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Ergodic control in a single product manufacturing system

徳大総科 大橋 守 (Mamoru Ohashi)

Abstract

We study the ergodic control problem related to stochastic production planning in a single product manufacturing system with production constraints. The existence of a solution to the corresponding Bellman equation and the optimal control are shown.

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1 Introduction

This paper deals with the following 1^{st} order differential equation:

$$\lambda = F(\frac{\partial v}{\partial x}(x,i)) - i\frac{\partial v}{\partial x}(x,i) + Av(x,i) + h(x), \quad x \in R^1, \ i = 1,2,...,d. \qquad (1)$$

Here λ is a constant, $F(x) = kx$ if $x < 0$, $= 0$ if $x \geq 0$ for some positive constant $k > 0$, h is convex function, and A denotes the infinitesimal generator of an irreducible Markov chain $(z(t), P)$ with state space $Z = \{1,2,...,d\}$:

$$Av(x,i) = \sum_{j \neq i} q_{ij}[v(x,j) - v(x,i)], \qquad (2)$$

where q_{ij} is the jump rate of $z(t)$ from i to j. The unknown are the pair (v, λ), where $v(\cdot, i) \in C^1(R^1)$ for every $i \in Z$.

Equation (1) arises in the ergodic control problem of stochastic production planning in a single product manufacturing system and is called the Bellman equation. The inventory level $x(t)$ of stochastic production planning modeled by Sethi and Zhang [11] is governed by the differential equation

$$\frac{dx(t)}{dt} = p(t) - z(t), \quad x(0) = x, \quad z(0) = i, \quad \text{P-a.s.}, \qquad (3)$$

for production rate $0 \leq p(t) \leq k$, in which $z(t)$ is interpreted as the demand rate. For ergodic control, the cost $J(p(\cdot) : x, i)$ associated with $p(\cdot)$ is given by

$$J(p(\cdot) : x, i) = \limsup_{T\to\infty} \frac{1}{T} E[\int_0^T h(x(t))dt \mid x(0) = x, z(0) = i], \tag{4}$$

where $h(x)$ represents the convex inventory cost.

The purpose of this paper is to show the existence of a solution of Bellman equation (1) and to present an optimal control minimizing the cost $J(p(\cdot) : x, i)$ subject to (3). In the control problem of manufacturing systems [5], [12] with discounted rate $\alpha > 0$, many authors have investigated the Bellman equation

$$\alpha u_\alpha(x, i) = F(\frac{\partial u_\alpha}{\partial x}(x, i)) - i\frac{\partial u_\alpha}{\partial x}(x, i) + Au_\alpha(x, i) + h(x). \tag{5}$$

Our method consists in studying the limit of (5) as α tends to 0. This approach develops the technique of Bensoussan-Frehse [2] concerning non-degenerate 2^{nd} order partial differential equations to our degenerate case. We also refer to Ghosh et al. [7], [8] in the case that the Brownian motion is added to (3) as sales returns and a bounded restriction on production rate p is made.

Section 2 is devoted to the existence problem of (1) under the convexity assumption and others on h, and properties of the solution are shown in § 3. In § 4 an optimal control for the ergodic control problem and the value are given. In § 5 we present an example of the solution to (1).

2 Existence

We are concerned with the equation

$$\alpha u_\alpha(x, i) = F(\frac{\partial u_\alpha}{\partial x}(x, i)) - i\frac{\partial u_\alpha}{\partial x}(x, i) + Au_\alpha(x, i) + h(x) \quad x \in R^1,\ i \in Z, \tag{6}$$

and make the following assumptions:

$$h(x) \text{ is nonnegative and convex on } R^1, \tag{7}$$

$$\exists C > 0;\ 0 \leq h(x) \leq C(1 + |x|^\kappa) \text{ for some positive integer } \kappa, \tag{8}$$

$$k - d > 0. \tag{9}$$

Theorem 2. 1 *We assume (7), (8) and (9). Then there exists a unique convex solution $u_\alpha(\cdot, i) \in C^1(R^1), i \in Z$ of equation (6) such that*

$$\alpha \|u_\alpha(\cdot, i)\|_{L^\infty(I_r)} \leq K_r, \tag{10}$$

$$\|\frac{\partial u_\alpha}{\partial x}(\cdot, i)\|_{L^\infty(I_r)} \leq K_r, \tag{11}$$

$$\|Au_\alpha(\cdot, i)\|_{L^\infty(I_r)} \leq K_r, \quad i \in Z, \tag{12}$$

where K_r is a positive constant depending only on r of $I_r = (-r, r)$.

Proof. According to [11, Theorem 3.1], equation (6) has a viscosity solution [6] given by

$$u_\alpha(x,i) = \inf_{p(\cdot)\in\mathcal{P}(x,i)} \{E[\int_0^\infty e^{-\alpha t} h(x(t))dt \mid x(0) = x, z(0) = i]\},$$

where $x(t)$ is as in (3), and the infimum is taken over the class $\mathcal{P}(x,i)$ of control processes $p(\cdot)$ such that $0 \le p(t) \le k$ and $p(t)$ is adapted to $\mathcal{F}_t = \sigma(z(s), s \le t)$. Moreover, $u_\alpha(x,i)$ is convex and hence a classical solution of (6) in $C^1(R^1)$. As is well-known [9], for the irreducible Markov chain $(z(t), P)$ there exists a unique equilibrium distribution $\pi = (\pi_1, \pi_2, \cdots, \pi_d) > 0$ such that

$$\pi A = 0 \quad \text{and} \quad \sum_{i\in Z} \pi_i = 1. \tag{13}$$

Now, multiplying (6) by π_i and summing up, we have

$$\alpha \sum_i \pi_i u_\alpha(x,i) = \sum_i \pi_i \{F(\frac{\partial u_\alpha}{\partial x}(x,i)) - i\frac{\partial u_\alpha}{\partial x}(x,i)\} + h(x). \tag{14}$$

Since $F(x) - ix \le 0$ under (9), we have

$$\begin{aligned} \alpha \sum_i \pi_i u_\alpha(x,i) &\le h(x) \\ &\le K_r \quad \text{on } I_r. \end{aligned}$$

Thus we can obtain (10) by $u_\alpha(x,i) \ge 0$.

Next, note that

$$F(x) - ix \le -a|x|, \tag{15}$$

where $a = \min\{k - d, 1\} > 0$. Hence, we have by (14)

$$a \sum_i \pi_i |\frac{\partial u_\alpha}{\partial x}(x,i)| \le h(x) - \alpha \sum_i \pi_i u_\alpha(x,i).$$

Thus we deduce $|\frac{\partial u_\alpha}{\partial x}(x,i)| \le K_r$ on I_r by (10) and then (11). Finally, (12) follows from (6), (10) and (11) immediately.

Next we show the behavior of a solution to equation(6) as $\alpha \to 0$.

Theorem 2. 2 *Under the assumptions of Theorem 2. 1 , there exists a subsequence $\alpha \to 0$ such that*

$$\begin{aligned} v_\alpha(x,i) &:= u_\alpha(x,i) - u_\alpha(0,i) &\to\quad& v_0(x,i) \in C^1(R^1), \\ \mu(\alpha) &:= \alpha u_\alpha(0,i) - Au_\alpha(0,i) &\to\quad& \mu_i \in R^1, \end{aligned}$$

uniformly on each $\bar{I}_r$. The limit $(v_0(\cdot,i),\mu_i)$, $i \in Z$, satisfies

$$\mu_i = F(\frac{\partial v_0}{\partial x}(x,i)) - i\frac{\partial v_0}{\partial x}(x,i) + Av_0(x,i) + h(x), \quad x \in R^1. \tag{16}$$

Proof. Let us note that $(v_\alpha(\cdot,i),\mu(\alpha))$ satisfies

$$\alpha v_\alpha(x,i) + \mu(\alpha) = F(\frac{\partial v_\alpha}{\partial x}(x,i)) - i\frac{\partial v_\alpha}{\partial x}(x,i) + Av_\alpha(x,i) + h(x). \tag{17}$$

By (11) it is obvious that

$$\|v_\alpha(\cdot,i)\|_{L^\infty(I_r)} + \|\frac{\partial v_\alpha}{\partial x}(\cdot,i)\|_{L^\infty(I_r)} \le K_r, \quad i \in Z. \tag{18}$$

Hence $\{v_\alpha(\cdot,i)\}$ is equicontinuous on $\bar{I}_r$.

Let us define

$$B_\alpha(x) = \alpha v_\alpha(x,i) + \mu(\alpha) - Av_\alpha(x,i) - h(x).$$

We recall that, by assumption, $h(x)$ is Lipschitz continuous on $\bar{I}_r$. Then, by (18)

$$|B_\alpha(x) - B_\alpha(y)| \le C|x-y|, \qquad (C > 0: \text{ indep. of } \alpha),$$

From (17) it follows that

$$B_\alpha(x) = F(\frac{\partial v_\alpha}{\partial x}(x,i)) - i\frac{\partial v_\alpha}{\partial x}(x,i),$$

Then we have

$$\frac{\partial v_\alpha}{\partial x} = \begin{cases} \frac{B_\alpha(x)}{k-i} & \text{if } \frac{\partial v_\alpha}{\partial x} < 0 \\ \\ -\frac{B_\alpha(x)}{i} & \text{if } \frac{\partial v_\alpha}{\partial x} \ge 0, \end{cases}$$

Since $\frac{\partial v_\alpha}{\partial x}$ is nondecreasing, we can see

$$|\frac{\partial v_\alpha}{\partial x}(x,i) - \frac{\partial v_\alpha}{\partial x}(y,i)| \le C|x-y|.$$

Thus $\{\frac{\partial v_\alpha}{\partial x}(\cdot,i)\}$ is also equicontinuous on $\bar{I}_r$. By the Ascoli-Arzelà theorem, there exists a subsequence $\alpha \to 0$ such that

$$v_\alpha(x,i) \to v_0(x,i), \tag{19}$$

$$\frac{\partial v_\alpha}{\partial x}(x,i) \to \frac{\partial v_0}{\partial x}(x,i), \quad \text{uniformly on } \bar{I}_r. \tag{20}$$

By a standard argument, we can choose a subsequence $\alpha \to 0$, independent of r, such that (19) and (20) are fulfilled on every $\bar{I}_r$. Further, by (10) and (12)

$$\mu(\alpha) \to \mu_i.$$

Letting $\alpha \to 0$ in (17), we deduce (16). The proof is complete.

Now let us show the existence of a solution to equation (1).

Theorem 2. 3 *We assume (7), (8) and (9). Then there exists a solution (v, λ) of equation (1) such that $v(x,i)$ is convex on R^1 and $v(\cdot,i) \in C^1(R^1)$.*

Proof. Let us define

$$v(x,i) = v_0(x,i) + f(i),$$
$$\lambda = \sum_i \pi_i \mu_i,$$

where $(v_0(\cdot,i), \mu_i)$ is as in (16) and $f(i)$ is a solution of

$$Af(i) = -\mu_i + \lambda, \quad i \in Z. \tag{21}$$

Then it is easily seen that (v, λ) satisfies (1). The convexity of $v(x,i)$ and $v(\cdot,i) \in C^1(R^1)$ are immediate from Theorems 2.1 and 2.2.

To complete the proof, it is sufficient to check the existence of $f(i)$. By the irreducible Markov chain $(z(t), P)$ it follows that for any $g \in R^d$

$$E[g(z(s/\alpha))] \to \sum_i \pi_i g(i) \quad \text{as} \quad \alpha \to 0.$$

Hence

$$\begin{aligned} \alpha G_\alpha g(i) &= \alpha E[\int_0^\infty e^{-\alpha t} g(z(t))dt] \\ &= \int_0^\infty e^{-s} E[g(z(s/\alpha))]ds \\ &\to \sum_i \pi_i g(i) = \pi g, \end{aligned}$$

where G_α denotes the resolvent operator of the Markov chain $(z(t), P)$. According to [4, Lemma 7.3(c,d), p.39], we can obtain the relation:

$$\{g \in R^d | \ \pi g = 0\} = \{Ag \in R^d | \ g \in R^d\}.$$

We notice by (13) that

$$\pi \ (-\mu. + \lambda) = 0.$$

Therefore we conclude that equation (21) admits a solution $f(i)$.

3 Properties

We investigate properties of a solution to the Bellman equation (1). Now we make the assumption:

$$h(x)/|x| \quad \to \quad \infty \quad \text{as } |x| \to \infty. \tag{22}$$

Lemma 3. 1 *Under (22), the convex solution $v(\cdot, i) \in C^1(R^1)$ of equation (1) satisfies*

$$|\frac{\partial v}{\partial x}(x,i)| \quad \to \quad \infty \quad as\ |x| \to \infty. \tag{23}$$

Proof. It is sufficient to show (23) in the case $x \to -\infty$. By the convexity of $v(x,i)$, we can define M_i by

$$M_i = - \lim_{x \to -\infty} \frac{\partial v}{\partial x}(x,i).$$

For any sequence $x_n \to -\infty$, we can easily see

$$\frac{v(x_n, i)}{|x_n|} \quad \to \quad M_i.$$

Suppose that $M_i < \infty$ for some $i \in Z$. Then, dividing (1) by $|x_n|$ and passing to the limit, we have by (22)

$$\lambda/|x_n| = [F(\frac{\partial v}{\partial x}(x_n,i)) - i\frac{\partial v}{\partial x}(x_n,i) + \sum_{j \neq i} q_{ij} v(x_n, j)$$
$$- \sum_{j \neq i} q_{ij} v(x_n, i) + h(x_n)]/|x_n| \quad \to \infty,$$

since $v(x,j) \geq ax + b$ for some constants a and b. This is a contradiction. Hence $M_i = \infty$ for all $i \in Z$, and thus the assertion follows.

Lemma 3. 2 *For the convex solution $v(\cdot, i) \in C^1(R^1)$ of equation (1), there is a constant $C > 0$ such that*

$$|v(x,i)| \leq C(1 + |x|^{\kappa+1}). \tag{24}$$

Proof. From (1) and (15) it follows that

$$\lambda \leq -a|\frac{\partial v}{\partial x}(x,i)| + Av(x,i) + h(x).$$

If $\frac{\partial v}{\partial x}(x,i) < 0$ on some interval $(-\infty, x_1)$ with $x_1 < 0$, then by (8)

$$- \frac{\partial v}{\partial x}(x,i) \leq \frac{1}{a} Av(x,i) + C(1 + |x|^{\kappa}) \tag{25}$$

Multiplying (25) by π_i and summing up, we get by (13)

$$- \sum_i \pi_i \frac{\partial v}{\partial x}(x,i) \leq C(1 + |x|^{\kappa}).$$

Integrating over (x, x_1), we have

$$\sum_i \pi_i(v(x,i) - v(x_1,i)) \leq C(1 + |x|^{k+1}).$$

This relation can be obtained in the case that $\frac{\partial v}{\partial x} \geq 0$ on some interval (x_2, ∞) with $x_2 > 0$. Therefore we can obtain the desired result by $\pi > 0$.

Next, we consider the equation

$$\frac{dx^*(t)}{dt} = p^*(x^*(t), z(t)) - z(t), \quad x^*(0) = x, \quad z(0) = i, \quad P - a.s., \tag{26}$$

where

$$p^*(x,i) = \begin{cases} k & \text{if } \frac{\partial v}{\partial x}(x,i) < 0 \\ i & \text{if } \frac{\partial v}{\partial x}(x,i) = 0 \\ 0 & \text{if } \frac{\partial v}{\partial x}(x,i) > 0. \end{cases} \tag{27}$$

Lemma 3. 3 *Equation (26) admits a unique solution $x^*(t)$, which satisfies*

$$\sup_t \|x^*(t)\|_{L^\infty} < \infty.$$

Proof. Since $p^*(x,i)$ is nonincreasing in x, the differential equation (26) has a unique solution by [6, Theorem 6.2].

To complete the proof, let $\bar{x} = \sup\{x \in R^1 : \ p^*(x,i) \geq i \text{ for some } i \in Z\}$. Obviously, $\bar{x}$ is finite, because $p^*(x,i)$ is nonnegative. Similarly, let $\tilde{x} = \inf\{x \in R^1 : p^*(x,i) \leq i \text{ for some } i \in Z\}$. Suppose that $\tilde{x}$ is not finite. Then there exists $i \in$Z such that $\frac{\partial v}{\partial x}(x,i) \geq -2i$ for all $x \in R^1$. On the other hand, by Lemma 3.1, $\frac{\partial v}{\partial x}(x,i) \to -\infty$ as $x \to -\infty$. This is a contradiction.

Now, if $x^*(t) > \bar{x}$ (*resp.* $x^*(t) < \tilde{x}$), then $\frac{dx^*}{dt}(t) < 0$ (*resp.* $\frac{dx^*}{dt}(t) > 0$). Hence the interval $[\tilde{x}, \bar{x}]$ is an attracting set for (26). Thus the boundedness of $x^*(t)$ is immediate.

Lemma 3. 4 *The constant solution λ of equation (1) satisfies*

$$\lambda = \inf_{p(\cdot) \in \mathcal{P}(x,i)} \limsup_{\alpha \to 0} \alpha E\left[\int_0^\infty e^{-\alpha t} h(x(t))dt \mid x(0) = x, z(0) = i\right]. \tag{28}$$

Proof. For the convex solution $v(\cdot, i) \in C^1(R^1)$, let us apply an elementary rule and Dynkin's formula to the first and the second variables of $v(x(t), z(t))$ respectively. Then we have the relation:

$$\begin{aligned} &E[e^{-\alpha t} v(x(t), z(t)) \mid x(0) = x, z(0) = i] \\ &\quad = v(x,i) - \alpha E\left[\int_0^t e^{-\alpha s} v(x(s), z(s))ds \mid x(0) = x, z(0) = i\right] \\ &\quad\quad + E\left[\int_0^t e^{-\alpha s} \frac{\partial v}{\partial x}(x(s), z(s))dx(s) \mid x(0) = x, z(0) = i\right] \\ &\quad\quad + E\left[\int_0^t e^{-\alpha s} A v(x(s), z(s))ds \mid x(0) = x, z(0) = i\right] \end{aligned} \tag{29}$$

We notice that the minimum of

$$\min_{0\le p\le k} p\frac{\partial v}{\partial x} = F(\frac{\partial v}{\partial x})$$

is attained by $p^*(x,i)$. By (1), we have

$$\lambda \le \frac{\partial v}{\partial x}(x,i)(p-i) + Av(x,i) + h(x), \tag{30}$$

and the equality holds for $p = p^*(x,i)$. Clearly, by (3)

$$|x(t)| \le C(t+1) \quad \text{for all} \quad p(\cdot) \in \mathcal{P}(x,i).$$

By Lemma 3.2

$$\begin{aligned} E[e^{-\alpha t}|v(x(t),z(t))| \mid x(0)=x, z(0)=i] & \\ \le CE[e^{-\alpha t}(1+|x(t)|^{\kappa+1}) \mid x(0)=x, z(0)=i] & \\ \le Ce^{-\alpha t}(1+(t+1)^{\kappa+1}) \ \to\ 0 \ \text{ as } \ t\to\infty. & \end{aligned}$$

Hence, substituting (30) into (29), we get

$$\begin{aligned} \frac{\lambda}{\alpha} \ \le\ & -v(x,i) + \alpha E[\int_0^\infty e^{-\alpha s} v(x(s),z(s))ds \mid x(0)=x, z(0)=i] \\ & +E[\int_0^\infty e^{-\alpha s} h(x(s))ds \mid x(0)=x, z(0)=i]. \end{aligned}$$

We note that by Lemma 3.2 and 3.3

$$\begin{aligned} \alpha^2 E[\int_0^\infty e^{-\alpha s}|v(x(s),z(s))|ds \mid x(0)=x, z(0)=i] & \\ \le \alpha^2 C\int_0^\infty e^{-\alpha s}(1+|x^*(s)|^{\kappa+1})ds \ \to\ 0 \ \text{ as } \ \alpha\to 0. & \end{aligned}$$

Thus we deduce

$$\lambda \le \inf_{p(\cdot)\in\mathcal{P}(x,i)} \limsup_{\alpha\to 0} \alpha E[\int_0^\infty e^{-\alpha s} h(x(s))ds \mid x(0)=x, z(0)=i],$$

and the equality holds for $p(t) = p^*(x^*(t), z(t))$ of (27).

4 An application to ergodic control

We shall study the ergodic control problem to minimize the cost:

$$J(p(\cdot):x,i) = \limsup_{T\to\infty} \frac{1}{T} E[\int_0^T h(x(t))dt \mid x(0)=x, z(0)=i]$$

over all $p(\cdot) \in U$ subject to

$$\frac{dx(t)}{dt} = p(t) - z(t), \quad x(0) = x, \; z(0) = i, \quad \text{P-a.s.},$$

where U is the set of all nonnegative progressively measurable processes $p(t)$ such that

$$\begin{aligned}
&p(t) \text{ is adapted to } \mathcal{F}_t, \\
&0 \le p(t) \le k, \\
&\sup_t E[|x(t)|^{\kappa+1} \mid x(0) = x, z(0) = i] < \infty \quad \text{for } \kappa \text{ in (8).}
\end{aligned}$$

Theorem 4. 1 *We assume (7), (8), (9) and (22). Then the optimal control $p^*(t)$ is given by*

$$p^*(t) = p^*(x^*(t), z(t)),$$

and the value by

$$J(p^*(\cdot) : x, i) = \lambda,$$

where $p^*(x^*(t), z(t))$ is as in (27).

Proof. From the same formula as (29) it follows that

$$\begin{aligned}
&E[v(x(T), z(T)) \mid x(0) = x, z(0) = i] \\
&\quad = v(x,i) + E[\int_0^T \frac{\partial v}{\partial x}(x(s), z(s))dx(s) \mid x(0) = x, z(0) = i] \\
&\quad\quad + E[\int_0^T Av(x(s), z(s))ds \mid x(0) = x, z(0) = i].
\end{aligned}$$

We recall (30) to obtain

$$\begin{aligned}
&E[v(x(T), z(T)) \mid x(0) = x, z(0) = i] \\
&\quad\quad \ge v(x,i) + E[\int_0^T (\lambda - h(x(s)))ds \mid x(0) = x, z(0) = i],
\end{aligned}$$

where the equality holds for $x = x^*$ and $p = p^* of$ (27). By Lemma 3. 2 and the definition of U

$$\begin{aligned}
&\frac{1}{T} E[|v(x(T), z(T))| \mid x(0) = x, z(0) = i] \\
&\quad\quad \le \frac{C}{T} E[1 + |x(T)|^{k+1} \mid x(0) = x, z(0) = i] \quad \to \quad 0 \quad \text{as } T \to \infty.
\end{aligned}$$

Also, by Lemma 3. 3 , $p^*(t)$ belongs to U. Thus we deduce

$$\begin{aligned}
J(p(\cdot) : x, i) &= \limsup_{T \to \infty} \frac{1}{T} E[\int_0^T h(x(s))ds \mid x(0) = x, z(0) = i] \\
&\ge \lambda = J(p^*(\cdot) : x, i).
\end{aligned}$$

The proof is complete.

5 An Example

In this section we present the example of an solution to the Bellman equation:

$$\lambda = F(\frac{\partial v}{\partial x}(x,i)) - i\frac{\partial v}{\partial x}(x,i) + Av(x,i) + h(x), \quad x \in R^1,\ i \in Z, \tag{31}$$

in the case that

$$h(x) = x^2, k = 3, \tag{32}$$

$$Z = \{1,2\}, q_{12} = q_{21} = 1. \tag{33}$$

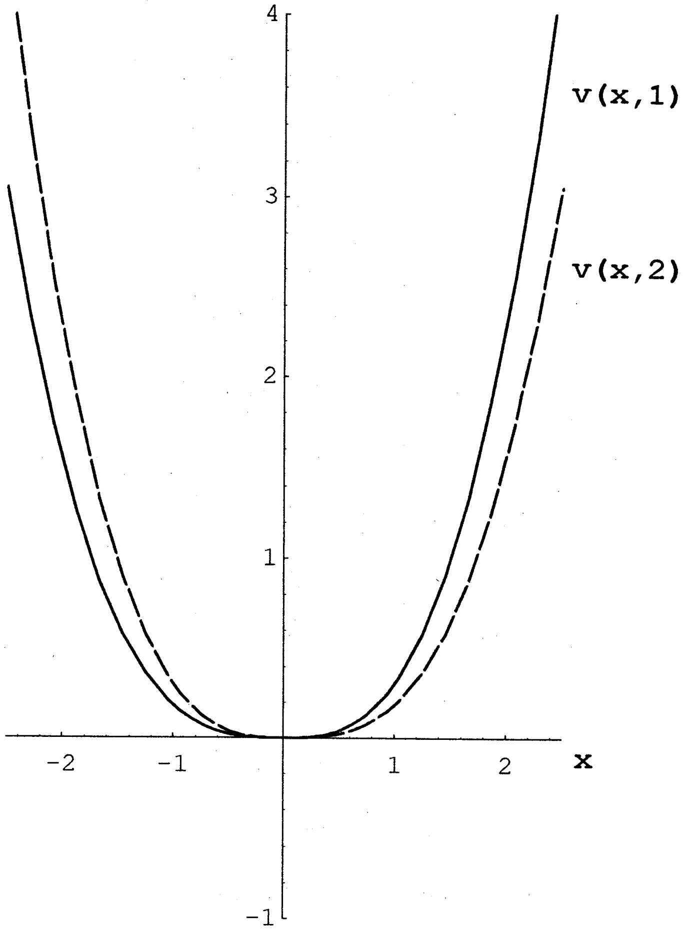

Figure Solution v(x,i), i=1,2, to the Bellman Equation(31)

We remark that the matrix induced by A is given by

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix},$$

and the equilibrium distribution π is

$$\pi = (\frac{1}{2}, \frac{1}{2}).$$

Therefore the assumptions of Theorem 4.1 are fulfilled.

Now, recalling the form of optimal control p^* and solving the Bellman equation (31) with (32) and (33), we have

$$\begin{aligned} &\lambda = 0, \\ &v(x,1) = \begin{cases} \frac{1}{81}(18x^3 + 18x^2 - 24x + 16 - 16e^{-\frac{3}{2}x}) & \text{if } x \geq 0 \\ -\frac{1}{81}(18x^3 + 9x^2 + 12x + 8 - 8e^{\frac{3}{2}x}) & \text{if } x < 0, \end{cases} \\ &v(x,2) = \begin{cases} \frac{1}{81}(18x^3 - 9x^2 + 12x - 8 + 8e^{-\frac{3}{2}x}) & \text{if } x \geq 0 \\ -\frac{1}{81}(18x^3 - 18x^2 - 24x - 16 + 16e^{\frac{3}{2}x}) & \text{if } x < 0. \end{cases} \end{aligned}$$

Then the optimal control p^* is given by

$$p^*(x,i) = \begin{cases} 0 & \text{if } \; x > 0 \\ i & \text{if } \; x = 0 \\ 3 & \text{if } \; x < 0. \end{cases}$$

The solution $v(x,i)$ with (23) - (24), i = 1, 2 can be shown in Figure.

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