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7.

Bimodular Type Simple K3 Singularities

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7.1 Introduction

Let $f_1, f_2, \ldots, f_r$ be holomorphic functions defined in an open set U of the complex space C^n. Let X be the analytic set $f_1^{-1}(0) \cap \ldots f_r^{-1}(0)$. Let $x \in X$, and let $g_1, g_2, \ldots, g_s$ be a system of generators of ideal $I(X)_{x_0}$ of the generators of the holomorphic functions which vanish identically on a neighborhood of x_0 in X. x_0 is called a simple point of X if the matrix $(\partial g_i/\partial x_j)$ attains its maximal rank. Otherwise, x_0 is called a singlar point(singularity) of X. For $r = 1, x_0$ is called a Hypersurface singularity of X. Let V be an analytic set in C^n. A singular point x_0 of V is said to be isolated if, for some open neighborhood W of x_0 in C^n, $W \cap V - \{x_0\}$ is a smooth submanifold of $W - \{x_0\}$.

Example

For a holomorphic function $f(z_0, \ldots, z_n)$ defined in a neighborhood U of the origin in C^{n+1}, let $X = \{(z_0, \ldots, z_n\} \in U | f(z_0, \ldots, z_n) = 0$. Then if

$$\{x = (0, \ldots, 0)\} = \{\frac{\partial f}{\partial z_0} = \ldots = \frac{\partial f}{\partial z_n} = 0\} \cap X,$$

X has an isolated singularity at x.

Let (X, x) be a germ of normal isolated singularity of dimension n. Suppose that X is a Stein space. Let $\pi : (M, E) \rightarrow (X, x)$ be a resolution of singularity. Then for $1 \leq i \leq n-1$, $dim(R^i\pi, \vartheta_M)_X$ is finite. $R^i\pi, \vartheta_M$ has support on x. hey are idependent of the resolution.
In fact

$$dim(R^i\pi, \vartheta_M)_X = dimH_X^{i+1}(X, \vartheta_M) \quad (1 \leq i \leq n-2)$$

and

$$dim(R^{n-1}\pi, \vartheta_M)_X = \frac{dim\Gamma(X - \{x\}, \vartheta K)}{L^2(X - \{x\})}.$$

where $L^2(X - \{x\})$ is the subspace of $\Gamma(X - \{x\}, \vartheta K)$ consisting of n-form on $X - \{x\}$ which are square integrable near x.

We denote them by

$$h^i(X, x) := dim(R^i\pi, \vartheta_M)_X \quad (1 \leq i \leq n-2)$$

and

$$P_g(X, x) := dim(R^i\pi, \vartheta_M)_X.$$

The invariant $P_g(X, x)$ is called the geometric genus of (X, x).

In the theory of two-dimensional singularities, simple elliptic singularities and cusp singularities are regarded as the next most reasonable class of singularities after rational singularities. What are natural generalizations in three-dimensional case of those singularities. They are purely elliptic singularities. We define the purely elliptic singularities.

Definition ([2])

For each positive integer m, the m-genus of a normal isolated singularity (X, x) in an n-dimensional analytic space is defined to be

$$\delta_m(X, x) = \frac{dim_c\Gamma(X - \{x\}, \vartheta(mK)}{L^{2/m}(X - \{x\})},$$

where K is the canonical line bundle on $X - \{x\}$, and $L^{2/m}(X - \{x\})$ is the set of all holomorphic m-ple n-forms on $X - \{x\}$ which are $L^{2/m}$-integrable at x.

The m-genus δ_m is finite and does not depend on the choice of a Stein neighborhood X.

Definition ([3])

A singularities (X, x) is said to be purely elliptic if $\delta_m(X, x) = 1$ for every positive integer m.

When X is a two-dimensional analytic space, purely elliptic singularities are quasi-Gorenstein singularities. In higher dimensions, however, purely elliptic singularities are not always quasi-Gorenstein.

Theorem ([4])

Let (X, x) be a quasi-Gorenstein normal isolated singularity of dimension 3, then

$$2\{P_g(X, x) - \frac{-K_M \cdot c_2(M)}{24} = dim_c H^1(M, \vartheta).$$

Consequently quasi-Gorenstein purely elliptic singularities of dimension 3 are classified into 6 classes.

(1) $h^1(X, x) = 2p$, (0,0)-type.

$p = 1 \rightarrow$ Hilbert modular cusp singularities.

$p > 1 \rightarrow$ Tsuchihashi cusp singularities ([1]).

(2) $h^1(X, x) = 2$, (0,1)-type.

(3) $h^1(X, x) = 2$, (0,2)-type.

(4) $h^1(X, x) = 0$, (0,0)-type.

(5) $h^1(X, x) = 0$, (0,1)-type.

(6) $h^1(X, x) = 0$, (0,2)-type.

Simple elliptic singularities and cusp singularities are characterized as two-dimensional purely elliptic singularities of (0,1)-type and of (0,0)-type, respectively.

Definition ([2])

A three-dimensional singularity (X, x) is a simple $K3$ singularity if the following two equivalent (Watanabe-Ishii[5]) conditions are satisfied:

(1) (X, x) is a Gorenstein purely elliptic singularity of (0,2)-type.

(2) The exceptional divisor D is a normal $K3$ surface for any Q-factorial terminal modification $\delta : (Y, D) \rightarrow (X, x)$.

The notion of a simple $K3$ singularity is defined as a three-dimensional isolated Gorenstein purely elliptic singularity of (0,2)-type.

Example

Let $f(x,y,z,w)$ be a quasi-homogeneous polynomial of type $(p,q,r,s:h)$ with $p+q+r+s=h$, and suppose $f(x,y,z,w)=0$ defines an isolated singularity at the origin in C^4. Then the origin is a simple $K3$ singularity.

Next we consider the case where (X,x) is a hypersurface singularity definred by a nondegenerate polynomial

$$f=\sum a_v x^v \in C[x_0,x_1,\ldots,x_n],$$

and $x=0\in C^{n+1}$. Recall that the Newton boundary $\Gamma(f)$ of f is the union of the compact faces of $\Gamma_+(f)$, where $\Gamma_+(f)$ is the convex hull of $\bigcup_{a_n\neq 0}(n+R_0^{n+1})$ in R^{n+1}. For any face Δ of $\Gamma_+(f)$, set $f_D:=\sum_{n\in\Delta} a_v x^v$. We say f to be nondegenerate, if

$$\frac{\partial f_\Delta}{\partial x_0}=\frac{\partial f_\Delta}{\partial x_1}=\ldots=\frac{\partial f_\Delta}{\partial x_n}=0$$

has no solution in $(C^*)^{n+1}$ for any face Δ.

When f is nondegenerate, the condition for (X,x) to be a purely elliptic singularity is given as follows:

Theorem ([3])

Let f be a nondegenerate polynomial and suppose $X=f=0$ has an isolated singularity at $x=0\in C^{n+1}$.

(1) (X,x) is purely elliptic if and only if $(1,1,\ldots,1)\in\Gamma(f)$.

(2) Let $n=3$ and let Δ_0 be the face of $\Gamma(f)$ containing $(1,1,1,1)$ in the relative interior of Δ_0. Then (X,x) is a simple $K3$ singularity if and only if $dim_R\Delta_0=3$.

Thus if f is nondegenerate and defines a simple $K3$ singularity, then f_{D_0} is a quasi-homogeneous polynomial of a uniquely determined weight a called the weight of f.

Yonemura([6]) classified nondegenerate hypersurface simple $K3$ singularities into ninety five classes in terms of the weight of f.

7.2 Parameters in a defining equation

Yonemura calculate the weights of hypersurface simple $K3$ singularities by nondegenerate polynomials and obtained examples such that the polynomial f is quasi-homogeneous and that $\{f = 0\} \subset C^4$ has a simple $K3$ singularity at the origin. The minimum number of parameters in the polynomial is less than or equal to 19 and is associated with the moduli of the $K3$ surface with singularities. The need for a unique form may be questioned. However, defining equations were not unique. So, in this section, we try to impose a condition to construct a unique form for quasi-homogeneous polynomials and decide conditions of their parameters.

We can take the following form for a weighted quasi-homogeneous polynomial f in C^{n+1} with the coordinate $[x_0, x_1, x_2, \ldots, x_n]$:

$$f = f_0 + f_1 + f_2 + \ldots + f_m$$

where $f_i (0 \leq i \leq m)$ is a homogeneous polynomial of degree i in C^{n+1}. And let $W = (w_0, w_1, \ldots, w_n)$ be the weight. Then we can take the following form for the homogeneous polynomial of each degree i:

$$\sum_{k_0+k_1+\ldots+k_n=1} a_{k_0 k_1 \ldots k_n} x_0^{k_0} x_1^{k_1} \ldots x_n^{k_n} \quad (k_i \in N_0, 0 \leq i \leq n).$$

Let $\ll$ be the lexical linear ordering of the terms of the homogeneous polynomials for $0 \leq i \leq m$ in turn from the minimal term to the maximal term given below:

Definition

Let $K = (k_0, k_1, \ldots, k_n)$ $(k_i \in N_0, 0 \leq i \leq n)$ and let $a_K X^K$ denote the term

$$a_K X^K = a_{k_0 k_1 \ldots k_n} x_0^{k_0} x_1^{k_1} \ldots x_n^{k_n}.$$

Then $a_K X^K \ll b_L X^L$ if there exists an integer $s (0 \leq s \leq n)$ such that $k_i = l_i$ for $m = 0, 1, \ldots, s-1$ and $k_s < l_s$.

Example

$$x_0^3 \ll x_0^2 x_1 \ll x_0^4 \ll x_2^4 \ll x_0^5$$

Hereafter, for the sake of simplicity, we shall sometimes omit the coefficients in indicating terms.

We will consider the following procedure by using this ordering.

Step 1

We try a term X^{K_i} to eliminate by a suitable analytic transformation with respect to X. We find a condition of the coefficient of term X^{K_i} where we can make the term X^{K_i} to eliminate without generating the term $X^{K_j} \ll X^{K_i}$. We classify the following two cases by the above condition.

Case 1 : We can make the term X^{K_i} to eliminate without generating the term $X^{K_j} \ll X^{K_i}$.

Case 2 : Otherwise for case 1.

For the condition of case 1, we make the term X^{K_i} to eliminate without generating the term $X^{K_j} \ll X^{K_i}$. For the condition of case 2, we don't use the anayltic transformation and go to next step.

Step 2

We make the coefficient of the term which determine the weight $(w_0, w_1, \ldots, w_{n-1})$ for the quasi-homogeneous polynomial equal to 1 by the magnification of the coordinate.

Let W_4 be the set of defining equations which has a nondegenerate hypersurface simple $K3$ singularity at the origin and let $\#m(f)$ be the minimum number of parameters of defining equation for any $f \in W_4$. Then for $\#m(f) = i (1 \leq i \leq 3)$, there exists 3 types, 8 types, 7 types, respectively. In general, the relation of the parameters is a simultaneous equation of them.

7.3 Relation of parameters

For $\#m(f) = 2$, we consider the relation of parameters in a defining equation of nondegenerate hypersurface simple $K3$ singularity which is constructed by the procedure in section 2. Then we obtain the following results:

Result

$f_{84} : x_0^3 + \lambda x_0 x_1 x_2 x_3 + x_0 x_2^3 + x_1^3 x_2 + x_1 x_3^4 + \mu x_2^2 x_3^3 = 0 \qquad ((\lambda^3 + 27\mu)\mu - \lambda^2)^2 \neq (2(9\lambda\mu - 8))^2,$

$f_{86} : x_0^2 x_1 + x_1^3 x_3 + x_0 x_3^4 + x_2^5 + \lambda x_1 x_2^2 x_3^2 + \mu x_2 x_3^5 = 0$

$$16\mu(125\lambda^2 + 8\lambda 4\mu + 225\lambda\mu^2 + 4\lambda^3\mu^3 + 108\mu^4) \neq 3125 - 64\lambda^5.$$

The number n of f_n denotes the number of the defining equation in the classification by Yonemura.

We show a file(program) of Mathematica for the above computation. The file name is the same name as the f_n in the above result

File: f84

```
f84=x0^3+p x0 x1 x2 x3+x0 x2^3+x1^3 x2+x1 x3^4+q x2^2 x3^3;
dx0 = D[f84,x0]; dx1 = D[f84,x1]; dx2 = D[f84,x2]; dx3 = D[f84,x3];
e0=dx0; e1=Expand[(9dx0 x0-8dx1 x1-3dx2 x2+2dx3 x3)/27];
e2=Expand[(-8dx1 x1+6dx2 x2+2dx3 x3)/(-18x2)];
e3=Expand[(dx1 x1-3dx2 x2+2dx3 x3)/9];
f0=Expand[e0 x2^2]; f1=Expand[e1 x2^8]; f2=Expand[e2 x2^9]; f3=Expand[e3 x2^3];
g0=3a^2+c+p b d; g1=a^3 c-b^3; g2=b^3-q c^2 d^3-a c^2; g3=b d^4-a c;
a=(b d^4)/c; j0=y^3+3x^2 z+p x y^2; j1=y^2-z^2; j2=x^3-x y z-q y^2 z;
y=z; s0=Expand[j0/(z^3)]; s1=Expand[j2/(z^3)];
u1=3t^2+p t+1; u2=t^3-t-q;
y=-z; t0=Expand[j0/(z^3)]; t2=Expand[j2/(z^3)];
v1=3t^2+p t-1; v2=t^3+t-q;
Timing[Reduce[{u1, u2}=={0, 0},t]]
{0.1333333333332121*Second, p*(-18 + p^2)*q + 27*q^2 == -16 + p^2 &&
t == (-p + 9*q)/(-12 + p^2)}
Timing[Reduce[{v1, v2}=={0, 0},t]]
{0.1333333333332121*Second, p*(18 + p^2)*q + 27*q^2 == 16 + p^2 &&
t == (p + 9*q)/(12 + p^2)}
File 1: f84
f86=x0^2 x1+x1^3 x3+x0 x3^4+x2^5+p x1 x2^2 x3^2+q x2 x3^5;
dx0 = D[f86,x0]; dx1 = D[f86,x1]; dx2 = D[f86,x2]; dx3 = D[f86,x3];
g0=dx0; g1=dx1; g2=Expand[(dx3 x3-5dx2 x2+8dx1 x1-4dx0 x0)/25]; g3=dx3;
x0=(-(x3^4)/(2x1)); x3=(x2^5)/(x1^3)];
h1=Expand[(g1 4x1^26)/(x2^5)]; h2=Expand[g3 x1^22];
u1=12a^5+4p a^4 b+b^5; u2=a^5+2p a^4 b+5q a^2 b^3-2b^5;
v1=12t^5+4p t^4+1; v2=t^5+2p t^4+5q t^2-2;
Timing[Reduce[{v1, v2}=={0, 0}, t]]
{2.966666666667152*Second,
2000*p^2*q + 128*p^4*q^2 + 3600*p*q^3 + 64*p^3*q^4 +
1728*q^5 == 3125 - 64*p^5 &&
t == (-369140625*p - 585000*p^6 - 2048*p^11 +
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7312500*p^3*q + 960*p^8*q - 379687500*q^2 +
924000*p^5*q^2 - 2048*p^10*q^2 + 4050000*p^2*q^3 +
704*p^7*q^3 + 1512000*p^4*q^4)/
(791015625 + 4212500*p^5 + 4096*p^10)}
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参考文献

[1]H. Tsuchihashi, Higher dimensional analogues of periodic continued fractions and cusp singularities, Tohoku Math. J. 35 (1983), pp.607-639.

[2]K. Watanabe, On plurigenera of normal isolated singularities, I. Math. Ann. 250 (1980), pp.30-3865-94.

[3]K. Watanabe, On plurigenera of normal isolated singularities II, in Complex Analytic Singularities (T. Suwa and P. Wagrech eds.) Advanced Studies in Pure Math. 8 Kinokuniya, Tokyo and North-Holand, Amsterdam, New York Oxford (1986), pp.671-685.

[4]K. Watanabe, Riemann-Roch Theorem for normal isolated singularities(preprit) (1989).

[5]K. Watanabe and S. Ishii, On simple K3 singularities (in Japanese), in Proc. of Con. on Algebraic Geometry at Tokyo Metropolitan Univ. 1988 (N. Sasakura, ed.), pp.20-31.

[6]T. Yonemura, Hypersurface Simple K3 Singularities, Tohoku Math. J, 42 (1990), pp.351-380.