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The Throughput Rate of Interchangeable Parallel Two-Stage Tandem Queue with Correlated Service Times

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ABSTRACT. Parallel two-stage tandem queueing system with no intermediate waiting room is considered. An arriving customer according to a homogeneous Poisson process, when his first stage service is finished, can enter both of the second stage channels. We call this model as an interchangeable queueing system. The service times are not independent but depend upon each other. It is assumed that their service distribution is the bivariate exponential distribution of Marshall and Olkin. The standard representation of the multivariate exponential distribution of Marshall and Olkin is derived and the throughput of this system is obtained using a matrix-geometric approach. Finally, comparison with the throughput of an ordinary parallel two-stage tandem queueing system is discussed.

1. Introduction

In most of queueing systems, it has been assumed that the service times of all servers are mutually independent. However, this assumption is not realistic, because the competition or cooperation among servers occurs.

Mitchell, Paulson and Weswick [5] simulated a correlated two-stage tandem queueing model in which their service distribution is a special case of the bivariate gamma distribution of Paulson [14] and discussed the performance of the system for both the positively and negatively correlated cases. Their simulation indicates that the mean waiting time at the second stage with infinite intermediate buffer spaces is smaller than the case of mutually independent service times. For this model Wolff [16] briefly showed the same result for the positively correlated case by means of usual stochastic ordering since two service time random variables have positively quadrant dependent property.

Yoneyama [17] considered a correlated two-stage tandem queueing model in which their service distribution is the bivariate exponential distribution (abbreviated as BVE) of Marshall and Olkin [4], and obtained the mean number of customers in the system and showed analytically that it decreases according to the increasing of the correlation parameter.

To the best of our knowledge, Nishida, Watanabe and Tahara [7] first discussed the application of the BVE in the queueing theory. They considered a two-server queueing

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model in which their service distribution is the BVE and obtained the steady state probabilities. As an extension of their model, Nishida and Yoneyama [8] considered a multiserver queueing model in which their sevice distribution is the multivariate exponential distribution (abbreviated as MVE) of Marshall and Olkin [4] and showed how to get the steady state probabilities. For this model, Yoneyama [18] obtained both the queueing time and the waiting time distribution and showed that Little's law holds for their mean.

Assaf et al. [1] showed that the BVE is a bivariate phase type (abbreviated as BPH) distribution and its multivariate extension is a multivariate phase type (abbreviated as MPH). Raftery [15] introduced a continuous multivariate exponential distribution and O'Cinneide and Raftery [10] showed that it is MPH and derived its MPH representation.

On the other hand, Nishida [9] considered an interchangeable parallel two-stage tandem queueing system without waiting room. He called the discipline that a customer who finished the first stage service can enter both of the second stage channels as an interchangeable queueing system. He found the optimal allocation of service rates for the first and second stage channels in the sense of minimizing the rate of loss calls.

In this paper, as an extension of the model introduced by Nishida [9], we consider an interchangeable parallel two-stage tandem queueing system in which the service times of each two channels in the first and second stage are assumed to follow the BVE. For this system we get the throughput of the system using a matrix-geometric approach of Neuts [6], and compare the throughput of the system with those of an ordinary parallel two-stage tandem queue and an interchangeable parallel two-stage tandem queue whose service distribution is an ordinary exponential distribution.

A closely related model is that by Latouche and Neuts [3]. An approach is closely related to Heavey, Papadopoulos and Browne [2], Papadopoulos, Heavey and O'Kelly [11,12] and Papadopoulos and O'Kelly [13]. Similarly to the definition by the above four papers, the throughput is defined as a critical input rate on the assumption that the first queue is never empty.

Before we describe the model fully, we derive the standard MPH representation of the MVE and get the BVE for the simplest case.

2. Standard MPH representation of MVE

The random vector $\mathbf{Y} = (Y_1, \cdots, Y_n)$ is said to have the multivariate exponential distribution (MVE) of Marshall and Olkin [4] if there exist independent exponential random variables $X_1, \cdots, X_k$ such that for $i = 1, \cdots, n$, $Y_i = \min_{j \in J_i} X_j$ where $J_i \subset \{1, \cdots, k\}$.

Assaf et al. [1] first formulated a multivariate phase type (MPH) distribution in the following way. Suppose $\{V(t) : t \geq 0\}$ is a regular Markov chain with finite state-space E. Let $\Gamma_1, \cdots, \Gamma_n$ be n non-empty subsets of E such that once V enters Γ_i it never leaves. Suppose that $\bigcap_{i=1}^{n} \Gamma_i$ consists of one state Δ, into which absorption is certain. Let β be an initial probability vector on E, which puts all its mass on states in $E \setminus \{\Delta\}$.

The infinitesimal generator Q of V is of the form

$$\mathbf{Q} = \begin{bmatrix} \mathbf{T} & -\mathbf{Te} \\ \mathbf{0} & 0 \end{bmatrix} \tag{1}$$

where $\mathbf{T}$ is a square matrix, $\mathbf{e}$ is a column vector of ones, and $\mathbf{0}$ is a column vector of zeros. Define $Y_i = \inf\{t : V(t) \in \Gamma_i\}$ $(i = 1, \cdots, n)$. Then the distribution of $(Y_1, \cdots, Y_n)$ is MPH.

Assaf et al. [1] showed that the MVE is MPH since each X_i is PH. It is of interest to give the standard MPH representation of the MVE, using the results of Assaf et al. [1].

To derive the standard MPH representation of the MVE, we specify explicitly the ingredients $E, \Gamma_1, \cdots, \Gamma_n, \mathbf{T}$ and β. The state-space is $E = \{1, \cdots, m, \Delta\}$, with 2^n elements $\mathbf{q} = (q_1, \cdots, q_n)$, where $q_i \in \{0, 1\}$. We have $\Gamma_i = \{\mathbf{q}, \Delta\}$ $(i = 1, \cdots, n)$, where $q_i = 0$, so that $\{\Delta\} = \bigcap_{i=1}^n \Gamma_i$. $\mathbf{T}$ has the following block patitioned structure:

$$\mathbf{T} = \begin{bmatrix} d_0 & \mathbf{B_1} & \mathbf{B_2} & \mathbf{B_3} & \cdots & \mathbf{B_k} & \cdots & \mathbf{B_l} & \cdots & \mathbf{B_{n-1}} \\ & \mathbf{D_1} & \mathbf{C_{12}} & \mathbf{C_{13}} & \cdots & \mathbf{C_{1k}} & \cdots & \mathbf{C_{1l}} & \cdots & \mathbf{C_{1n-1}} \\ & & \mathbf{D_2} & \mathbf{C_{23}} & \cdots & \mathbf{C_{2k}} & \cdots & \mathbf{C_{2l}} & \cdots & \mathbf{C_{2n-1}} \\ & & & \mathbf{D_3} & & & & & & \\ & & & & & \vdots & & \vdots & & \vdots \\ & & & & & \mathbf{D_k} & \cdots & \mathbf{C_{kl}} & \cdots & \mathbf{C_{kn-1}} \\ & & & & & & & \mathbf{D_l} & \cdots & \mathbf{C_{ln-1}} \\ & & & & & & & & & \vdots \\ & & & & & & & & & \mathbf{C_{n-2n-1}} \\ & & & & & & & & & \mathbf{D_{n-1}} \end{bmatrix} \tag{2}$$

where

$$d_0 = -\left(\sum_{i=1}^n \mu_i + \sum_{i<j}^n \mu_{ij} + \sum_{i<j<k}^n \mu_{ijk} + \cdots + \mu_{123\cdots n} \right)$$

and all the unmarked entries are zeros.

The submatrices are defined as below. The dimensionality of $\mathbf{B}$ is $1 \times \binom{n}{l}$ $(1 \le l \le n-1)$, $\mathbf{C}$ is $\binom{n}{k} \times \binom{n}{l}$ $(1 \le k \le n-2, 2 \le l \le n-1)$ and $\mathbf{D}$ is $\binom{n}{k} \times \binom{n}{k}$ $(1 \le k \le n-1)$. If $\mathbf{q} = (q_1, \cdots, q_n)$ and $\mathbf{r} = (r_1, \cdots, r_n)$ are two states in $E \setminus \{\Delta\}$, we denote by $b_{\mathbf{qr}}$, $c_{\mathbf{qr}}$ and $d_{\mathbf{qr}}$ the corresponding element of submatrices $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$, respectively. $b_{\mathbf{qr}}$, $c_{\mathbf{qr}}$ and $d_{\mathbf{qr}}$ will be zero unless one of the following holds:

(i) $r_{i_1} = r_{i_2} = \cdots = r_{i_u} = 0$ $(1 \le u \le l)$.
Then

$$b_{\mathbf{qr}} = \mu_{i_1 i_2 \cdots i_l} \quad (1 \le l \le n-1).$$

(ii) there is a u such that $q_{i_u} = r_{i_u} = 0$ $(1 \le u \le k)$, and $q_{j_v} \ne r_{j_v}, r_{j_v} = 0$ $(1 \le v \le l-k)$.
Then

$$c_{\mathbf{qr}} = \mu_{j_1 \cdots j_{l-k}} + \sum_{u=1}^{k} \mu_{i_u j_1 \cdots j_{l-k}} + \mu_{i_1 \cdots i_k j_1 \cdots j_{l-k}} \quad (1 \le k \le n-2, 2 \le l \le n-1).$$

(iii) there is a u such that $q_{i_u} = r_{i_u} = 0$ $(1 \le u \le k)$.

Then

$$d_{\mathbf{qr}} = d_0 + \sum_{u=1}^{k} \mu_{i_u} + \sum_{u<v}^{k} \mu_{i_u i_v} + \cdots + \mu_{i_1 \cdots i_k} \qquad (1 \le k \le n-1).$$

where $i_u \subset \{1, \cdots, n\}$ and $j_v \subset \{1, \cdots, n\}$.

To define β, let $\mathbf{Y} = (Y_1, \cdots, Y_n)$ be a random vector taking values in $\{1, \cdots, m\}$ and $p_{j_1 \cdots j_n} = P[Y_1 = j_1, \cdots, Y_n = j_n]$, where j_h ranges corresponding to the state $\mathbf{q}$, we have

$$\beta_{\mathbf{q}} = \begin{cases} p_{q_1 \cdots q_n} & \text{if } q_i \neq 0 \ (i = 1, \cdots, n), \\ 0 & \text{otherwise.} \end{cases}$$

This completes the specification of the standard MPH representation of the MVE of Marshall and Olkin [4].

Consider the simplest, bivariate case, where $n = 2$ and $m = 3$. Then the state-space consists of the four elements $(1,1), (1,0), (0,1)$ and $(0,0)$, where $\Gamma_1 = \{(0,1), (0,0)\}$, and $\Gamma_2 = \{(1,0), (0,0)\}$, so that $\Delta = (0,0)$. Then $\mathbf{T}$ is 3×3 matrix

$$\mathbf{T} = \begin{bmatrix} -(\mu_1 + \mu_2 + \mu_{12}) & \mu_2 & \mu_1 \\ 0 & -(\mu_1 + \mu_{12}) & 0 \\ 0 & 0 & -(\mu_2 + \mu_{12}) \end{bmatrix}. \tag{3}$$

The initial distribution may be written in the form $\beta = (1,0,0,0)$. From the results of Assaf et al. [1], $\overline{F}(t_1, t_2) = P(Y_1 > t_1, Y_2 > t_2)$ has the following closed form

$$\begin{aligned} \overline{F}(t_1, t_2) &= \alpha\, e^{\mathbf{T}\, t_2}\, \mathrm{g}_2\, e^{\mathbf{T}\, (t_1 - t_2)}\, \mathrm{g}_1\, \mathbf{e} && \text{if} \quad t_1 \ge t_2 \ge 0, \\ &= \alpha\, e^{\mathbf{T}\, t_1}\, \mathrm{g}_1\, e^{\mathbf{T}\, (t_2 - t_1)}\, \mathrm{g}_2\, \mathbf{e} && \text{if} \quad t_2 \ge t_1 \ge 0, \end{aligned} \tag{4}$$

where

$$\alpha = (1,0,0), \quad \mathrm{g}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathrm{g}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{e} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

They yield

$$\begin{aligned} \overline{F}(t_1, t_2) &= e^{-(\mu_1 + \mu_{12})\, t_1 - \mu_2 t_2} && \text{if} \quad t_1 \ge t_2 \ge 0, \\ &= e^{-\mu_1 t_1 - (\mu_2 + \mu_{12})\, t_2} && \text{if} \quad t_2 \ge t_1 \ge 0, \end{aligned} \tag{5}$$

consequently we get

$$\overline{F}(t_1, t_2) = e^{-\mu_1 t_1 - \mu_2 t_2 - \mu_{12} \max\, (t_1, t_2)}, \tag{6}$$

which gives the bivariate exponential distribution (BVE) of Marshall and Olkin [4].

3. Description of model

We consider a two-stage tandem queueing system with no intermediate buffers, in which each stage consists of two channels in parallel. Customers arrive at the queue, which is assumed to be infinite, according to a homogeneous Poisson process with rate λ. After completion of the first stage service, each customer can enter both of the second stage channels. Namely, when the second stage channel which is located on as the same line as his first stage channel is busy, he can enter another channel in the second stage if it is free. We call this operation *interchangeable*. On the other side, after completion of the first stage service, each customer can enter only the second stage channel which is located on as the same line as his first stage channel if it is free. We call this operation *ordinary*. If a customer has already completed service of two stages, then he departs the system. But if he has not completed service of the second stage and his second stage channel is not free, he has to stay there, that is to say, this channel is blocked, and when his second stage channel has completed service, he can enter that channel. In the interchangeable case, a blocking occurs only when both of the second stage channels are busy, whereas in the ordinary case, it always occurs when the second stage channel which is located on as the same line as his first stage channel is busy.

The service times of each two channels in the first and second stages are not independent but depend upon each other. It is assumed that their service distribution is the BVE given by (6). We call this model *correlated*. On the other side, we call a model in which their service distribution is an usual exponential distribution *independent*.

It is assumed that customers can transfer between channels instantaneously. Service to a customer at any channel, once initiated, is completed without interruptions. The queueing discipline is first-come first-served.

As shown in Fig. 1, the model I is interchangeable and correlated. We denote the throughput of this system by $\lambda^*_{I,C}$. Similarly, the model II, III and IV are interchangeable and independent, ordinary and correlated and ordinary and independent, respectively. We denote the throughput of the model II, III and IV by $\lambda^*_{I,I}$, $\lambda^*_{O,C}$ and $\lambda^*_{O,I}$, respectively.

4. Analysis of model I

We give numbers 1, 2, 3 and 4 to channels in the first and second stage as shown in Fig. 1. The joint service distribution of channel 1 and 2 and channel 3 and 4 are respectively given as follows:

$$\begin{aligned} \overline{F_1}(t_1,t_2) &= e^{-\mu_1\ (t_1+t_2)-\alpha\ \max\ (t_1,t_2)}, \\ \overline{F_2}(t_3,t_4) &= e^{-\mu_2\ (t_3+t_4)-\alpha\ \max\ (t_3,t_4)}. \end{aligned} \tag{7}$$

The model I under consideration can be studied as a continuous time Markov chain with state-space $\{(0,j) : 1 \leq j \leq m_0\} \cup \{(i,j) : i \geq 0, 1 \leq j \leq m\}$. The state $(0,j)$ denotes that the system is in the boundary state and m_0 denotes the number of the boudary states. In the state (i,j), i denotes the number of customers in the queue, whereas j denotes the state of the network consisting of four channels and m denotes the number of states in the network.

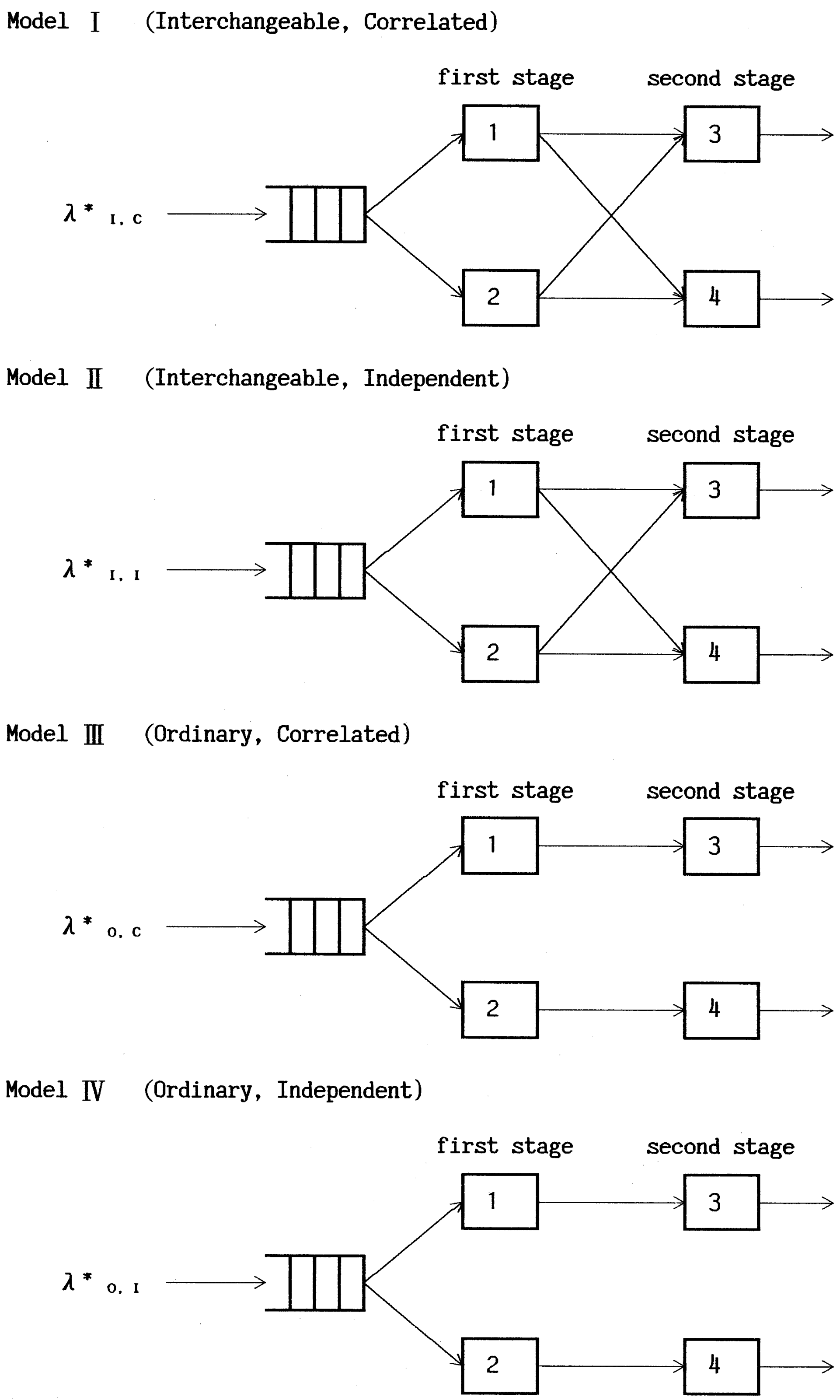
Model Ⅰ (Interchangeable, Correlated)
first stage
second stage
$\lambda^*_{I,C}$
1
2
3
4
Model Ⅱ (Interchangeable, Independent)
first stage
second stage
$\lambda^*_{I,I}$
1
2
3
4
Model Ⅲ (Ordinary, Correlated)
first stage
second stage
$\lambda^*_{O,C}$
1
2
3
4
Model Ⅳ (Ordinary, Independent)
first stage
second stage
$\lambda^*_{O,I}$
1
2
3
4

Fig. 1 Models

The states of the network are described by the vector:

$$(s_1, s_2, s_3, s_4)$$

where s_i $(i = 1, 2, 3, 4)$ can take any value from 0 to 2 with

$$s_i = \begin{cases} 0 & \text{-the } i\text{-th channel is idle,} \\ 1 & \text{-the } i\text{-th channel is in service,} \\ 2 & \text{-the } i\text{-th channel is blocked.} \end{cases}$$

Then the boundary states are

$$\begin{gathered}(0,0,0,0),(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1),\\(0,0,1,1),(1,0,1,0),(1,0,0,1),(0,1,1,0),(0,1,0,1),\\(1,0,1,1),(0,1,1,1),(0,2,1,1),(2,0,1,1).\end{gathered}$$

The states of the network are

$$\begin{gathered}(1,1,0,0),(1,1,1,0),(1,1,0,1),(1,1,1,1),(2,1,1,1),\\(1,2,1,1),(2,2,1,1).\end{gathered}$$

By ordering the states as described above, the infinitesimal generator of the continuous time Markov chain has the following block patitioned structure:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{A_{01}} & \mathbf{A_{04}} & & & & \\ \mathbf{A_{02}} & \mathbf{A_1} & \mathbf{A_0} & & & \\ \mathbf{A_{03}} & \mathbf{A_2} & \mathbf{A_1} & \mathbf{A_0} & & \\ & \mathbf{A_3} & \mathbf{A_2} & \mathbf{A_1} & \mathbf{A_0} & \\ & & \mathbf{A_3} & \mathbf{A_2} & \mathbf{A_1} & \mathbf{A_0} \\ & & & \mathbf{A_3} & \mathbf{A_2} & \mathbf{A_1} \\ & & & & & \vdots & \ddots \end{bmatrix} \tag{8}$$

where all the unmarked entries are zeros.

The dimensionality of $\mathbf{A_{01}}$ is 14×14, $\mathbf{A_{02}}$ and $\mathbf{A_{03}}$ are 7×14, $\mathbf{A_{04}}$ is 14×7, $\mathbf{A_0}$, $\mathbf{A_1}$, $\mathbf{A_2}$ and $\mathbf{A_3}$ are 7×7.

Let $\mathbf{p}$ equal the steady state probabilities of the network, assuming that the queue is never empty, which has elements $p(j)$ $(1 \le j \le 7)$. We can determine $\mathbf{p}$, by solving the system

$$\mathbf{pA} = \mathbf{0}, \qquad \mathbf{pe} = 1 \tag{9}$$

where the conservative matrix given by

$$\mathbf{A} = \mathbf{A_0} + \mathbf{A_1} + \mathbf{A_2} + \mathbf{A_3} \tag{10}$$

and $\mathbf{e}$ is (7×1) column vector with all elements equal 1. The equilibrium condition is given (see Neuts [6]) by

$$\mathbf{pA_0e} < \mathbf{p}\left(\mathbf{A_2} + 2\mathbf{A_3}\right)\mathbf{e}. \tag{11}$$

From this relationship the critical input rate, λ^*, to the system can be determined. In the steady state, this critical input rate is identical to the maximum throughput rate of the system.

The conservative matrix of the model I becomes

$$\mathbf{A} = \begin{bmatrix} -a_1 & 2\mu_1 & 0 & \alpha & 0 & 0 & 0 \\ a_2-\mu_2 & -b_1 & 0 & 2\mu_1 & 0 & \alpha & 0 \\ a_2-\mu_2 & 0 & -b_1 & 2\mu_1 & 0 & \alpha & 0 \\ \alpha & \mu_2 & \mu_2 & -b_1-\mu_2 & \mu_1 & \mu_1 & \alpha \\ 0 & \alpha & 0 & 2\mu_2 & -b_2 & 0 & a_1-\mu_1 \\ 0 & \alpha & 0 & 2\mu_2 & 0 & -b_2 & a_1-\mu_1 \\ 0 & 0 & 0 & \alpha & \mu_2 & \mu_2 & -a_2 \end{bmatrix} \tag{12}$$

where

$$\begin{aligned} a_1 &= 2\mu_1 + \alpha, \\ a_2 &= 2\mu_2 + \alpha, \\ b_1 &= 2\mu_1 + \mu_2 + 2\alpha, \\ b_2 &= \mu_1 + 2\mu_2 + 2\alpha. \end{aligned}$$

When $\mu_1 = \mu_2 = \mu$ and $\theta = \alpha/\mu$, the equilibrium condition (11) in this case is given by

$$\lambda < \mu\,(2+\theta)\,[\,1-p(4)\,] + \mu\,\theta\,[\,p(1)+p(7)\,]. \tag{13}$$

By solving the system (9) and substituting $p(j)$ into (13), if $\mu = 1$, we obtain

$$\lambda < \frac{4(1+\theta)^2(6+\theta)}{16+21\theta+3\theta^2} = \lambda^*_{I,C}. \tag{14}$$

When $\theta = 0$, that is, the service times of each two channels in the first and second stage are independent, the maximum throughput rate (14) yields

$$\lambda^*_{I,I} = \frac{3}{2}, \tag{15}$$

which gives the maximum throughput rate of the model II.

5. Analysis of model III

As additional boundary states, the states $(2,0,1,0)$ and $(0,2,0,1)$ are combined with the boundary states of the model I. Similarly, the states $(2,1,1,0)$ and $(1,2,0,1)$ are combined with the states of the network of the model I. The infinitesimal generator of the continuous time Markov chain has the same block patitioned structure as that of the model I. The dimensionality of $\mathbf{A_{01}}$ is 16×16, $\mathbf{A_{02}}$ and $\mathbf{A_{03}}$ are 9×16, $\mathbf{A_{04}}$ is 16×9, $\mathbf{A_0}$, $\mathbf{A_1}$, $\mathbf{A_2}$ and $\mathbf{A_3}$ are 9×9.

Let $\mathbf{p}$ equal the steady state probabilities of the network, assuming that the queue is never empty, which has elements $p(j)$ $(1 \le j \le 9)$. The relationship (9), (10) and (11) also hold for the model III.

The conservative matrix of the model III becomes
$\mathbf{A} =$

$$
\begin{bmatrix}
-a_1 & \mu_1 & \mu_1 & \alpha & 0 & 0 & 0 & 0 & 0 \\
a_2-\mu_2 & -b_1 & 0 & \mu_1 & \mu_1 & 0 & \alpha & 0 & 0 \\
a_2-\mu_2 & 0 & -b_1 & \mu_1 & 0 & \mu_1 & 0 & \alpha & 0 \\
\alpha & \mu_2 & \mu_2 & -b_1-\mu_2 & 0 & 0 & \mu_1 & \mu_1 & \alpha \\
0 & a_2-\mu_2 & 0 & 0 & -b_{12} & 0 & a_1-\mu_1 & 0 & 0 \\
0 & 0 & a_2-\mu_2 & 0 & 0 & -b_{12} & 0 & a_1-\mu_1 & 0 \\
0 & \alpha & 0 & \mu_2 & \mu_2 & 0 & -b_2 & 0 & a_1-\mu_1 \\
0 & 0 & \alpha & \mu_2 & 0 & \mu_2 & 0 & -b_2 & a_1-\mu_1 \\
0 & 0 & 0 & \alpha & 0 & 0 & \mu_2 & \mu_2 & -a_2
\end{bmatrix}
\tag{16}
$$

where $\quad b_{12} = \mu_1 + \mu_2 + 2\alpha.$

When $\mu_1 = \mu_2 = \mu$ and $\theta = \alpha/\mu$, the equilibrium condition (11) in this case is given by

$$\lambda < \mu\,(1+\theta)\,[\,1 + p(1) + p(5) + p(6) + p(9) - p(4)\,]. \tag{17}$$

By solving the system (9) and substituting $p(j)$ into (17), if $\mu = 1$, we obtain

$$\lambda < \frac{4(1+\theta)}{3} = \lambda^*_{O,C}. \tag{18}$$

When $\theta = 0$, that is, the service times of each two channels in the first and second stage are independent, the maximum throughput rate (18) yields

$$\lambda^*_{O,I} = \frac{4}{3}, \tag{19}$$

which gives the maximum throughput rate of the model IV.

6. Comparison of throughput

If

$$\theta = \theta_0 = 0.125, \tag{20}$$

then

$$\lambda^*_{I,I} = \lambda^*_{O,C}.$$

Consequently, since both $\lambda^*_{I,C}$ and $\lambda^*_{O,C}$ are increasing function with respect to $\theta > 0$, we obtain the following result.

Theorem.

$$
\begin{aligned}
&\lambda^*_{O,I} < \lambda^*_{O,C} \le \lambda^*_{I,I} < \lambda^*_{I,C} \qquad && \textit{if} \quad 0 < \theta \le \theta_0, \\
&\lambda^*_{O,I} < \lambda^*_{I,I} < \lambda^*_{O,C} < \lambda^*_{I,C} \qquad && \textit{if} \quad \theta_0 < \theta.
\end{aligned}
\tag{21}
$$

The following explanation is derived from the above theorem. When the correlation parameter is small, an interchangeable parallel two-stage tandem queueing model with independent service times improves the throughput of the system better than an ordinary parallel two-stage tandem queueing model with correlated service times, by utilizing an empty channel. When the correlation parameter is large, an ordinary parallel two-stage tandem queueing model with correlated service times improves it better than an interchangeable parallel two-stage tandem queueing model with independent service times, by the frequent occurrence of the event that two channels finish service at the same time.

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