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# 4.

# The affine algebraic curve of Newton maps in Przytycki's cubic rational maps

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## 4.1 Introduction

Przytycki studies a family of cubic rational maps:

$$\mathcal{PC}(a,b,c) = \{f(z) = z^2 + c + \frac{b}{z-a} \ : a,b,c \in \mathbf{C}\},$$

and defines the exotic map to be a map of $\mathcal{PC}(a,b,c)$ having two super-attracting fixed points and a critical point of period 2. An example of exotic map is given in Przytycki [Prz94], obtained by computer experiment.

**Example:** $f(z) = z^2 + c + b/(z-a) : c = -3.121092, a = 1.719727, b = 0.3142117.$

He considers 1-parameter families joining exotic examples with Newton maps for degree 3 polynomials. The subfamily with a super-attracting fixed point except of $\infty$ can be parametrized by two parameters $(k,w)$ as follows:

let $w$ be super-attracting fixed point, and $a = kw$. Then

$$a = kw,\ b = 2w^3(1-k)^2,\ c = w^2(2k-3) + w,$$

and

$$\mathcal{PC}(w,k) = \left\{ f(z) = z^2 + w^2(2k-3) + w + \frac{2w^3(1-k)^2}{z-kw} \right\}.$$

Four critical points are $\infty$, $w$ and

$$u = \frac{w(-1+2k-\sqrt{4k-3})}{2},\quad v = \frac{w(-1+2k+\sqrt{4k-3})}{2}$$

.

Przytycki gives two questions in [Prz94]:

**Question 1** In the set of Newton maps for the polynomials $P_\lambda = z^3 + (\lambda - 1)z - \lambda$, there exist Mandelbrot-like sets where the free critical point converges to a periodic attrcting orbit. For a critical point $v = g(k,w)$, what happens to $\mathcal{M}(v)$-sections of the set of exotic maps when we change parameters from Newton maps to the exotic ones ? : namely, do these sets move to $\mathcal{M}(v)$ (or $\mathcal{M}(u)$)-sections of the set of exotic maps when we change parameters from Newton maps to the exotic ones ?

**Question 2** Describe precisely how does the dynamics bifurcate for real parameters $k, w$.

Concerning about these questions, we obtain following results: to the first question, there is an affine algebraic curve consisting of Newton maps for degree 3 polynomials, and to the second one, we partly give an answer. Namely,for any fixed parameter $k$, we consider the bifurcation of this family as the parameter w varies monotonely. We observe complex bifurcations for $3/4 < k < 1$.

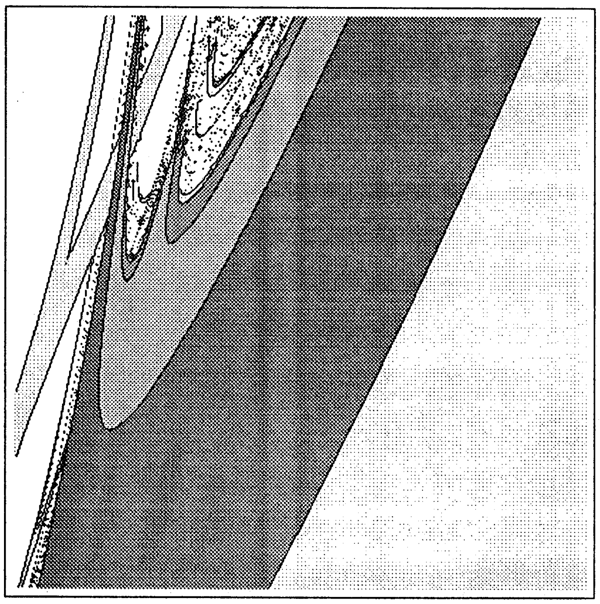


図 1 Bifurcations in real $(k,w)$-plane

# 4.2 Cubic Newton Curve

We show that there is an affine algebraic curve consisted by all Newton maps in $\mathcal{PC}(a, b, c)$. We call hereafter this curve **cubic Newton curve**, denoted by $\mathcal{N}(w, k)$.

**Proposition 1** *The defining equation of the cubic Newton curve is the follwing:*

$$\mathcal{N}(w,k): \; k^2w^2 - 6kw^2 + 9w^2 - 2kw - 3w + 1 = 0.$$

*The cubic Newton curve is irreducible of genus one with two singular points*

$$(0,0,1), \quad (0,1,0)$$

*on* $L_\infty : Z = 0$ *of* $\mathbf{P}^2\mathbf{C} : (Z, k, w)$.

Outline of proof:

$$f(k,w,x) = x^2 + w^2\,(2k-3) + w + \frac{2\,w^3\,(1-k)^2}{(x-kw)},$$

$$f'(k,w,x) = 2\,x - \frac{2\,(1-k)^2\,w^3}{(x-k\,w)^2}.$$

The critical points are $\infty$ , $w$, and

$$u = \frac{-w\,\sqrt{4k-3} - (1-2k)\,w}{2}, \quad v = \frac{w\,\sqrt{4k-3} - (1-2k)\,w}{2}.$$

For $f$ to be a Newton map, we claim $f(k,w,u) = u$, or $f(k,w,v) = v$.

Therefore the equation of the Newton curve is

$$(k^2w^2 - 6kw^2 + 9w^2 - 2kw - 3w + 1) = 0.$$

Let

$$F(Z,k,w) = Z^4 - 3wZ^3 - 2z^2kw + 9Z^2w^2 - 6Zkw^2 + k^2w^2 = 0,$$

and $P_k$ : $(0,0,1)$, $P_w$ : $(0,1,0)$. The singular points are $P_k$, $P_w$. The principal part of F is $(3Z-k)^2$. Therefore by Plücker's formula, we can calculate the genus one.

We define dynamical curves in the parameter space:

**Definition** Let $\mathrm{Per}_p(\mu)$ consist of all parameter pairs $(k, w)$ for which the associated cubic rational map $f(k, w, x)$ has a periodic orbit of period $p$ with multiplier $(f^p)'$ equal to $\mu$.

In particular, $\mathrm{Per}_1(0)$ consists of all parameter pairs with a super-attracting fixed point. The real part of the curve $\mathrm{Per}_1(1)$ consists of all parameter pairs for which the graph of $f$ is tangent to the diagonal. Such points of tangency are called **saddle nodes** of period 1. On $\mathrm{Per}_1(-1)$ attracting period one orbits bifurcate into attracting period two orbits.

It is clear that $(k, w)$ of $\mathrm{Per}_1(-1)$ belongs to $\mathrm{Per}_2(1)$, but it is not known whether inverse inclusion holds. In the quadratc rational maps, we have $\mathrm{Per}_2(1) = \mathrm{Per}_1(-1)$.

**Proposition 2** *We obtain defining equations of two dynamical curves:*

$Per_1(1):\ k^2w^2 - 6kw^2 + 9w^2 - 2kw - 2w + 1 = 0,$

$Per_1(-1):\ 3k^2w^2 - 18kw^2 + 27w^2 - 6kw - 14w + 3 = 0.$

See Figures for plots of these curves in the real case.

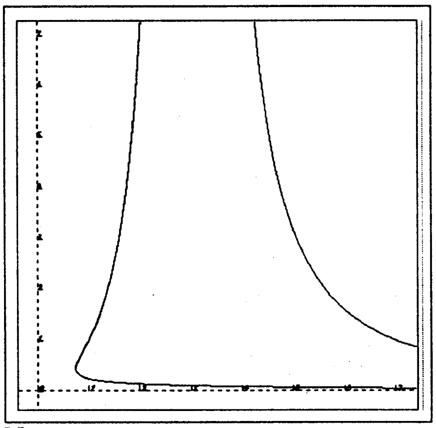


図 2 **Newton curve** $:(k^2 - 6k + 9)w^2 - (2k + 3)w + 1 = 0.$

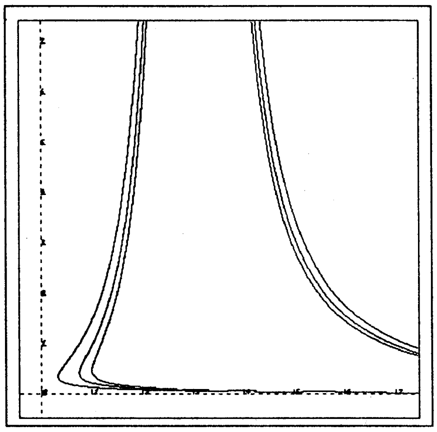


図 3 **Dynamical curves** $:Per_1(1),\ Per_1(-1).$

Outline of proof:

The fixed points of $f(k, w, x)$ are $\infty$, $w$ and

$$x_1 = \frac{(1-k)\,w - 1 - \sqrt{(k^2 - 6k + 9)w^2 + (-2k - 2)w + 1}}{2},$$

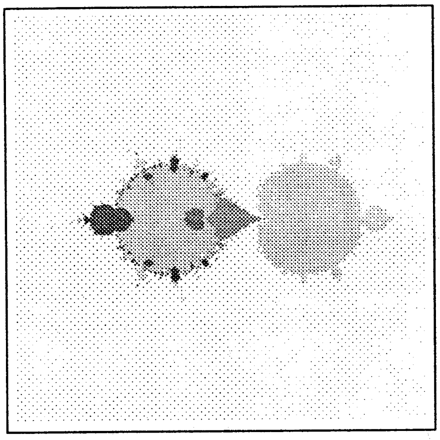
$$x_2 = \frac{-(1-k)\,w + 1 + \sqrt{(k^2 - 6k + 9)w^2 + (-2k - 2)w + 1}}{2}.$$

From $f'(k, w, x_i) = 1$, ( resp. $= -1$ ) for $i = 1$, or 2, we obtain the equation of $Per_1(1)$ ( resp. $Per_1(-1)$ ).

# 4.3 bifurcations

**Proposition 3** *For the real parameter $k$, we can rougly divide $(k, w)$-plane into four distinct classes as follows: (1) $k < 3/4$ , (2) $3/4 \le k < 1$ , (3) $1 < k < 3/2$, (4) $3/2 \le k$.*

Outline of Proof:



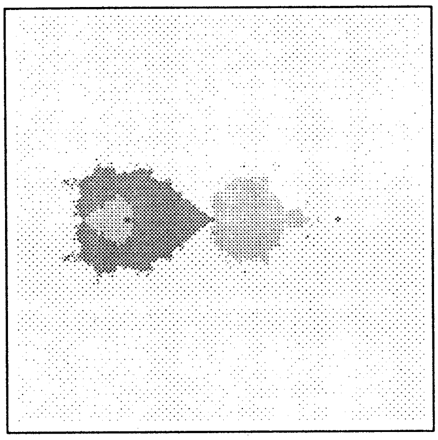


図4 **Bifurcation:**$k = 1.15$, $w : (-1.5, -2) - (2.5, 2)$ 図5 **Bifurcation:** $k = 1.5$, $w : (-2, -4) - (6, 4)$

• For $k$ : $3/4 \leq k \leq 1$, the critical points $u$, $v$, $w$ except $\infty$ are ordered as $u < v < a < w$.

• For $k$ : $1 \leq k < 3/2$, $u < w < a < v$.

• For $k$ : $3/2 \leq k$, $w < u < a < v$. in case (3) ( resp.(4)), the dynamics of $u$ ( resp. $v$ )is analyzed as the dynamics of a quadratic map by Douaddy-Hubbard-Sullivan's theory( [D-H85]). In case (2), the dynamical behavior is very complex. In case (1), two critical points $u, v$ are compex numbers, mutually complex conjugate. The dynamical behavior is very complex: namley we can observe "swallow" and "tri-corn" configulations ([Mil90]).

# 参考文献

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