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A WAVELET-BASED NUMERICAL METHOD AND APPLICATIONS TO CFD

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1 Introduction

Wavelet methods have been around for about 10 years, and there are actually very few useful numerical methods which have come from the wavelet field. This paper introduces a numerical method which was created about 3 years ago in an attempt to finally produce a method which is useful to a wide range of applied scientists. The numerical method is named the Wavelet Optimized Finite Difference Method (WOFD). First a review of wavelet analysis will be given followed by the definition of WOFD.

2 Definition of Daubechies-based Wavelets

To define Daubechies-based wavelets, see [2] for the original work, consider the two functions $\phi(x)$, the scaling function, and $\psi(x)$, the wavelet. The scaling function is the solution of the dilation equation,

$$\phi(x) = \sqrt{2}\sum_{k=0}^{L-1} h_k \phi(2x-k), \tag{1}$$

where $\phi(x)$ is normalized $\int_{-\infty}^{\infty} \phi(x)dx = 1$, and the wavelet $\psi(x)$ is defined in terms of the scaling function,

$$\psi(x) = \sqrt{2} \sum_{k=0}^{L-1} g_k \phi(2x - k). \tag{2}$$

One builds an orthonormal basis from $\phi(x)$ and $\psi(x)$ by dilating and translating to get the following functions:

$$\phi_k^j(x) = 2^{-\frac{j}{2}} \phi(2^{-j}x - k), \tag{3}$$

and

$$\psi_k^j(x) = 2^{-\frac{j}{2}} \psi(2^{-j}x - k), \tag{4}$$

where $j, k \in Z$. j is the dilation parameter and k is the translation parameter. The coefficients $H = \{h_k\}_{k=0}^{L-1}$ and $G = \{g_k\}_{k=0}^{L-1}$ are related by $g_k = (-1)^k h_{L-k}$ for $k = 0, ..., L-1$. All wavelet properties are specified through the parameters H and G. If one's data is defined on a continuous domain such as $f(x)$ where $x \in R$ is a real number then one uses $\phi_k^j(x)$ and $\psi_k^j(x)$ to perform the wavelet analysis. If, on the other hand, one's data is defined on a discrete domain such as $f(i)$ where $i \in Z$ is an integer then the data is analyzed, or filtered, with the coefficients H and G. In either case, the scaling function $\phi(x)$ and its defining coefficients H detect localized low frequency information, i.e., they are low-pass filters (LPF), and the wavelet $\psi(x)$ and its defining coefficients G detect localized high frequency information, i.e., they are high-pass filters (HPF). Specifically, H and G are chosen so that dilations and translations of the wavelet, $\psi_k^j(x)$, form an orthonormal basis of $L^2(R)$ and so that $\psi(x)$ has M vanishing moments which determines the accuracy. In other words, $\psi_k^j(x)$ will satisfy

$$\delta_{kl}\delta_{jm} = \int_{-\infty}^{\infty} \psi_k^j(x)\psi_l^m(x)dx, \tag{5}$$

where δ_{kl} is the Kronecker delta function, and the accuracy is specified by requiring that $\psi(x) = \psi_0^0(x)$ satisfy

$$\int_{-\infty}^{\infty} \psi(x)x^m dx = 0, \tag{6}$$

for $m = 0, ..., M-1$. Under the conditions of the previous two equations, for any function $f(x) \in L^2(R)$ there exists a set $\{d_{jk}\}$ such that

$$f(x) = \sum_{j \in Z} \sum_{k \in Z} d_{jk} \psi_k^j(x), \tag{7}$$

where

$$d_{jk} = \int_{-\infty}^{\infty} f(x) \psi_k^j(x) dx. \tag{8}$$

The two sets of coefficients H and G are known in signal processing literature as quadrature mirror filters [4]. For Daubechies wavelets the number of coefficients in H and G, or the length of the filters H and G, denoted by L, is related to the number of vanishing moments M by $2M = L$. For example, the famous Haar wavelet is found by defining H as $h_0 = h_1 = 1$. For this filter, H, the solution to the dilation equation (1), $\phi(x)$, is the box function: $\phi(x) = 1$ for $x \in [0, 1]$ and $\phi(x) = 0$ otherwise. The Haar function is very useful as a learning tool, but because of its low order of approximation accuracy and lack of differentiability it is of limited use as a basis set. The coefficients H needed to define compactly supported wavelets with a higher degree of regularity can be found in [2]. As is expected, the regularity increases with the support of the wavelet. The usual notation to denote a Daubechies-based wavelet defined by coefficients H of length L is D_L.

It is usual to let the spaces spanned by $\phi_k^j(x)$ and $\psi_k^j(x)$ over the parameter k, with j fixed, be denoted by V_j and W_j respectively,

$$V_j = \underset{k \in Z}{span}\ \phi_k^j(x), \tag{9}$$

$$W_j = \underset{k \in Z}{span}\ \psi_k^j(x). \tag{10}$$

The spaces V_j and W_j are related by,

$$... \subset V_1 \subset V_0 \subset V_{-1} \subset ..., \tag{11}$$

and

$$V_j = V_{j+1} \bigoplus W_{j+1}, \tag{12}$$

where the notation $V_0 = V_1 \bigoplus W_1$ indicates that the vectors in V_1 are orthogonal to the vectors in W_1 and the space V_0 is simply decomposed into these two component subspaces.

The previously stated condition that the wavelets form an orthonormal basis of $L^2(R)$ can now be written as,

$$L^2(R) = \bigoplus_{j \in Z} W_j. \tag{13}$$

Two final properties of the spaces V_j are that,

$$\bigcap_{j \in Z} V_j = \{0\}, \tag{14}$$

and

$$\overline{\bigcup_{j \in Z} V_j} = L^2(R). \tag{15}$$

3 The Wavelet-Optimized Finite Difference Method

The localized and multiscale nature of wavelets appeals to one's intuition for analysis of functions which are composed of local features. However, a complete numerical method using Daubechies-based wavelets has difficulties with nonlinear terms and boundary conditions. That is, nonlinear terms are currently being dealt with in the wavelet community by either transforming back to the physical space for evaluation, or by some expensive approximation to the wavelet coefficients of nonlinear terms. Likewise, wavelets at a boundary and boundary conditions imposed on these boundary functions is an incipient area of research. In fact, it is perhaps not possible to obtain a sufficient order of accuracy at a boundary in order to maintain the global accuracy of the scheme [7]. If, however, one utilizes the Daubechies wavelets in their finite-difference form then one can avoid the complications at boundaries and with nonlinear terms by performing all calculations in the physical space. That is, the goal is to create a numerical method which imitates a wavelet method while avoiding the complications.

The proposed method is named the Wavelet-Optimized Finite Difference Method, or WOFD, and it works by allowing the wavelets to place the degrees-of-freedom at the same location and at the same resolution as a complete wavelet method. These degrees-of-freedom, however, now take the form of point values instead of wavelet coefficients, and with these point

values one can perform all calculations in the physical space by applying finite-difference operators on the arbitrarily chosen wavelet-based grid. That is, wavelets can detect oscillations in a function at any location and scale. Given a function $f(x)$ for $x \in I$, where I is some interval, one decomposes $f(x)$ into a set of wavelet coefficients which depend on two parameters, one for location and one for scale, say d_k^j, where k is the location parameter and j is the scale parameter. If a wavelet coefficient is large in magnitude,

$$|d_k^j| > T, \tag{16}$$

or large in energy (In practice the two criteria yield roughly the same grid.),

$$(d_k^j)^2 > T, \tag{17}$$

where T is a coefficient threshold chosen by the user, then WOFD adds a grid point, or two, at location k and at a grid density corresponding to the scale j. That is, WOFD defines a grid which will completely resolve a function across the entire domain without over resolving it where it is relatively smooth, or composed only of large scale structure. For the specific case of the D_4 wavelet outlined in the previous section, the D_4 wavelet decomposition provides the optimal grid for 4th-order finite differencing.

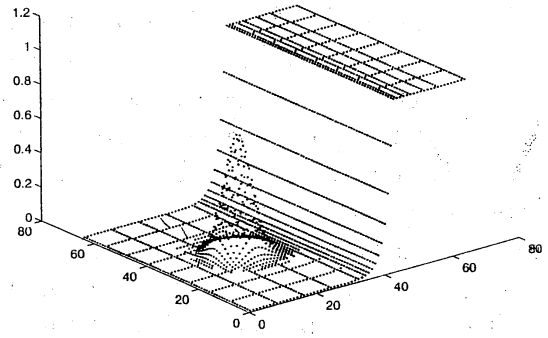
The grid definition should be made by a Daubechies wavelet which corresponds in terms of superconvergence accuracy to the accuracy of the finite difference operator. That is, it was proven in [6] that the differentiation matrix for the Daubechies wavelet D_{2M}, where M is the number of vanishing moments, displays differentiation accuracy of order $2M$ under the assumptions of periodicity and a uniform grid. Recall, that this wavelet subspace can only represent exactly the first M polynomials as determined by the number of vanishing moments. This order of accuracy $2M$ should equal the order of accuracy of the finite difference operator for optimal grid selection.

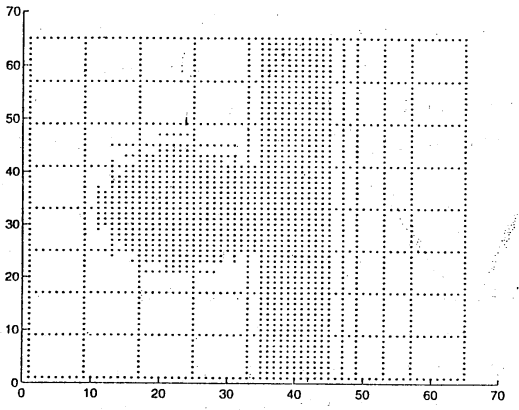
In other words, suppose a calculation begins with N evenly-spaced samples of a function $\vec{f}$ and that some quadrature method produces N scaling function coefficients on the finest scale denoted by V_0. If the spacing between adjacent values in the vector $\vec{f}$ is Δx then this is also the physical-space resolution of any calculation done in V_0. Now, decompose V_0 once to get $V_0 = V_1 \oplus W_1$. Similarly speaking, the physical space resolution of V_1 is $2\Delta x$ and the refinement from the $2\Delta x$ physical-space resolution to the Δx physical-space resolution is dictated by the wavelet coefficients in W_1, see [10].

Perhaps the best way to see the power of WOFD is to examine a couple of examples of the grid selection mechanism.

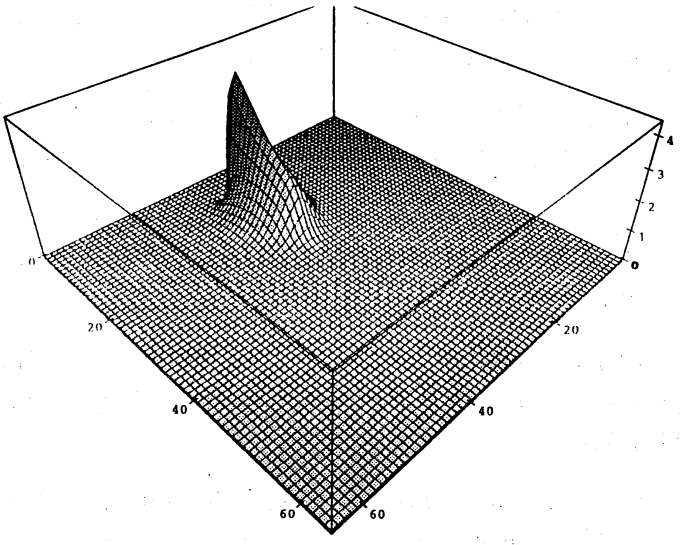
4 Examples of Grid Selection

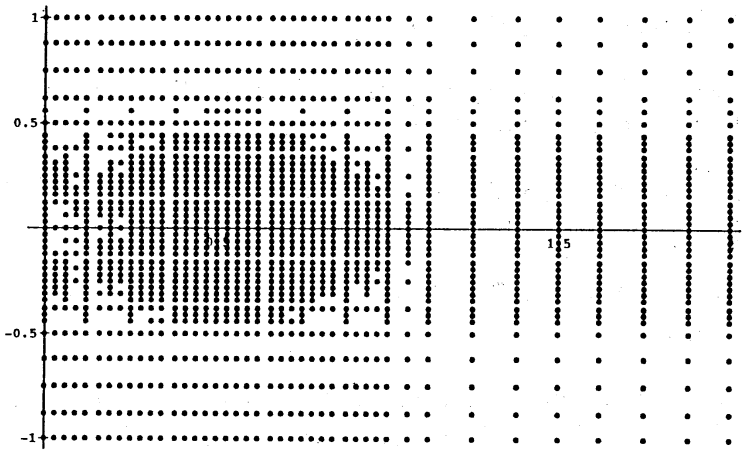
The first example has been used many times because it illustrates the most salient features of the grid selection mechanism. The function is composed of a steep gradient region with a Gaussian-shaped pulse. Obviously one needs many points in the steep gradient region to resolve the physical structure as well as many points in the region of the Gaussian pulse. The second picture shows the grid which is generated by the wavelet grid selection mechanism.





This second example is of a pulse in an isolated region of the domain. The grid selected for this pulse is very dense in the region of the pulse and sparse in the remainder of the domain.





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