U}, \\ &y(0;v)=y_0\in V, \quad \dot{y}(0;v)=y_1\in H, \end{aligned}\right\} \tag{1.4}$$

where $A_1(t)$ is the operator defined by a bilinear form on V, and $A_2(t)$ is the operator defined by another bilinear form on V_2. We assume that inclusions $V\subset V_2\subset H$ are continuous embeddings. The quadratic cost subject to the system (1.4) is given by (1.2). The optimal control theory for the system (1.4) is not devoloped in Lions [5], Barbu[2], Lasiecka and Triggiani[4]. The following five practical partial differential equations having damping terms are covered by the system (1.4).

Example 1.1 (Damped wave equation)
Let $\Omega\subset\mathbf{R}^n$ be a bounded domain and let $\Gamma=\partial\Omega$ be a smooth boundary of Ω. We denote $Q=(0,T)\times\Omega$ and $\Sigma=(0,T)\times\Gamma$. Then the damped wave equation is described by

$$\left.\begin{aligned} &\frac{\partial^2}{\partial t^2}y(t,x)-\kappa\Delta\frac{\partial}{\partial t}y(t,x)-\Delta y(t,x)=f(t,x) \text{ in } Q, \\ &\frac{\partial}{\partial \mathbf{n}}\left[y(t,x)+\kappa\frac{\partial}{\partial t}y(t,x)\right]=g(t,x) \text{ on } \Sigma, \\ &y(0,x)=y_0(x), \quad \frac{\partial}{\partial t}y(0,x)=y_1(x) \text{ in } \Omega, \end{aligned}\right\}$$

where κ is a positive constant.

Example 1.2 (Air damped wave equation)
The wave equation having air damped effect is described by

$$\left.\begin{aligned} &\frac{\partial^2}{\partial t^2}y(t,x)+\alpha\frac{\partial}{\partial t}y(t,x)-\Delta y(t,x)=f(t,x) \text{ in } Q, \\ &y(t,x)=0 \text{ on } \Sigma, \\ &y(0,x)=y_0(x), \quad \frac{\partial}{\partial t}y(0,x)=y_1(x) \text{ in } \Omega, \end{aligned}\right\}$$

where $\alpha>0$.

Example 1.3 (Euler-Bernoulli beam equation)
The transverse vibration of a beam of length 1 is described by the following Euler-Bernoulli

beam equation with damping

$$\left.\begin{aligned}
&\frac{\partial^2}{\partial t^2}y(t,x)+\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2}{\partial x^2}y(t,x)+C_DI(x)\frac{\partial^3}{\partial x^2\partial t}y(t,x)\right]\\
&=\frac{x}{I_h}f(t) \ \text{ in } \ (0,T)\times(0,1),\\
&\frac{\partial^2}{\partial x^2}y(t,x)\Big|_{x=1}=\frac{\partial^3}{\partial x^3}y(t,x)\Big|_{x=1}=0, \ \ t\in(0,T),\\
&y(t,0)=\frac{\partial}{\partial x}y(t,x)\Big|_{x=0}=0, \ \ t\in(0,T),\\
&y(0,x)=y_0(x), \ \ \frac{\partial}{\partial t}y(0,x)=y_1(x) \ \text{ in } \ (0,1),
\end{aligned}\right\}$$

where I_h, EI, C_DI are physical constants.

Example 1.4 (Periodic Viscous damping of beam equation)
In the dynamics of vibrating beams we consider the case where the damping effect is viscous and periodic in time. This vibration equation is described by

$$\left.\begin{aligned}
&\frac{\partial^2}{\partial t^2}y(t,x)+\alpha_1\frac{\partial^4}{\partial x^4}y(t,x)+p(t)\frac{\partial}{\partial t}y(t,x)=\frac{x}{I_h}f(t) \ \text{ in } \ (0,T)\times(0,1),\\
&\frac{\partial^2}{\partial x^2}y(t,x)\Big|_{x=1}=\frac{\partial^3}{\partial x^3}y(t,x)\Big|_{x=1}=0, \ \ t\in(0,T),\\
&y(t,0)=\frac{\partial}{\partial x}y(t,x)\Big|_{x=0}, \ \ t\in(0,T),\\
&y(0,x)=y_0(x), \ \ \frac{\partial}{\partial t}y(0,x)=y_1(x) \ \text{ in } \ (0,1),
\end{aligned}\right\}$$

where $p(t)$ is a periodic function in t and $I_h, \alpha_1 > 0$ are physical constants.

Example 1.5 (Structural damped plate equation)
The structural damped plate equation is described by

$$\left.\begin{aligned}
&\frac{\partial^2}{\partial t^2}y(t,x)-\alpha_2\Delta\frac{\partial}{\partial t}y(t,x)+\Delta^2y(t,x)=f(t,x) \ \text{ in } \ Q,\\
&y(t,x)=0, \ \ \tfrac{\partial}{\partial \mathbf{n}}y(t,x)=0 \ \text{ on } \ \Sigma,\\
&y(0,x)=y_0(x), \ \ \frac{\partial}{\partial t}y(0,x)=y_1(x) \ \text{ in } \ \Omega,
\end{aligned}\right\}$$

where $\alpha_2 > 0$ is a constant.

The purpose of this paper is to extend the general quadratic optimal control theory for the hyperbolic (undamped) system (1.1) with (1.2) in Lions [6] to the damped second order system (1.4) with (1.2), which includes the examples 1.1-1.5. Further discussions and results for (1.4) are explained in Ha [3]. For related researches of damped second order systems, we refer to Banks, Ito and Wang [1], Lions [5, 8], and Lions and Magenes [7].

2 Damped second order evolution equations

Let X be a Hilbert space. $(\cdot,\cdot)_X$ denotes an inner product on X with the induced norm $\|\cdot\|_X$. X' denotes the dual space of X and $\langle\cdot,\cdot\rangle_{X',X}$ denotes a dual pairing between X' and X. Λ_X denotes the canonical isomorphism from X onto X'. Let us introduce underlying Hilbert spaces to describe damped second order system equations. Let H be a real pivot Hilbert space, its norm $\|\cdot\|_H$ is denoted simply by $|\cdot|_H$. For $i=1,2$, let V_i be a real separable Hilbert space. Assume that each pair (V_i,H) is a Gelfand triple space with the notation, $V_i \hookrightarrow H \equiv H' \hookrightarrow V_i'$, which means that the embedding $V_i \subset H$ is continuous and V_i is dense in H, so that the embedding $H \subset V_i'$ is also continuous and the identified $H \equiv H'$ is dense in V_i'. From now on, we write $V_1 = V$ for notational convenience. We shall give the exact description of damped second order evolution equation. Let $0 < T < \infty$ be a fixed terminal time.

Let $a_1(t;\phi,\varphi), t \in [0,T]$ be a family of bilinear forms on $V \times V$ satisfying

i) $a_1(t;\phi,\psi) = a_1(t;\psi,\phi)$ for all $\phi,\psi \in V$ and $t \in [0,T]$, (2.1)

ii) there exists $c_{11} > 0$ such that

$$|a_1(t;\phi,\varphi)| \le c_{11}\|\phi\|_V\|\varphi\|_V \text{ for all } \phi,\psi \in V \text{ and } t \in [0,T] \tag{2.2}$$

and there exist $\alpha_1 > 0$ and $\lambda_1 \in \mathbf{R}$ such that

$$a_1(t;\phi,\phi) + \lambda_1|\phi|_H^2 \ge \alpha_1\|\phi\|_V^2 \text{ for all } \phi \in V \text{ and } t \in [0,T], \tag{2.3}$$

iii) the function $t \to a_1(t;\phi,\varphi)$ is continuously differentiable in $[0,T]$ and there exists $c_{12} > 0$ such that

$$|\dot a_1(t;\phi,\varphi)| \le c_{12}\|\phi\|_V\|\varphi\|_V \text{ for all } \phi,\psi \in V \text{ and } t \in [0,T]. \tag{2.4}$$

where $\dot{} = \frac{d}{dt}$. Then we can define the operator $A_1(t) \in \mathcal{L}(V,V'), t \in [0,T]$ defined by the relation

$$a_1(t;\phi,\varphi) = \langle A_1(t)\phi,\varphi\rangle_{V',V} \text{ for all } \phi,\varphi \in V. \tag{2.5}$$

Similarly by (2.4) we have the operator $\dot A_1(t) \in \mathcal{L}(V,V'), t \in [0,T]$ defined by

$$\dot a_1(t;\phi,\varphi) = \langle \dot A_1(t)\phi,\varphi\rangle_{V',V} \text{ for all } \phi,\varphi \in V. \tag{2.6}$$

In order to consider a class of damping operators we introduce the second family of bilinear forms $a_2(t;\phi,\varphi), t \in [0,T]$ on $V_2 \times V_2$. It is assumed that

i) $a_2(t;\phi,\psi) = a_2(t;\psi,\phi)$ for all $\phi,\psi \in V_2$ and $t \in [0,T]$, (2.7)

ii) there exists $c_{21} > 0$ such that

$$|a_2(t;\phi,\varphi)| \le c_{21}\|\phi\|_{V_2}\|\varphi\|_{V_2} \quad \text{for all} \quad \phi,\psi \in V_2 \quad \text{and} \quad t \in [0,T] \tag{2.8}$$

and there exist $\alpha_2 > 0$ and $\lambda_2 \in \mathbf{R}$ such that

$$a_2(t;\phi,\phi) + \lambda_2|\phi|_H^2 \ge \alpha_2\|\phi\|_{V_2}^2 \quad \text{for all} \quad \phi \in V_2, \tag{2.9}$$

iii) the function $t \to a_2(t;\phi,\varphi)$ is continuously differentiable in $[0,T]$ and there exists $c_{22} > 0$ such that

$$|\dot a_2(t;\phi,\varphi)| \le c_{22}\|\phi\|_{V_2}\|\varphi\|_{V_2} \quad \text{for all} \quad \phi,\psi \in V_2 \quad \text{and} \quad t \in [0,T]. \tag{2.10}$$

Then we have a family of the operators $A_2(t) \in \mathcal{L}(V_2, V_2'), t \in [0,T]$ by the relation

$$a_2(t;\phi,\varphi) = \langle A_2(t)\phi,\varphi\rangle_{V_2',V_2} \quad \text{for all} \quad \phi,\varphi \in V_2. \tag{2.11}$$

Also by (2.10) we have $\dot A_2(t) \in \mathcal{L}(V_2, V_2'), t \in [0,T]$ defined by

$$\dot a_2(t;\phi,\varphi) = \langle \dot A_2(t)\phi,\varphi\rangle_{V_2',V_2} \quad \text{for all} \quad \phi,\varphi \in V_2. \tag{2.12}$$

We suppose that V is continuously embeded in V_2. Then we see that $V \hookrightarrow V_2 \hookrightarrow H \equiv H' \hookrightarrow V_2' \hookrightarrow V'$ and $\langle\phi,\varphi\rangle_{V',V} = \langle\phi,\varphi\rangle_{V_2',V_2}$ for $\phi \in V_2', \varphi \in V$ and $\langle\phi,\varphi\rangle_{V',V} = (\phi,\varphi)_H$ for $\phi \in H, \varphi \in V$.

We consider the following abstract damped second order evolution equation

$$\left.\begin{aligned} &\ddot y + A_2(t)\dot y + A_1(t)y = f \quad \text{in} \quad (0,T),\\ &y(0) = y_0 \in V,\\ &\dot y(0) = y_1 \in H, \end{aligned}\right\} \tag{2.13}$$

where $f \in L^2(0,T;V_2')$ and $\ddot{} = \frac{d^2}{dt^2}$.

We define a Hilbert space, which will be a solution space, as

$$W(0,T) = \{g | g \in L^2(0,T;V), \dot g \in L^2(0,T;V_2), \ddot g \in L^2(0,T;V')\}$$

with inner product

$$(g_1,g_2)_{W(0,T)} = \int_0^T \{(g_1(t),g_2(t))_V dt + (\dot g_1(t),\dot g_2(t))_{V_2} + (\ddot g_1(t),\ddot g_2(t))_{V'}\}dt$$

and induced norm

$$\|g\|_{W(0,T)} = \left(\|g\|^2_{L^2(0,T;V)} + \|\dot g\|^2_{L^2(0,T;V_2)} + \|\ddot g\|^2_{L^2(0,T;V')}\right)^{\frac{1}{2}}.$$

Definition 2.1 A function $y \in W(0,T)$ is a variational solution of (2.13) if y satisfies the following equation for every $t \in [0,T]$

$$\left.\begin{aligned} &\langle \ddot{y}(t), \phi\rangle_{V',V} + a_2(t;\dot{y}(t),\phi) + a_1(t;y(t),\phi) = \langle f(t),\phi\rangle_{V_2',V_2}, \quad \forall \phi \in V,\\ &y(0) = y_0 \in V,\\ &\dot{y}(0) = y_1 \in H. \end{aligned}\right\} \tag{2.14}$$

We shall state the existence and uniqueness result of solutions of (2.13).

Theorem 2.1 Assume that a_1 and a_2 satisfy (2.1)-(2.4) and (2.7)-(2.10), respectively and $f \in L^2(0,T;V_2')$. Then the equation (2.13) has a unique variational solution y in $W(0,T)$. Moreover, the solution y depends continuously on the data, that is, the map

$$(f, y_0, y_1) \to y$$

is continuous from $L^2(0,T;V_2') \times V \times H$ into $W(0,T)$.

A proof of Theorem 2.1 is given in Ha [3]. The next regularity of solution is important.

Theorem 2.2 Assume that the conditions in Theorem 2.1 hold. Then $y \in C([0,T];V)$ and $\dot{y} \in C([0,T];H)$.

The following energy equality for (2.13) is essential in proving Theorem 2.2 (cf.[3]).

Lemma 2.1 Assume that all conditions in Theorem 2.1 hold. Let y be the solution of (2.13). Then, for each $t \in [0,T]$ we have the following energy equality

$$\begin{aligned} & a_1(t;y(t),y(t)) + |\dot{y}(t)|_H^2 + 2\int_0^t a_2(\sigma;\dot{y}(\sigma),\dot{y}(\sigma))d\sigma \\ = \ & a_1(0;y_0,y_0) + |y_1|_H^2 + \int_0^t \dot{a}_1(\sigma;y(\sigma),y(\sigma))d\sigma + 2\int_0^t \langle f(\sigma),\dot{y}(\sigma)\rangle_{V_2',V_2}d\sigma. \end{aligned} \tag{2.15}$$

3 Optimal control problems and adjoint systems

Let $\mathcal{U}$ be a Hilbert space of control variables. Let B be an operator satisfying

$$B \in \mathcal{L}(\mathcal{U}, L^2(0,T;V_2')), \tag{3.1}$$

which is called as a controller. For each $v \in \mathcal{U}$, we consider the following controlled damped second order system:

$$\left\{\begin{aligned} &\ddot{y}(v) + A_2(t)\dot{y}(v) + A_1(t)y(v) = f + Bv \quad \text{in} \quad (0,T),\\ &y(0;v) = y_0 \in V, \ \dot{y}(0;v) = y_1 \in H. \end{aligned}\right. \tag{3.2}$$

Here in (3.2) $A_1(t), A_2(t)$ and f are operators and a forching function satisfying the assumptions given in Section 2. By virtue of Theorem 2.1 and (3.1), we can define the affine solution map $v \to y(v)$ of $\mathcal{U}$ into $W(0,T)$. We shall call $y(v)$ the state of the controlled system (3.2), where $y(v)$ is the solution of (3.2). The observation of the state is assumed to be given by $z(v) = Cy(v)$, where $C \in \mathcal{L}(W(0,T), M)$ is an operator called the observer, and M is a Hilbert space of observation variables. The cost function associated with the controlled system (3.2) is given by

$$J(v) = \|Cy(v) - z_d\|_M^2 + (Rv, v)_{\mathcal{U}} \quad \text{for all} \quad v \in \mathcal{U}, \tag{3.3}$$

where $z_d \in M$ is a desired value of $z(v)$ and $R \in \mathcal{L}(\mathcal{U}) = \mathcal{L}(\mathcal{U},\mathcal{U})$ is symmetric and positive, i.e.,

$$(Rv, v)_{\mathcal{U}} = (v, Rv)_{\mathcal{U}} \geq \gamma \|v\|_{\mathcal{U}}^2, \tag{3.4}$$

for some $\gamma > 0$. Let $\mathcal{U}_{ad}$ be a closed convex subset of $\mathcal{U}$, which is called the admissible set. The quadratic cost optimal control problems for (3.3) subject to (3.2) are:

i) Find an element $u \in \mathcal{U}_{ad}$ such that

$$\inf_{v \in \mathcal{U}_{ad}} J(v) = J(u). \tag{3.5}$$

ii) Give a characterization of such the u.

We shall call u the optimal control for the optimal control problem. It is easily verified as in the proof of Lions [6, Chap.1] that under the assumption (3.4), there exists a unique optimal control u for the cost (3.3) enjoying (3.5). Thus the problem i) is solved and the problem ii) is solved generally in [6, Chap. 1] as that the optimality condition for u is given by the variational inequality

$$J'(u)(v - u) \geq 0 \quad \text{for all} \quad v \in \mathcal{U}_{ad}, \tag{3.6}$$

where $J'(u)$ denotes the Gateaux derivative of $J(v)$ in (3.3) at $v = u$. The objective of this section is to write down formally the optimality condition (3.6) in terms of adjoint state systems. By Theorem 2.2 we know that $y(v) \in C([0,T]; V)$ and $\dot{y}(v) \in C([0,T]; H)$. Therefore, in order to avoid the complexity of setting up observation spaces, we consider the following four types of distributive and terminal value observations. That is, the following cases:

1. We take $C_1 \in \mathcal{L}(L^2(0,T;V), M)$ and observe $z(v) = C_1 y(v)$.
2. We take $C_2 \in \mathcal{L}(L^2(0,T;V_2), M)$ and observe $z(v) = C_2 \dot{y}(v)$.

3. We take $C_3 \in \mathcal{L}(V; M)$ and observe $z(v) = C_3 y(T; v)$.

4. We take $C_4 \in \mathcal{L}(H; M)$ and observe $z(v) = C_4 \dot{y}(T; v)$.

For each case we can introduce an adjoint state system, and formally calculate the condition (3.6) and derive necessary optimality conditions, which solves the problem ii) in a satisfactory manner. Further the justifications of such conditions under stronger assumptions on observers $C_i (i = 1, 2, 3)$ can be given. Because of the lack of space we consider two cases of $C_1 \in \mathcal{L}(L^2(0, T; V), M)$ and $C_2 \in \mathcal{L}(L^2(0, T; V_2), M)$.

3.1 Case of $C_1 \in \mathcal{L}(L^2(0, T; V), M)$

If we choose $z(v) = C_1 y(v)$, then the cost function is given by

$$J(v) = \|C_1 y(v) - z_d\|_M^2 + (Rv, v)_{\mathcal{U}}, \quad v \in \mathcal{U}_{ad}. \tag{3.7}$$

Then it is verified easily that the optimality condition (3.6) is written as

$$(C_1 y(u) - z_d, C_1(y(v) - y(u)))_M + (Ru, v - u)_{\mathcal{U}} \geq 0 \quad \forall v \in \mathcal{U}_{ad}, \tag{3.8}$$

where u is the optimal control for (3.7). Using the canonical isomorphism Λ_M, we can transform the condition (3.8) to

$$\int_0^T \langle C_1^* \Lambda_M (C_1 y(u; t) - z_d(t)), y(v; t) - y(u; t) \rangle_{V', V} dt + (Ru, v - u)_{\mathcal{U}} \geq 0, \quad \forall v \in \mathcal{U}_{ad}. \tag{3.9}$$

We want to write down the condition (3.9) in terms of adjoint state equations. For this, we introduce the adjoint system by

$$\left.\begin{array}{l} \ddot{p}(u) - A_2(t)\dot{p}(u) + (A_1(t) - \dot{A}_2(t))p(u) = C_1^* \Lambda_M (C_1 y(u) - z_d) \text{ in } (0, T), \\ p(u; T) = \dot{p}(u; T) = 0, \end{array}\right\} \tag{3.10}$$

where $p(u)$ denotes an adjoint state depending on the optimal control u.

Now we proceed the formal calculation. Multiply both sides of the equation in (3.10) by $y(v; t) - y(u; t)$ and integrate them on $[0, T]$. Then we have

$$\begin{aligned} & \int_0^T (\ddot{p}(u; t), y(v; t) - y(u; t))dt - \int_0^T (A_2(t)\dot{p}(u; t), y(v; t) - y(u; t))dt \\ & + \int_0^T (A_1(t)p(u; t), y(v; t) - y(u; t))dt - \int_0^T (\dot{A}_2(t)p(u; t), y(v; t) - y(u; t))dt \\ = & \int_0^T (C_1^* \Lambda_M (C_1 y(u; t) - z_d(t)), y(v; t) - y(u; t))dt \\ = & (C_1 y(u) - z_d, C_1(y(v) - y(u)))_M. \end{aligned} \tag{3.11}$$

Using integration by parts and using the symmetricity of $A_1(t)$ and $A_2(t)$ and $p(u;T) = \dot{p}(u;T) = y(v;0) - y(u;0) = \dot{y}(v;0) - \dot{y}(u;0) = 0$, the left hand side of (3.11) is calculated formally as

$$
\begin{aligned}
& -\int_0^T (\dot{p}(u;t), \dot{y}(v;t) - \dot{y}(u;t))dt - \int_0^T (\dot{p}(u;t), A_2(t)(y(v;t) - y(u;t)))dt \\
& + \int_0^T (p(u;t), A_1(t)(y(v;t) - y(u;t)))dt \\
& - \int_0^T (\frac{d}{dt}(A_2(t)p(u;t)) - A_2(t)\dot{p}(u;t), y(v;t) - y(u;t))dt \\
= & \int_0^T (p(u;t), \ddot{y}(v;t) - \ddot{y}(u;t))dt + \int_0^T (p(u;t), A_1(t)(y(v;t) - y(u;t)))dt \\
& - \int_0^T (\frac{d}{dt}(A_2(t)p(u;t)), y(v;t) - y(u;t))dt \\
= & \int_0^T (p(u;t), \left(\frac{d^2}{dt^2} + A_1(t)\right)(y(v;t) - y(u;t)))dt \\
& + \int_0^T (A_2(t)p(u;t), \dot{y}(v;t) - \dot{y}(u;t))dt \\
= & \int_0^T (p(u;t), \left(\frac{d^2}{dt^2} + A_2(t)\frac{d}{dt} + A_1(t)\right)(y(v;t) - y(u;t))dt \\
= & \int_0^T (p(u;t), B(v-u)(t))dt \\
= & \langle B^*p(u), v - u\rangle_{\mathcal{U}',\mathcal{U}} = (\Lambda_{\mathcal{U}}^{-1}B^*p(u), v-u)_{\mathcal{U}}.
\end{aligned} \tag{3.12}
$$

Thus, by (3.11) and (3.12), the condition (3.9) is established formally as

$$(\Lambda_{\mathcal{U}}^{-1}B^*p(u) + Ru, v - u)_{\mathcal{U}} \geq 0 \quad \forall v \in \mathcal{U}_{ad}. \tag{3.13}$$

Here in (3.13) we do not know that B^* can apply to $p(u)$, i.e., $p(u) \in L^2(0,T;V_2)$ or not. The above calculations suggest us that if C_1

$$C_1 \in \mathcal{L}(L^2(0,T;V_2), M), \tag{3.14}$$

then by $C_1^*\Lambda_M(Cy(u) - z_d) \in L^2(0,T;V_2')$ and $\dot{A}_2 \in L^\infty(0,T;\mathcal{L}(V_2,V_2'))$ (by (2.10)), we know that the adjoint system (3.10) is well-posed and permits a unique solution $p(u)$ in $W(0,T)$. Thus the above calculations have exact meanings under the assumption (3.14). Hence we have the following theorem.

Theorem 3.1 Assume that all conditions of Theorem 2.1 hold. Assume further that C_1 satisfy (3.14). Then the optimal control u for (3.7) subject to (3.2) is characterized by the following system of equations and inequality:

$$
\left.
\begin{aligned}
& \ddot{y}(u) + A_2(t)\dot{y}(u) + A_1(t)y(u) = Bu + f \ \text{ in } \ (0,T), \\
& y(u;0) = y_0 \in V, \quad \dot{y}(u;0) = y_1 \in H,
\end{aligned}
\right\} \tag{3.15}
$$

$$\left.\begin{array}{l}\ddot{p}(u) - A_2(t)\dot{p}(u) + (A_1(t) - \dot{A}_2(t))p(u) = C_1^* \Lambda_M (C_1 y(u) - z_d) \text{ in } (0,T), \\ p(u;T) = 0, \quad \dot{p}(u;T) = 0,\end{array}\right\} \tag{3.16}$$

$$(\Lambda_{\mathcal{U}}^{-1} B^* p(u) + Ru, v - u)_{\mathcal{U}} \geq 0, \quad \forall v \in \mathcal{U}_{ad}, \tag{3.17}$$

with

$$\left.\begin{array}{l} y(u), \quad p(u) \in L^2(0,T;V), \\ \dot{y}(u), \quad \dot{p}(u) \in L^2(0,T;V_2). \end{array}\right\} \tag{3.18}$$

For a detailed proof of Theorem 3.1, see Ha [3].

3.2 Case of $C_2 \in \mathcal{L}(L^2(0,T;V_2), M)$

When the observation $z(v)$ is given by $z(v) = C_2 \dot{y}(v)$, the cost function is defined as

$$J(v) = \|C_2 \dot{y}(v) - z_d\|_M^2 + (Rv, v)_{\mathcal{U}}, \quad v \in \mathcal{U}_{ad}. \tag{3.19}$$

Let u be the optimal control for (3.19) and assume that A_1 satisfies

$$\dot{A}_1 \in L^\infty(0,T;\mathcal{L}(V_2, V')). \tag{3.20}$$

Then we have the following theorem.

Theorem 3.2 Assume that (3.20) and all conditions of Theorem 2.1 hold. Then the optimal control u for (3.19) subject to (3.2) is characterized by the following system of equations and inequality:

$$\left.\begin{array}{l}\ddot{y}(u) + A_2(t)\dot{y}(u) + A_1(t)y(u) = Bu + f \text{ in } (0,T), \\ y(u;0) = y_0 \in V, \quad \dot{y}(u;0) = y_1 \in H,\end{array}\right\} \tag{3.21}$$

$$\left.\begin{array}{l}\ddot{p}(u) - A_2(t)\dot{p}(u) + A_1(t)p(u) + \int_t^T \dot{A}_1(\sigma)p(\sigma)d\sigma = C_2^* \Lambda_M (C_2 \dot{y}(u) - z_d) \text{ in } (0,T), \\ p(u;T) = 0, \quad \dot{p}(u;T) = 0,\end{array}\right\} \tag{3.22}$$

$$(-\Lambda_{\mathcal{U}}^{-1} B^* \dot{p}(u) + Ru, v - u)_{\mathcal{U}} \geq 0, \quad \forall v \in \mathcal{U}_{ad}, \tag{3.23}$$

with

$$\left.\begin{array}{l} y(u), \quad p(u) \in L^2(0,T;V), \\ \dot{y}(u), \quad \dot{p}(u) \in L^2(0,T;V_2). \end{array}\right\} \tag{3.24}$$

4 Applications to optimal control problems

In this section we develop the optimal control theory for practical damped second order partial differential equations. Let us consider the bilinear forms defined by

$$a_1(t;\phi,\psi) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}(t,x)\frac{\partial\phi(x)}{\partial x_i}\frac{\partial\psi(x)}{\partial x_j}dx + \int_\Omega a_0(t,x)\phi(x)\psi(x)dx, \quad \forall\phi,\psi \in V \tag{4.1}$$

and

$$a_2(t;\phi,\psi) = \sum_{i,j=1}^{n} \int_\Omega b_{ij}(t,x)\frac{\partial\phi(x)}{\partial x_i}\frac{\partial\psi(x)}{\partial x_j}dx + \int_\Omega b_0(t,x)\phi(x)\psi(x)dx, \quad \forall\phi,\psi \in V_2, \tag{4.2}$$

where a_{ij}, b_{ij}, a_0, b_0 are the functions satisfying

$$\begin{cases} \text{(i)} & a_{ij} = a_{ji}, \quad b_{ij} = b_{ji}, \\ \text{(ii)} & a_{ij}, b_{ij}, a_0, b_0 \in C^1([0,T]; L^\infty(\Omega)), \\ \text{(iii)} & \sum_{i,j=1}^n a_{ij}(t,x)\xi_i\xi_j \geq \alpha_1(\xi_1^2 + \cdots + \xi_n^2), \quad \alpha_1 > 0, \quad \xi_i \in \mathbf{R}, \\ \text{(iv)} & \sum_{i,j=1}^n b_{ij}(t,x)\xi_i\xi_j \geq \alpha_2(\xi_1^2 + \cdots + \xi_n^2), \quad \alpha_2 > 0, \quad \xi_i \in \mathbf{R}. \end{cases} \tag{4.3}$$

Let $f \in L^2(0,T;V_2')$ and $B \in \mathcal{L}(\mathcal{U}, L^2(0,T;V_2'))$. Since the bilinear forms given in (4.1) and (4.2) satisfy all conditions of Theorem 2.1, we have a unique solution $y \in W(0,T)$ of

$$\left.\begin{aligned} &\langle \ddot{y}(v;t),\phi\rangle_{V',V} + a_2(t;\dot{y}(v;t),\phi) + a_1(t;y(v;t),\phi) = \langle f(t) + Bv(t),\phi\rangle_{V_2',V_2} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. in } (0,T), \quad \forall v \in \mathcal{U}, \quad \forall\phi \in V, \\ &y(v;0) = y_0 \in V, \quad \dot{y}(v;0) = y_1 \in H. \end{aligned}\right\} \tag{4.4}$$

In order to consider distributed observation and control for the Dirichlet problem, we set $V = V_2 = H_0^1(\Omega)$, $H = L^2(\Omega)$. We choose a control variable space $\mathcal{U} = L^2(Q) = L^2(0,T;L^2(\Omega))$ to treat a distributed control. Then it is clear that $\Lambda_{\mathcal{U}} = I$. Let $f \in L^2(Q)$ and let us take $B = I$, the identity operator. From (4.4) we find a unique solution $y(v)$ of

$$\left.\begin{aligned} &\frac{\partial^2}{\partial t^2}y(v) + A_2(t)\frac{\partial}{\partial t}y(v) + A_1(t)y(v) = f + v \quad \text{in} \quad Q, \\ &y(v) = 0 \quad \text{on} \quad \Sigma, \\ &y(v;0,x) = y_0(x), \quad \frac{\partial}{\partial t}y(v;0,x) = y_1(x) \quad \text{on} \quad \Omega \end{aligned}\right\} \tag{4.5}$$

and also the solution y satisfies

$$y(v), \quad \frac{\partial y(v)}{\partial x_i}, \quad \frac{\partial y(v)}{\partial t}, \quad \frac{\partial^2 y(v)}{\partial t\partial x_i} \in L^2(Q), \tag{4.6}$$

where $A_1(t) = A_1(t,x,\frac{\partial}{\partial x}), A_2(t) = A_2(t,x,\frac{\partial}{\partial x})$ are operators given as

$$\left.\begin{aligned} A_1(t) = A_1\left(t,x,\frac{\partial}{\partial x}\right) &= -\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(a_{ij}(t,x)\frac{\partial}{\partial x_i}\right) + a_0(t,x), \\ A_2(t) = A_2\left(t,x,\frac{\partial}{\partial x}\right) &= -\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(b_{ij}(t,x)\frac{\partial}{\partial x_i}\right) + b_0(t,x), \quad (t,x) \in Q. \end{aligned}\right\} \tag{4.7}$$

In what follows we assume $R \in \mathcal{L}(L^2(Q))$. Here we study only the case where $C_1 = I : L^2(0,T;H_0^1(\Omega)) \to L^2(Q)$ is an identity distributive observation. Since $M = L^2(Q)$, we have $\Lambda_M = I$ and the cost functional $J(v)$ is given by

$$J(v) = \int_Q (y(v;t,x) - z_d(t,x))^2 dxdt + \int_Q Rv(t,x)v(t,x)dxdt, \quad v \in \mathcal{U}_{ad} \subset L^2(Q), \quad (4.8)$$

where $z_d \in L^2(Q)$. Then the optimal control u subject to (4.5) with (4.8) is characterized by

$$\begin{aligned} &\int_Q (y(u;t,x) - z_d(t,x))(y(v;t,x) - y(u;t,x))dxdt \\ + &\int_Q Ru(t,x)(v(t,x) - u(t,x))dxdt \geq 0, \quad \forall v \in \mathcal{U}_{ad}. \end{aligned} \quad (4.9)$$

For the optimal control u satisfying (4.9) we introduce an adjoint state system in accordance with Theorem 3.1 as follows:

$$\left.\begin{aligned} &\frac{\partial^2}{\partial t^2}p(u) - A_2(t)\frac{\partial}{\partial t}p(u) + \left[A_1(t) - \dot{A}_2(t)\right]p(u) = y(u) - z_d \text{ in } Q, \\ &p(u) = 0 \text{ on } \Sigma, \\ &p(u;T,x) = 0, \quad \frac{\partial}{\partial t}p(u;T,x) = 0 \text{ in } \Omega, \end{aligned}\right\} \quad (4.10)$$

where

$$\dot{A}_2(t) = \frac{\partial}{\partial t}A_2\left(t,x,\frac{\partial}{\partial x}\right) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial t}b_{ij}(t,x)\frac{\partial}{\partial x_i}\right) + \frac{\partial}{\partial t}b_0(t,x). \quad (4.11)$$

Since $V = V_2$, by Theorem 2.1 there exists a unique solution $p(u)$ of (4.10) in the sense of distribution on Q and the solution $p(u)$ satisfies

$$p(u), \quad \frac{\partial p(u)}{\partial x_i}, \quad \frac{\partial p(u)}{\partial t}, \quad \frac{\partial^2 p(u)}{\partial t \partial x_i} \in L^2(Q). \quad (4.12)$$

Consequently, by Theorem 3.1 we have the following optimality condition

$$\int_Q (p(u;t,x) + Ru(t,x))(v(t,x) - u(t,x))dxdt \geq 0, \quad \forall v \in \mathcal{U}_{ad}. \quad (4.13)$$

Note that we can derive this optimality condition (4.13) directly by multiplying (4.10) by $y(v) - y(u)$ and integrating it on Q.

Example 4.1 We consider the case without any constraint, that is, $\mathcal{U}_{ad} = \mathcal{U} = L^2(Q)$. In this case it follows from (4.13) that

$$p(u) + Ru = 0 \text{ in } Q.$$

Thus, the optimal control $u = -R^{-1}p(u)$ is obtained by solving the following system of partial differential eqautions:

$$\left.\begin{aligned}
&\frac{\partial^2}{\partial t^2}y(u) + A_2(t)\frac{\partial}{\partial t}y(u) + A_1(t)y(u) = f - R^{-1}p(u) \text{ in } \mathcal{Q},\\
&\frac{\partial^2}{\partial t^2}p(u) - A_2(t)\frac{\partial}{\partial t}p(u) + \left[A_1(t) - \dot{A}_2(t)\right]p(u) = y(u) - z_d \text{ in } \mathcal{Q},\\
&y(u) = 0, \quad p(u) = 0 \text{ on } \Sigma,\\
&y(u;0,x) = y_0(x), \quad \frac{\partial}{\partial t}y(u;0,x) = y_1(x) \text{ in } \Omega,\\
&p(u;T,x) = 0, \quad \frac{\partial}{\partial t}p(u;T,x) = 0 \text{ in } \Omega.
\end{aligned}\right\} \tag{4.14}$$

Example 4.2 We shall consider the unilateral problem in this example. Let us consider the case where

$$\mathcal{U}_{ad} = \{v | v \geq 0 \text{ almost everywhere in } \mathcal{Q}\}.$$

Then we can deduce from (4.13) that

$$\left.\begin{aligned}
p(u) + Ru \geq 0 &\quad \text{almost everywhere in } \mathcal{Q},\\
u \geq 0 &\quad \text{almost everywhere in } \mathcal{Q},\\
(p(u) + Ru)u = 0 &\quad \text{almost everywhere in } \mathcal{Q}.
\end{aligned}\right\} \tag{4.15}$$

Accordingly, the optimal control u is characterized by the solution of the system of unilateral equations:

$$\left.\begin{aligned}
&\frac{\partial^2}{\partial t^2}y(u) + A_2(t)\frac{\partial}{\partial t}y(u) + A_1(t)y(u) - f \geq 0 \text{ in } \mathcal{Q},\\
&\frac{\partial^2}{\partial t^2}p(u) - A_2(t)\frac{\partial}{\partial t}p(u) + [A_1(t) - \dot{A}_2(t)]p(u) = y(u) - z_d \text{ in } \mathcal{Q},\\
&p(u) + R\left(\frac{\partial^2}{\partial t^2}y(u) + A_2(t)\frac{\partial}{\partial t}y(u) + A_1(t)y(u) - f\right) \geq 0 \text{ in } \mathcal{Q},\\
&\left[p(u) + R\left(\frac{\partial^2}{\partial t^2}y(u) + A_2(t)\frac{\partial}{\partial t}y(u) + A_1(t)y(u) - f\right)\right] \times\\
&\qquad\left[\frac{\partial^2}{\partial t^2}y(u) + A_2(t)\frac{\partial}{\partial t}y(u) + A_1(t)y(u) - f\right] = 0 \text{ in } \mathcal{Q},\\
&y(u) = p(u) = 0 \text{ on } \Sigma,\\
&y(u;0,x) = y_0(x), \quad \frac{\partial}{\partial t}y(u;0,x) = y_1(x) \text{ in } \Omega,\\
&p(u;T,x) = 0, \quad \frac{\partial}{\partial t}p(u;T,x) = 0 \text{ in } \Omega.
\end{aligned}\right\} \tag{4.16}$$

Let us study (4.15) in details. From the last condition of (4.15) there are three possibilities such that

$$\text{(i)} \quad u = 0 \text{ and } p(u) + Ru = p(u) > 0 \text{ almost everywhere in } \mathcal{Q},$$

(ii) $p(u) + Ru = 0$ and $u > 0$, almost everywhere in Q,

(iii) there exists a region $Q_1 \subset Q$ such that $\text{meas}(Q_1) > 0$ and

$u = 0$ and $p(u) = 0$ in Q_1.

If the condition (iii) hold, then we have $y(u) = z_d$ in Q_1, which implies

$$\frac{\partial^2}{\partial t^2} z_d + A_2(t) \frac{\partial}{\partial t} z_d + A_1(t) z_d = f \quad \text{in} \quad Q_1. \tag{4.17}$$

Hence we assume that

$$\frac{\partial^2}{\partial t^2} z_d + A_2(t) \frac{\partial}{\partial t} z_d + A_1(t) z_d \neq f \quad \text{almost everywhere in} \quad Q. \tag{4.18}$$

Then the case (i) and (ii) are possible. Thus, the optimal control u satisfies either $u = 0$ or $u = -R^{-1} p(u)$. In particular, when $R = \nu \times I, \nu > 0$, we have

$$u = -\frac{1}{\nu} \inf\{0, p(u)\} \quad \text{almost everywhere in} \quad Q. \tag{4.19}$$

At this time, such the optimal control u is determined by the solutions of following equations:

$$\left.\begin{array}{l}
\dfrac{\partial^2}{\partial t^2} y(u) + A_2(t) \dfrac{\partial}{\partial t} y(u) + A_1(t) y(u) + \dfrac{1}{\nu} \inf\{0, p(u)\} = f \quad \text{in} \quad Q, \\
\dfrac{\partial^2}{\partial t^2} p(u) - A_2(t) \dfrac{\partial}{\partial t} p(u) + [A_1(t) - \dot{A}_2(t)] p(u) = y(u) - z_d \quad \text{in} \quad Q, \\
y(u) = 0, \quad p(u) = 0 \quad \text{on} \quad \Sigma, \\
y(u; 0, x) = y_0(x), \quad \dfrac{\partial}{\partial t} y(u; 0, x) = y_1(x) \quad \text{in} \quad \Omega, \\
p(u; T, x) = 0, \quad \dfrac{\partial}{\partial t} p(u; T, x) = 0 \quad \text{in} \quad \Omega.
\end{array}\right\} \tag{4.20}$$

Finally we note that other types of observation operators can be treated as in Example 4.1 and Example 4.2.

References

[1] H. T. Banks, K. Ito and Y. Wang, Well Posedness for damped second order systems with unbounded input operators, *Center for Research in Scientific Computation, North Carolina State university*, 1993.

[2] V. Barbu, Optimal control of variational inequalities, *Research Notes in Mathematics, Pitman Advanced Publishing Program,* Vol. 100, 1984.

[3] Junhong Ha, Quadratic optimal control problems for damped second order systems, *PhD-thesis, Kobe University, Japan.*

[4] Lasiecka and Triggiani, Differential and Algebraic Riccati Equations with Application to Boundary/ Point Control Problems: Continuous Theory and Approximation Theory, *Lecture Notes in Control and Information Sciences, Springer-Verlag,* Vol. 164, 1991.

[5] J. L. Lions, Equations Differentielles · Operationnelles et problémes aux limits, *Springer-Verlag Berlin · Gotteningen · Heidelberg*, 1961.

[6] J. L. Lions, Optimal Control of Systems Governed by Partial Differential Equations, *Springer-Verlag Berlin Heidelberg New York*, 1971.

[7] J. L. Lions and E. Magenes, Non-Homogeneous Boundary Value Problems and Applications I, II, *Springer-Verlag Berlin Heidelberg New York*, 1972.

[8] J. L. Lions, Contrôlabilité exacte perturbations et stabilisation de systemes distribués, Tome 1, *RMA*, Vol 9, 1988.