数理解析研究所講究録
947 巻 1996 年 137-142

Two-Person Hi-Lo Poker —— Stud and Draw

名古屋商科大 坂口 実 (Minoru Sakaguchi)
Inst. of Nat. Res, RUSSIA V. V. Mazalov

Abstract This paper analyses a continuous version of a class of two-person Hi-Lo poker. Stud-poker and draw-poker versions are discussed in each of which simultaneous-move and bilateral-move one-round games are formulated and explicit solutions are derived. It is shown that in bilateral-move games the first-mover innevitably gives his opponent some information about his true hand, and so the second-mover is able to utilize this information in deciding his best response in the optimal play. A connection between Hi-Lo poker and simple exchange games is mentioned.

References [3] R.A.Epstein, The theory of gambling and statistical logic, Academic Press, New York 1977. [6] V.V.Mazalov, Game theoretic model of preference, Intern. Year-Book of Game Theory and Applications, 2 (1994), 26-34, Nova Science Publishers, Inc. [9] M. Sakaguchi, and S.Sakai, Solutions of some three-person stud and draw poker, Math. Japonica, 37 (1992), 1147-1160. [10] M.Sakaguchi, On two and three person exchange games, Math. Japonica, 38 (1993), 791-801.

1. A Hi-Lo Stud Poker

Suppose that two players I and II, receive cards x and y respectively, the values of which we consider as *iid* random variable distributed according to $U_{[0,1]}$. I(II) observes the value of $x(y)$ only.

Let A and B given real numbers with $0 \leqq A \leqq B$. Players are requested to choose either one of Hi or Lo. Choices should be made simultaneously and independently of the rival's choice. Then the players make show-down. If choices are Hi-Hi (Hi-Lo or Lo-Hi) a player with higher hand wins and gets $B(A)$ from the opponent. If choices are Lo-Lo a player with lower hand wins and gets unity from the opponent. If hands are equal, *i.e.*, $x=y$, there is a draw. The payoff table is shown as:

$$(1.1) \qquad \begin{array}{cc} & \begin{array}{cc} \text{Hi} & \text{Lo} \end{array} \\ \begin{array}{c} \text{Hi} \\ \text{Lo} \end{array} & \begin{bmatrix} B\,\mathrm{sgn}(x-y) & A\,\mathrm{sgn}(x-y) \\ A\,\mathrm{sgn}(x-y) & \mathrm{sgn}(y-x) \end{bmatrix} \end{array}$$

Payoff function under the strategy-pair $\alpha(\cdot)-\beta(\cdot)$ is

$$(1,2)\quad M(\alpha,\beta)\equiv E_{x,y}\left\{(\alpha(x),\bar{\alpha}(x))\begin{bmatrix} B & A \\ A & -1 \end{bmatrix}^{\operatorname{sgn}(x-y)}\begin{bmatrix}\beta(y)\\ \bar{\beta}(y)\end{bmatrix}\right\}$$

Theorem 1. *The optimal strategy for the game with payoff function*(1. 2) *is common to the players and takes the form* $\alpha^*(x)=I(x \geqq b)$, *where* $b=(B-A)/(B+1)$. *The value of the game is zero.*

2. *A* Bilateral-Move Hi-Lo Stud Poker

In the Hi-Lo poker discussed in Section 1 the players must move independently and simultaneously. The poker discussed in the following is played by bilateral moves by the players. The choices and payoffs are given as the same as in (1. 1). The only difference is that player I(II) should move first (second). So the game is played as described by:

Player's hand	1st move	2nd move	Player I's payoff
I : x	Hi	Hi	$B\operatorname{sgn}(x-y)$
		Lo	$A\operatorname{sgn}(x-y)$
II : y	Lo	Hi	$A\operatorname{sgn}(x-y)$
		Lo	$\operatorname{sgn}(y-x)$

Player II, at his move, knows which choice was made by player I in the previous move, and therefore he can utilize the information in deciding his own choice.

Payoff ft. under the str.-pair $\alpha(\cdot)-(\beta(\cdot),\gamma(\cdot))$ is

$$(2.1)\quad M(\alpha,\beta,\gamma)\equiv E_{x,y}\Big[\big\{\alpha(x)\big(B\beta(y)+A\bar{\beta}(y)\big) + \bar{\alpha}(x)\big(A\gamma(y)-\bar{\gamma}(y)\big)\big\}\operatorname{sgn}(x-y)\Big].$$

Theorem 2. *The optimal strategy-pair for the game with payoff function* (2. 1) *is:*

$$\alpha^*(x)=\begin{cases} 0 & \text{if } 0\leqq x<b_0 \\ \text{arbitrary, but satisfies the requirements } 0\leqq\alpha^*(x)\leqq 1 \text{ and } \int_{b_0}^{b_1}\alpha^*(x)dx=\frac{1}{2}-b_0 & \text{if } b_0<x\leqq b_1 \\ 1, & \text{if } b_1<x\leqq 1 \end{cases}$$

$\beta^*(y)=I(y\geqq b_1)$, and $\gamma^*(y)=I(y\geqq b_0)$,

where $b_0=(B-A)/2(B+1)$ and $b_1=b_0+1/2$.

The value of the game is $-(1/4)(B-A)(A+1)/(B+1)$.

Concerning the above theorem we observe some interesting points. (1) The value of the game is negative. This reflects that player I has a unfavorable condition that he has to move first and inevitably gives some information about his hand to his opponent. (2) Player I has an infinitely many optimal strategies, but player II has a unique one. (3) Player I, the first mover, has the possibility of bluffing. That is, he can take, for example,

$$\alpha^*(x) = 1, \quad \text{in } (b_0, 1/2)\,; \quad = 0, \quad \text{in } (1/2, b_1).$$

After I has chosen Hi(Lo), II has to guess whether I's hand is truly high(low) and so he has chosen Hi(Lo), or I's hand is low-(high) and he wants to mislead II's choice. (4) We always have $b_1 - b_0 = 1/2$, independently of A and B.

3. A Hi-Lo Draw Poker

Suppose that in the Hi-Lo poker discused in Section 2 each player may draw another card from the pile, and use the card with the larger value (than one delivered in the beginning of play) throwing away the card with the smaller value. This choice, the players may take, we call "bet" in this section. The other choice each player may take is to "pass", that is, he doesn't draw a new card. Thus if the new cards are denoted by *iid* $z, w \sim U_{[0,1]}$ the payoff table is shown as:

$$(3.1) \qquad \begin{array}{cc} & \overbrace{\begin{array}{cc} \text{Bet} & \text{Pass} \end{array}}^{(\mathrm{II})} \\ (\mathrm{I})\begin{cases} \text{Bet} \\ \text{Pass} \end{cases} & \begin{bmatrix} a_{11}(x,y) & a_{10}(x,y) \\ a_{01}(x,y) & a_{00}(x,y) \end{bmatrix} \end{array}$$

where

$$a_{11}(x,y) \equiv E_{z,w}\{B\,\mathrm{sgn}(x \vee z - y \vee w)\} = B(x \vee y)^2\,\mathrm{sgn}(x-y),$$

$$a_{10}(x,y) \equiv E_z\{A\,\mathrm{sgn}(x \vee z - y)\} = A(\bar{y} + y\,\mathrm{sgn}(x-y)),$$

$$a_{01}(x,y) \equiv E_w\{A\,\mathrm{sgn}(x - y \vee w)\} = -A(\bar{x} - x\,\mathrm{sgn}(x-y)),$$

$$a_{00}(x,y) \equiv \mathrm{sgn}(y-x)$$

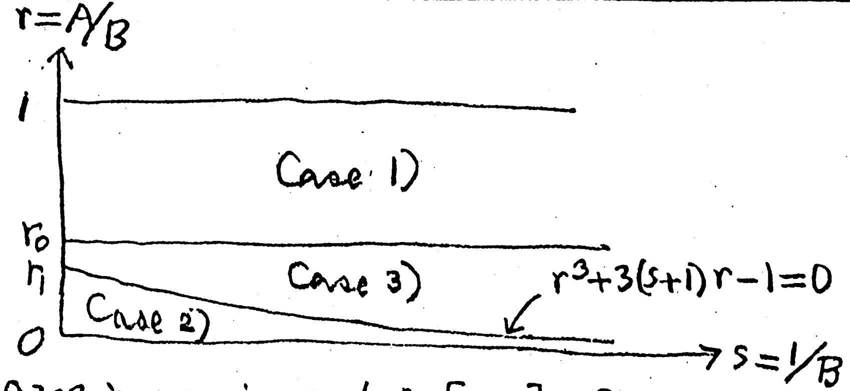
Payoff ft. under the str.-pair $\alpha(\cdot)$-$\beta(\cdot)$ is

$$(3.2) \qquad M(\alpha, \beta) \equiv E_{x,y}\left\{(\alpha(x), \bar{\alpha}(x)) \begin{bmatrix} a_{11} & a_{10} \\ a_{01} & a_{00} \end{bmatrix} \begin{bmatrix} \beta(y) \\ \bar{\beta}(y) \end{bmatrix}\right\}$$

Theorem 3 can be rewritten as:

Theorem 3: The solution to the game with payoff function (3.2) is as follows: Let $r = A/B \in [0,1]$, $s = 1/B > 0$ and $r_0 \equiv \frac{1}{2}(\sqrt{3}-1) \doteqdot 0.366$. The value of the game is 0. The optimal strategy is common to the players and is (see Figure 2):

Case	Optimal strategy
1) $r_0 \leqq r \leqq 1$	$\alpha^*(x) \equiv 1$ ("always" Bet")
2) $0 \leqq r < r_0$ and $r^3+3(s+1)r-1 \leqq 0$	$\alpha^*(x) = I(x \geqq b)$ ("Bet" if and only if $x \geqq b$, where b is given by (3.5))
3) Otherwise	$\alpha^*(x)$ is given by (3.8)~(3.13)



($r_1 \doteqdot 0.322$ is a unique root in $[0,1]$ of the equation $r^3+3r-1=0$.)

Figure 2. Common optimal strategy in Theorem 3

4. A Bilateral-Move Hi-Lo Draw Poker

Now we go to discuss about the bilateral-move version of the poker solved in the previous section. The game is played as described by:

Players' hands	1st move	2nd move	Player I's payoff
I: x	Bet		
	Pass		
	Bet →	Bet	$a_{11}(x,y)$
II: y		Pass	$a_{10}(x,y)$
	Pass →	Bet	$a_{01}(x,y)$
		Pass	$a_{00}(x,y)$

Payoff ft. under the str.-pair $\alpha(\cdot)$-$(\beta(\cdot),\gamma(\cdot))$ is

$$(4.1)\quad M(\alpha,\beta,\gamma)=E_{x,y}\Big[\alpha(x)\big(a_{11}\beta(y)+a_{10}\bar{\beta}(y)\big)+\bar{\alpha}(x)\big(a_{01}\gamma(y)+a_{00}\bar{\gamma}(y)\big)\Big].$$

Theorem 4. *Let $A=0<B$. The optimal strategy-pair for the game with payoff function* (4. 1) *is:*

$$\alpha^*(x)=\begin{cases} 0, & \text{if } 0\leqq x<b_0 \\ \text{arbitrary, but satisfies the requirement} & \\ 0\leqq\alpha^*(x)\leqq 1 \ \text{and} \ \int_{b_0}^{b_1}\alpha^*(x)\,dx=b_1-2b_0, & \text{if } b_0<x<b_1 \\ 1, & \text{if } b_1<x\leqq 1. \end{cases}$$

$$\beta^*(y)=I(y\geqq b_1) \quad \text{and} \quad \gamma^*(y)=I(y\geqq b_0),$$

where $b_0=(1/3)B(1-b_1^3)$ *and* b_1 *is a unique root in* (0, 1) *of the equation*

$$(4.13)\qquad b_1^2(Bb_1^3+2b_1-B)=1/2.$$

The value of the game is $-b_0(b_1-b_0)$.

The value of the game is negative. This again reflects that player I, the first mover, stands at a unfavorable condition since he leaks some information about his true hand to his opponent by moving first. He is able to make bluf by taking $\alpha^*(x)=I(b_0<x<b_1-b_0 \text{ or } x>b_1)$. After I has chosen Bet (Pass), II has to guess whether I's hand is truly high (low) and he has made the choice, or I's hand is truly low (high) and he wants to mislead his opponent's choice.

Example 2.

Let A=0 and B= $3/2$. In the simultaneous-move version (Theorem 3) the optimal strategy common to players is to Bet (Pass) if his hand is $>(<)\,b\fallingdotseq 0.455$, where b is the unique root of the equation $b^3+2b-1=0$. The value of the game is zero.

In the bilateral-move version, the solution to the game is described by Theorem 4. with the values

$b_0 \fallingdotseq 0.256$, $b_1 \fallingdotseq 0.787$, $m=b_1-2b_0 \fallingdotseq 0.275$ and $V=-b_0(b_1-b_0) \fallingdotseq -0.136$

where $b_0=(1/2)(1-b_1^3)$ and b_1 is the unique root of the equation $b_1^2(3b_1^3+4b_1-3)=1$.

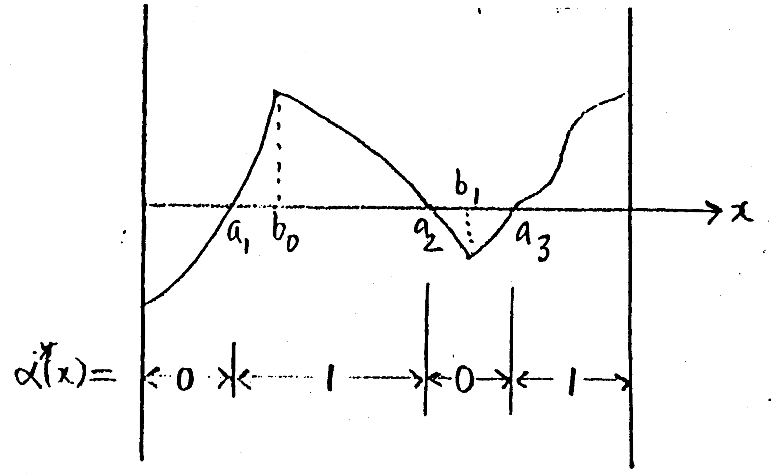
If $0<A \leqq B$, the analysis is far more tedius. Under II's strategy $\beta(y)=I(y \geqq b_1)$, $\gamma(y)=I(y \geqq b_0)$, with $0<b_0<b_1<1$, we find

$$M(\alpha,\beta,\gamma)=E_x\{K(x|\beta,\gamma)\alpha(x)\}+[\text{term indep. of } \alpha(\cdot)]$$

where

$$K(x|\beta,\gamma) \equiv E_y\{(a_{11}\beta(y)+a_{10}\bar{\beta}(y))-(a_{01}\gamma(y)+a_{00}\bar{\gamma}(y))\}$$

Among a lot of possible behavior of the function $K(x|\beta,\gamma)$, depending on the two parameters A and B, the following is most plausible one, *i.e.*,



although determining the five constants $0<a_1<b_0<a_2<b_1<a_3<1$ such that $\alpha^*(x)$ and $\beta^*(y)$-$\gamma^*(y)$ with these decision thresholds constitute a saddle point of (4. 1) will be very difficult.

[The full paper will appear in Math. Japonica, Vol. 43(1966), no. 3.]