数理解析研究所講究録
949 巻 1996 年 118-127

Generic Weakly-Nonlinear Model Equations for Density Waves in Two-Phase Fluids

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Abstract

Whitham's linear theory of traffic flows, which is applicable also to void waves, is extended to include dispersion and nonlinearity. As a result we obtain some KdV-like model equations with non-conservative terms of novel form such as $\partial_T \partial_X \Psi$. The model equations are rigorously derived by means of an improved multiple-scale expansion similar to the Padé approximation. It is shown, numerically and analytically, that the novel terms incorporate not only linear dispersion relation but also some higher nonlinearity, which we call 'baseline effect'.

1 Introduction

Seemingly there is a certain kind of wave phenomenon which is commonly observed in several non-conservative systems subject to one-dimensional continuity equation. One of such observations has been known as void waves in two-phase flows [1, 2, 3]. The void waves represent the generic dynamical feature of two-fluid systems such as gas-powder mixture flows, bubbly liquid flows and gas-droplet flows. Since many kinds of flows of nearly uniform two-phase fluids are modeled by quite similar sets of equations [4], a universal discussion of two-fluid systems based on a generic model set of equations should be justified.

Recently notice has been taken of phenomenological resemblance between granular pipe flows and traffic flows [5, 6]. In granular pipe flows the presence of fluid (air, water etc.) is believed to be essential, so this is again a two-phase system. On the other hand, it was more than a decade ago that the behaviour of linearized waves in two-phase fluids was explained in terms of Whitham's "wave hierarchies", which were originally proposed in the context of traffic flows [7, 8, 2, 3]. On these evidences we may identify the wave evolutions in traffic flows with those in various systems of two-phase fluids.

In fact, Whitham's equations of traffic flow are quite similar to the governing equations of nearly uniform two-phase fluids. These governing equations consist of the continuity equation

$$\partial_t \phi + \partial_x (\phi v) = 0 \tag{1}$$

and the momentum balance equation

$$\partial_t (\phi v) + \partial_x \sum [\text{momentum flux terms}] = \sum [\text{non-conservative force terms}] \tag{2}$$

in which non-conservative force terms are dominant and nearly balanced among themselves. ($1 - \phi$ stands for so called void fraction.) The momentum balance equation, together with some constitutive equations, is rewritten in the form

$$F(\phi, v) + \epsilon F_1(\partial_x \phi, \partial_t v, \cdots) + \cdots = 0, \tag{3}$$

which is reduced to a velocity-density relation

$$F(\phi, v) = 0 \tag{4}$$

at the lowest order of approximation. For this reason eq. (3) may be called *velocity-density conjuncting equation.* In this paper we establish a procedure to deal with non-linear, non-conservative waves subject to eqs. (1) and (3) as an extension of Whitham's linear theory.

In Section 2 we relate our problem to Whitham's idea of *wave hierarchies.* Then we introduce nonlinearity to obtain a KdV-like equation

$$[\partial_T + \partial_X - \partial_T \partial_X^2]\Psi + \Psi \partial_X \Psi - \mu \Psi^2 \partial_X \Psi - \gamma \partial_X [\partial_T + \partial_X]\Psi = 0 \tag{5}$$

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with a novel type of non-conservative terms (the terms with γ). This equation is rigorously derived in Section 3 by means of an improved multiple-scale expansion method. Results of numerical simulations are shown in Section 4, both for the KdV-like equation and for an original set of the two-phase model equations. The properties of the new equation (5) are discussed in Section 5.

The authors believe that eq. (5) is ubiquitous, in the sense that it is common to void waves, traffic congestion waves and generally to waves in systems subject to the continuity equation (1) and the velocity-density conjuncting equation (3). Because the zero wave number mode plays an important role in these waves, the Ginzburg-Landau equation is not relevant. Our equation is rather related to the Benney equation [9, 10]. The main difference is that the latter explicitly adopts the fourth derivative, while the former avoids it and therefore is free from the artifacts caused by it.

2 Nonlinear Model Equations

2.1 Wave Hierarchies

To begin with, we review the idea of "wave hierarchies" after Whitham [7].

Let us think of a one-dimensional system governed by the continuity equation (1) and the velocity-density conjuncting equation (3). At the lowest order of approximation in regard to a smallness parameter ϵ, eq. (3) is reduced to the relation (4). Provided $F(\bar{\phi}, \bar{v}) = 0$ with constant $\bar{\phi}$ and $\bar{v}$, these equations are linearized as

$$[\partial_t + \bar{v}\partial_x]\psi + \bar{\phi}\partial_x w = 0, \quad A\psi - w = 0 \tag{6}$$

where $\phi = \bar{\phi} + \psi$, $v = \bar{v} + w$. Elimination of w yields a first-order linear hyperbolic equation

$$[\partial_t + a\partial_x]\psi = 0. \tag{7}$$

Proceeding to the higher order of approximation in regard to ϵ, we obtain by a similar procedure

$$[\partial_t + a\partial_x]\psi + \tau[\partial_t + b_1\partial_x][\partial_t + b_2\partial_x]\psi = 0 \tag{8}$$

with $\tau \sim \epsilon$ and $\tau > 0$. This postulation comes from the linearized form of (3),

$$\tau\partial_t w = -w + A\psi - B\partial_x\psi + \cdots, \tag{9}$$

where τ must be positive so that w will be "slaved" to ψ.

The first-order equation (7) should be a good approximation to the second-order equation (8) for time scales much longer than τ. Meanwhile under the second-order hyperbolic equation (8) signals propagate at finite speeds, namely at b_1 and b_2. Therefore, in order that the two levels of descriptions (7) and (8) shall be consistent, the *wave hierarchy condition*

$$b_1 \leq a \leq b_2 \tag{10}$$

must be satisfied; otherwise the initial value problem is ill-posed. The term *wave hierarchy* implies that the characteristics of the lower-order waves should be between the characteristics of the higher-order waves.

The criterion of well-posedness (10) is verified by substituting the elementary solution

$$\psi = \psi_0 \exp(\sigma t + ikx) \tag{11}$$

into eq. (8) and seeking the condition for the real part of σ to be non-positive for any real value of k. A straightforward calculation is possible, but it would be wiser to begin with obtaining the neutrally stable cases where $\sigma = -i\omega$ is purely imaginary. The quadratic equation for σ is then decoupled into two simple equations of real variables

$$(\omega - b_1 k)(\omega - b_2 k) = \omega - ak = 0, \tag{12}$$

which has a solution only when $a = b_1$ or $a = b_2$. Obviously this leads to the stability criterion of the form (10).

Without loss of generality we can set $b_1 = -b_2 = b$ and rewrite eq. (8) as

$$[\partial_t + a\partial_x]\psi + \tau[\partial_t^2 - b^2\partial_x]\psi = 0 \tag{13}$$

with a and b being positive constants with the dimension of velocity.

2.2 Extended Idea of Wave Hierarchies

If $a > b$ in eq. (13), the initial value problem is ill-posed, and leads to instability in such a way that the growth is faster for shorter wave length. This behaviour does not reflect the real one of the physical system described by the original set of eqs. (1) and (3). Evidently higher derivative terms in eq. (3) prevent the short wave modes to grow.

Typically we think of the "momentum diffusion term" (usually called *viscosity term*) which takes the form $\partial_x^2 w$ appearing in the right-hand side of eq. (9). Inclusion of this term modifies eq. (13) so that we may have

$$[\partial_t + a\partial_x]\psi + \tau[\partial_t^2 - b^2\partial_x]\psi - \lambda^2\partial_t\partial_x^2\psi = 0. \tag{14}$$

Equation (14) is divided into two parts as $\hat{L}_1\psi + \hat{L}_2\psi = 0$, so that both of the equations

$$\hat{L}_1\psi = [\partial_t + a\partial_x - \lambda^2\partial_t\partial_x^2]\psi = 0, \quad \hat{L}_2\psi = [\partial_t^2 - b^2\partial_x]\psi = 0 \tag{15}$$

admit only neutrally stable waves, i.e. only purely imaginary $\sigma = -i\omega$. Their velocities are $a/(1+\lambda^2k^2)$, $\pm b$. The first-order wave is now dispersive. The neutrally stable modes of eq. (14) is then easily obtained. The criterion for the mode k not to grow is written in the form

$$-b < \frac{a}{1+\lambda^2k^2} < +b \tag{16}$$

which is an extension of the condition (10) for the dispersive case.

The left inequality of the condition (16) always holds. The right inequality becomes invalid for small wave-number modes when $a > b$. Even in that case the range of k for growing modes is finite. The short waves always damp, so the initial value problem is well-posed in the sense that $\Re\sigma$ is bounded as $k \to \infty$ [11, 12].

2.3 Extension to Nonlinear Cases

When $a > b$ and therefore the small wave-number modes have positive growth rate, nonlinearity must be included to limit the wave growth. We apply the method of frozen coefficients which gives deep results for nonlinear problems cheaply [11].

We recall that a, b, λ and τ in eq. (14) may all depend on $\bar{\phi}$. This is true when $\bar{\phi}$ is constant. We assume that locally eq. (14) still holds when $\bar{\phi}$ varies slowly in space and time. Then we have

$$[\partial_t + a(\bar{\phi})\partial_x]\phi + \tau(\bar{\phi})[\partial_t^2 - b(\bar{\phi})^2\partial_x]\phi - \lambda(\bar{\phi})^2\partial_t\partial_x^2\phi = 0 \tag{17}$$

with $\phi = \bar{\phi} + \psi$. As ψ is small, $\bar{\phi}$ in (17) may be replaced by ϕ.

Interested in the appearance of the growing modes, we define ϕ^* by the critical condition $a(\phi^*) = b(\phi^*)$, around which we perform an expansion

$$a = a(\bar{\phi}) = a_0 + a_1 \cdot (\bar{\phi} - \phi^*) + a_2 \cdot (\bar{\phi} - \phi^*)^2 + \cdots \tag{18}$$

and similarly for b, λ and τ. The dominance of long wave modes suggests, however, that the coefficients of the higher derivative terms are less influential to the behaviour of eq. (17). Therefore it may be allowed, at least in a heuristic discussion, to regard b, λ and τ as constants. We adopt only the expansion (18) of a, which is substituted into eq. (14) with $b = \text{const.} = a_0$ (by definition of ϕ^*). The unidirectionality leads to the rewriting

$$\tau[\partial_t^2 - b^2\partial_x]\psi = \tau(\partial_t - b\partial_x)(\partial_t + b\partial_x)\psi \simeq -2\tau a_0\partial_x(\partial_t + a_0\partial_x)\psi, \tag{19}$$

because for long waves $\partial_t\psi \simeq -a_0\partial_x\psi$ at the lowest approximation. By suitable rescaling we obtain a new weakly-nonlinear equation

$$[\partial_T + \partial_X - \partial_T\partial_X^2]\Psi + \Psi\partial_X\Psi - \gamma\partial_X[\partial_T + \partial_X]\Psi = 0 \tag{20}$$

with $\Psi \propto \phi - \phi^*$, when a is expanded till a_1. Later we will show that inclusion of a_2 is indispensable. This inclusion yields an MKdV-term $-\mu\Psi^2\partial_X\Psi$ so that we arrive at eq. (5).

In analogy to (15), eq. (20) is divided into two parts as $\hat{M}\Psi + \hat{N}\Psi = 0$ where

$$\hat{M}\Psi = [\partial_T + \partial_X - \partial_T\partial_X^2]\Psi + \Psi\partial_X\Psi, \quad \hat{N}\Psi = -\gamma\partial_X[\partial_T + \partial_X]\Psi. \tag{21}$$

Each operator corresponds to a wave equation whose solution can travel in a constant shape, not growing nor damping. If these two equations have a common solution traveling with a common velocity c, eq. (20) also admits the steady traveling solution. A family of cnoidal wave solutions

$$\Psi = \frac{12}{l^2}\left[m^2\,\mathrm{cn}^2(\frac{x-ct}{l}, m) + \frac{1}{3}(1-2m^2)\right], \quad c = 1 \tag{22}$$

is found to meet this demand. Later we will show that the condition $c = 1$ is not only sufficient but also necessary for admitting steady traveling solutions to equations such as (5) or (20).

3 Rigorous Derivation of Model Equations

3.1 Improved Multiple-Scale Expansion Method

The newly derived equation (5) has terms of novel form, such as $\partial_T \partial_X \Psi$ and $\partial_T \partial_X^2 \Psi$. The latter has been known in the Regularized Long-Wave equation [13, 14]. The merit of such terms has been thought to be an improved expression of the linear dispersion relation. In Section 5 we will show, however, that also some part of the higher-order nonlinear effect is expressed by these terms.

However, it may be questionable whether the nonlinearity to the degree both sufficient and necessary is included or not. The heuristic derivation is not free from the suspicion that approximations are arbitrary and maybe inconsistent with each other. Evidently we must resort to systematic and justifiable analysis. We propose an improved method of multiple-scale expansion, which, fortunately, legitimates eq. (5).

Before describing the new expansion method, we would like to clarify why the usual reductive perturbation expansion is not good enough. Let us follow the usual method in multiple-scale notation. The Gardner-Morikawa transform $\partial_t = \epsilon\partial_{t_1} + \epsilon^3\partial_{t_3}$, $\partial_x = \epsilon\partial_{x_1}$, $\partial_{t_1} = -c\partial_{x_1}$ and scaling of the far-field variables $\psi_{\rm wave} \sim w_{\rm wave} \sim \epsilon^2$ yield the KdV equation at the fifth order of ϵ. In the next order ϵ^6, the Benney equation with an additional non-conservative term $\partial_x^2(\psi^2)$ is obtained [10]. The additional term is necessary in order to describe the influence of the "baseline" mode upon the emergence of positive growth. This nonlinear destabilizing term, however, cannot be balanced till we proceed to ϵ^8 to pick up $\partial_x^2(\psi^3)$ and $\partial_x^4(\psi^2)$. Such a high-order expansion would be ridiculous, because there would be too many terms and no guarantee of convergence.

It is possible, however, to obtain a less intractable equation. In principle we can perform the expansion procedure till the eighth order of ϵ, and then put some higher terms together into the form $\partial_t\partial_x\phi$, $\partial_t\partial_x^2\phi$ etc. reducing the number of terms. Practically, the tedious expansion procedure is skipped by the following technique. We define a linear differential operator

$$\hat{L} = 1 + A^{(1)}\partial_x + A^{(2)}\partial_x^2 + \cdots \tag{23}$$

with adjustable constants $A^{(j)}$. Then a "distorted" time derivative

$$\partial_s = \hat{L}\partial_t \tag{24}$$

is introduced. Multiple-scale expansion is performed in regard to (∂_s, ∂_x) instead of (∂_t, ∂_x). The adjustable parameters are defined so that higher-order terms may vanish. This procedure is an operator analogue of the Padé approximation.

3.2 Calculation Procedure of New Expansion Method

Suppose that an explicit form of the velocity-density conjuncting equation (3) is given. For concreteness we assume the following form:

$$\mathcal{R}[\partial_t + v\partial_x]v = (V_{\rm ex} - v)I(\phi) - 1 - \mathcal{R}\mathcal{M}^{-2}\partial_x P(\phi) + \partial_x^2\phi. \tag{25}$$

which is just a rewriting of the generic model equation proposed by Kawahara [4]. We express $I(\phi)$ and $P(\phi)$ as expansions around some $\phi_0 = \text{const.}$ for later convenience:

$$I(\phi) = V_0^{-2}\left[V_0 + a(\psi/\phi_0) + \alpha(\psi/\phi_0)^2 + \alpha^{(3)}(\psi/\phi_0)^3 + \cdots\right], \tag{26}$$

$$\mathcal{M}^{-2}P(\phi) = \left[b^2 + \beta(\psi/\phi_0) + \cdots\right]\phi_0^{-1}\partial_x\psi \tag{27}$$

where $\psi = \phi - \phi_0$. Note that the expansion coefficients depend on ϕ_0.

If we linearize the governing set of equations (1) and (25) around the uniform state $(\phi, v) = (\phi_0, 0)$ with $V_{\rm ex} = V_0$, we obtain (14) together with the coefficients $\tau = \mathcal{R}V_0$, $\lambda^2 = V_0$, finding a and b to be given by the expansions (26) and (27). By assuming an elementary solution (11), $\sigma = \sigma_\pm$ is given in an explicit form. The real part of σ_- is always negative, while that of σ_+ can be positive when $a > b$. Interested in the emergence of positive $\Re\sigma_+$, we set $\phi_0 = \phi^*$ so that $a = b$. At this point (2,2)-Padé approximant to σ_+ is calculated as

$$\sigma_+\big|_{a=b} \simeq \frac{-iak - 2a^2\mathcal{R}V_0k^2}{1 - 2ia\mathcal{R}V_0k + V_0k^2}. \tag{28}$$

Let us formulate the long-wave expansion. The differential operators and the variables are expanded as

$$\partial_s = \epsilon\partial_{s_1} + \epsilon^2\partial_{s_2} + \epsilon^3\partial_{s_3} + \cdots, \tag{29}$$

$$\partial_x = \epsilon\partial_{x_1} + \epsilon^2\partial_{x_2} + \epsilon^3\partial_{x_3} + \cdots, \tag{30}$$

$$\phi = \phi_0 + \psi = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \epsilon^3\phi_3 + \cdots, \tag{31}$$

$$v = \epsilon v_1 + \epsilon^2 v_2 + \epsilon^3 v_3 + \cdots, \tag{32}$$

where ϕ_1 and v_1 are independent of s_1, s_2, x_1, x_2. This means that $\phi_0 + \epsilon\phi_1$ varies so slowly that $\partial_x(\phi_0 + \epsilon\phi_1) \sim \epsilon^4 \partial_{x_3}\phi_1$ is negligible in comparison with $\partial_x\phi \sim \epsilon^3\partial_{x_1}\phi_2$. These ϵ^1-order variables are introduced so that higher order nonlinear terms, such as $\phi_1\partial_{x_1}^2\phi_2$ and $\phi_1^2\partial_{x_1}\phi_2$, will appear at the same order as $\partial_{x_1}^3\phi_2$ and $\phi_2\partial_{x_1}\phi_2$. Due to (25), (26) and (32), the control parameter $V_{\rm ex}$ should be proximate to V_0, so we write $V_{\rm ex} = V_0 + \epsilon V_1$.

The adjustable constants in $\hat{L}$ should be determined after all the calculation, but provisionally we set

$$\hat{L} = 1 - 2\mathcal{R}Va\partial_x - V\partial_x^2 \tag{33}$$

in accordance with the denominator in the Padé approximant (28). The governing equations are now rewritten as

$$\partial_s\phi + \hat{L}\partial_x(\phi v) = 0, \tag{34}$$

$$\mathcal{R}\partial_s v = \hat{L}\left\{-\mathcal{R}\partial_x(v^2/2) + (V_{\rm ex} - v)I(\phi) - 1 - \mathcal{R}\mathcal{M}^{-2}\partial_x P(\phi) + \partial_x^2\phi\right\}, \tag{35}$$

into which we substitute (29), (30), (31) and (32).

At the first and the second order of ϵ we obtain

$$v_1 = V_1 + a\phi_1/\phi_0, \tag{36}$$

$$v_2 = a\phi_2/\phi_0 + a^{(2)}(\phi_1/\phi_0)^2, \tag{37}$$

where $a^{(2)} = \alpha - a^2/V_0$. The next order ϵ^3 yields

$$[\partial_{s_1} + a\partial_{x_1}]\phi_2 = 0, \tag{38}$$

$$v_3 = a\phi_3/\phi_0 + 2a^{(2)}\phi_1\phi_2/\phi_0^2 + a^{(3)}(\phi_1/\phi_0)^3 \tag{39}$$

with $a^{(3)} = \alpha^{(3)} - 2a\alpha/V_0 + a^3/V_0^2$. Hereafter $\partial_{s_1} + a\partial_{x_1}$ is always equated to zero, which is just the Gardner-Morikawa transform.

At the fourth order we use a secular condition for ϕ_2, noting that ϕ_1 is independent of x_1, x_2, s_1 and s_2. Then we obtain

$$[\partial_{s_2} + a\partial_{x_2} + V_1\partial_{x_1}]\phi_2 + 2(a + a^{(2)})(\phi_1/\phi_0)\partial_{x_1}\phi_2 - 2a^2\mathcal{R}V_0\partial_{x_1}^2\phi_2 = 0, \tag{40}$$

$$[\partial_{s_3} + a\partial_{x_3}]\phi_1 = 0, \tag{41}$$

$$v_4 = a\phi_4/\phi_0 + a^{(2)}(2\phi_1\phi_3 + \phi_2^2)/\phi_0^2 + 3a^{(3)}\phi_1^2\phi_2/\phi_0^3 + a^{(4)}(\phi_1/\phi_0)^4 + \mathcal{R}V_0[4aa^{(2)} + a^2 - \beta]\phi_0^{-2}\phi_1\partial_{x_1}\phi_2. \tag{42}$$

The constant $a^{(4)}$ is composed of $\alpha^{(4)}$, $\alpha^{(3)}$, α, a and V_0.

We then move on to the fifth order to collect all that is needed. The result is

$$\begin{aligned}
&[\partial_{s_2} + a\partial_{x_2} + V_1\partial_{x_1}]\phi_3 + [\partial_{s_3} + a\partial_{x_3} + V_1\partial_{x_2}]\phi_2 + [\partial_{s_4} + a\partial_{x_4} + V_1\partial_{x_3}]\phi_1 \\
&+2(a + a^{(2)})\phi_0^{-1}\phi_1\partial_{x_1}\phi_3 + 2(a + a^{(2)})\phi_0^{-1}\phi_1\partial_{x_2}\phi_2 + 2(a + a^{(2)})\phi_0^{-1}\phi_1\partial_{x_3}\phi_1 + 3(a^{(2)} + a^{(3)})\phi_0^{-2}\phi_1^2\partial_{x_1}\phi_2 \\
&-\mathcal{R}V_0(3a^2 + \beta)\phi_0^{-1}\phi_1\partial_{x_1}^2\phi_2 - 2a\mathcal{R}V_0\left[a\partial_{x_1}^2\phi_3 + 2a\partial_{x_1}\partial_{x_2}\phi_2 + V_1\partial_{x_1}^2\phi_2\right] = 0.
\end{aligned} \tag{43}$$

Equation (43), combined with (38) and (40), can be rewritten as

$$\begin{aligned}
&[\partial_s + (a + \Delta V)\partial_x]\psi + (a + a^{(2)})\phi_0^{-1}\partial_x[\psi^2] + (a^{(2)} + a^{(3)})\phi_0^{-2}\partial_x[\psi^3] \\
&-2a\mathcal{R}V_0(a + \Delta V)\partial_x^2\psi - (3a^2 + \beta)\mathcal{R}V_0\phi_0^{-1}\partial_x^2[\psi^2/2] = o(\epsilon^5),
\end{aligned} \tag{44}$$

where

$$\partial_s = [1 - 2\mathcal{R}V_0 a\partial_x - V_0\partial_x^2]\partial_t, \tag{45}$$

$$\Delta V = V_{\rm ex} - V_0 = \epsilon V_1, \tag{46}$$

$$\psi = \phi - \phi_0 = \epsilon\phi_1 + \epsilon^2\phi_2 + \epsilon^3\phi_3 + \epsilon^4\phi_4 + O(\epsilon^5). \tag{47}$$

When the boundary condition allows Galilei transform, ΔV can be set equal to zero without loss of generality. Otherwise ΔV can be approximately cancelled out by an origin shift of ψ. By suitable rescaling of variables we obtain

$$[\partial_T + \partial_X - \partial_T\partial_X^2]\Psi + \Psi\partial_X\Psi - \mu\Psi^2\partial_X\Psi - \gamma'\partial_X[\partial_T + \partial_X]\Psi - \delta\partial_X^2[\Psi^2/2] = 0, \tag{48}$$

where γ' and μ are positive constants. But we have not yet reached the goal. By substituting $\Psi = \Psi_b + \epsilon\exp[\sigma T + ikX]$ with $\epsilon \ll 1$, eq. (48) leads to

$$\sigma = \frac{-i(1 + \Psi_b - \mu\Psi_b^2)k - (\gamma' + \delta\Psi_b)k^2}{1 - i\gamma' k + k^2} \rightarrow -(\gamma' + \delta\Psi_b) \qquad (k \rightarrow +\infty) \tag{49}$$

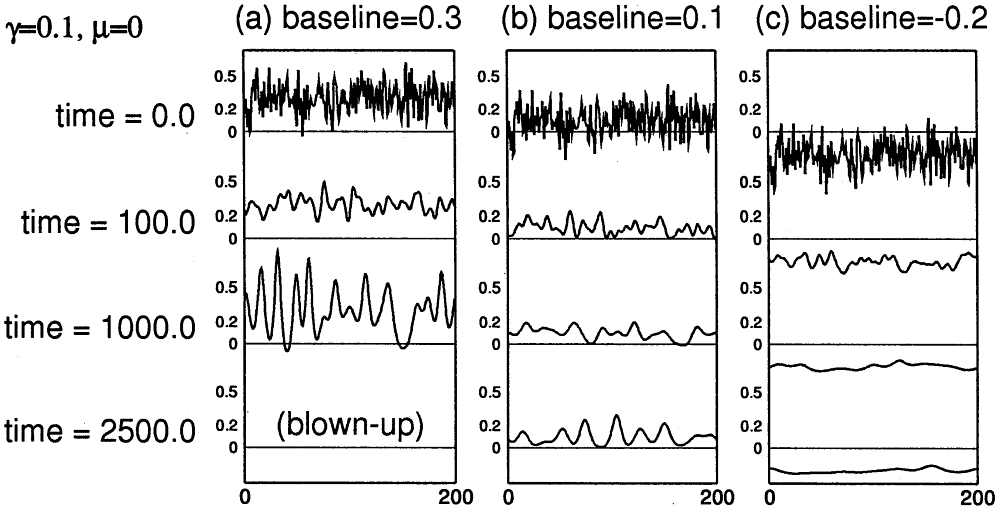

Figure 1: Time evolutions under eq. (20) with different baseline level.

and, alas, meet with a true ill-posedness for some values of Ψ_b. This difficulty is due to the term $\partial_X^2[\Psi^2/2]$, which is "regularized" by noting that

$$\begin{aligned} -\partial_X^2[\Psi^2/2] &= -\partial_X[\Psi\partial_X\Psi] \\ &= \partial_X\left[(\partial_T+\partial_X-\partial_X^2\partial_T)\Psi+O(\epsilon^5)\right] \\ &= \partial_X[\partial_T+\partial_X]\Psi+O(\epsilon^6). \end{aligned} \tag{50}$$

Setting $\gamma=\gamma'-\delta$, finally we obtain (5). This "regularization" is equivalent to setting

$$\hat{L}=1-\left(2a-\frac{3a^2+\beta}{a+a^{(2)}}\right)\mathcal{R}V\partial_x-V\partial_x^2. \tag{51}$$

4 Numerical Simulations

4.1 Description of Numerical Simulations

Initial value problems are numerically solved under the periodic boundary condition, both for the reduced equation (5) and for the original set of model equations (1) and (25). For both cases, the pseudo-spectral method by Fourier expansion is adopted. Time integration is performed by the 4-th order Runge-Kutta method. The adequacy of the numerical scheme, time step and mode number was checked by running solutions expected to travel in constant shapes. Such solutions (steady traveling solutions) can be obtained as eigensolutions, numerically or maybe analytically.

4.2 Dynamics of Reduced Equation

The newly derived equation (5) involves eq. (20) as a special case where $\mu=0$. Let us begin with this case.

In figure 1 three runs (a), (b) and (c) are compared. The parameters are common: $\gamma=0.1$, $\mu=0$. Also the initial data (of white-noise spectrum) are the same except for the zeroth Fourier mode ("baseline"). The baseline levels for (a), (b) and (c) are set at 0.3, 0.1 and -0.2, respectively.

In every case the highest modes rapidly damp away. The lower modes survive to form a rather irregular wave train. In the case (a) each peak in this irregular wave train tends to grow higher under the constraint of mass conservation. Finally the highest peaks are found to blow up due to self-focusing. On the contrary, the peaks in the case (c) are subject to diminution; all the structures seem to fade away till a uniform state. Something like a dispersive shock in small amplitude is observed at the final stage. The case (b) is intermediate. As far as $t<3000$, several peaks endure to the end, not blowing up nor damping away. We conclude that the zero-wavenumber mode is influential to the overall wave evolution. In this paper we call it "baseline effect".

The presence of positive μ suppresses the explosion of peaks, as is seen in figure 2. The long-time limit state is considered to be a separation into two levels.

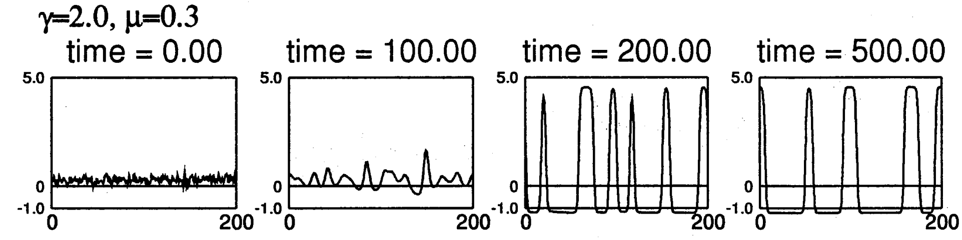

Figure 2: Peak growth saturation due to $\mu > 0$ in eq. (5).

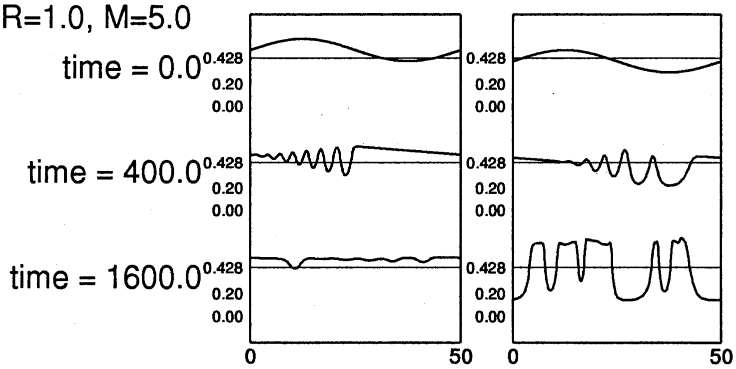

Figure 3: Fully nonlinear dynamics under eqs. (1) and (25).

4.3 Comparison with Original Dynamics

Some initial value problems for the set of eqs. (1) and (25) are solved numerically. Explicit form of I and P are assumed as $I(\phi) = (1-\phi)^{-1-m}$, $P(\phi) = (1-\phi)^{-n}$, with $m = 4$, $n = 1$. Then a and b are calculated explicitly, yielding $\phi^* = 0.428$.

Figure 3 shows the result of two runs for the same parameter values $\mathcal{R} = 1.0$, $\mathcal{M} = 5.0$. For both runs the initial condition for ϕ is given by a sinusoidal wave which is of the lowest mode and of the same amplitude 0.1. Only the baseline mode is different: 0.5 ($> \phi^*$) in (a) and 0.4 ($< \phi^*$) in (b). The initial condition for v (here given by a sinusoidal wave) is not important, because v soon becomes almost "slaved" to ϕ. For this reason we do not graph v in figure 3.

At the first stage of time evolution, both examples (a) and (b) show formation of pulses, seemingly due to the dispersion. In the case (a) the pulses damp, while in the case (b) they grow as long as the numerical scheme endures the amplitude of ϕ. The result of (b) is regarded as a separation into two phases of different density (i.e. of different void fraction).

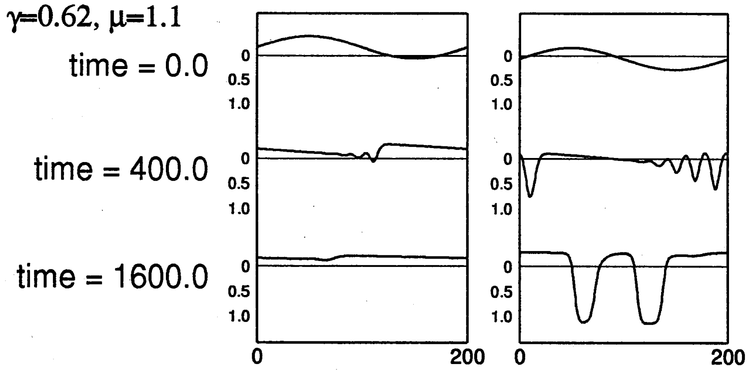

Figure 4: Time evolutions under eq. (5). The graphs are upside-down for comparison with Fig. 3.

Following the expansion recipe given in Section 3, we calculate the numerical setting for the reduced equation (5). When $\mathcal{R} = 1.0$, $\mathcal{M} = 5.0$, the following values are obtained:

$$\Psi = -2.325 \times (\phi - \phi^*), \quad \phi^* = 0.428, \quad dX/dx = 4.05, \quad dT/dt = 0.93,$$
$$\gamma = \gamma' - \delta = 0.113 - (-0.511) = 0.624, \quad \mu = 1.14.$$

We then perform numerical simulations of (5) under this setting, with initial conditions corresponding to those in figure 3. The result is seen in figure 4. Behaviour of the solutions is qualitatively reproduced, at least in regard to the pulse amplitude.

5 Discussion

5.1 The Significance of Novel Terms

The outstanding feature of the new equation (5), or (20) in a special case, is that it includes a term $\partial_T \partial_X \Psi$. This term seems to have been never considered before, at least in context of long wave model equations. As well as the term $\partial_T \partial_X^2 \Psi$, this term has two merits. On one hand it reproduces the linear σ-k relation for shorter waves. On the other hand it introduces a kind of higher-order nonlinear effect, which we call "baseline effect" in this paper.

Let us begin with the linear relation. For simplicity we set $\mu = 0$. When $\Psi = \Psi_0 \exp[\sigma t + ikx]$ is small, eq. (20) is linearized, yielding a (complex) dispersion relation

$$\sigma = \frac{-ik - \gamma k^2}{1 - i\gamma k + k^2}. \tag{52}$$

This is nothing other than a Padé approximant to the original dispersion relation under eq. (14). It reproduces the behaviour of σ not only for small values but also for large values of k. This is meaningful in the present case for two reasons. First, growth and damping lead to interaction between different scales, so we cannot limit ourselves to the long wave modes. Second, if description by pulse dynamics is possible, the tail structure of the pulses is important [16]; therefore the linear evanescent modes should be correctly expressed.

What seems more important is that the terms $\partial_T \partial_X \Psi$ and $\partial_T \partial_X^2 \Psi$ can incorporate nonlinearity. Suppose that

$$\Psi = \Psi_b + \epsilon \exp[\sigma T + ikX] \tag{53}$$

with constant Ψ_b. As $\epsilon \ll 1$ we obtain

$$\sigma = \frac{-i(1+\Psi_b)k - \gamma k^2}{1 - i\gamma k + k^2} = \begin{cases} -i(1+\Psi_b)k + \gamma \Psi_b k^2 + \cdots & \text{(for long waves)} \\ -\gamma + O(k^{-1}) & \text{(for short waves)} \end{cases} \tag{54}$$

to find that the sign of Ψ_b defines the sign of $\Re\sigma$ for long waves. The zero-wavenumber mode or "baseline mode" Ψ_b is influential through the implicit nonlinearity introduced by the novel terms. This is explained intuitively by recognizing that $\partial_T \simeq -(1+\Psi_b)\partial_X$ in (20) for long waves.

A similar discussion is possible for the case where $\mu > 0$. It is found that positive growth is confined within a finite range of baseline level, defined by the condition $\Psi_b - \mu \Psi_b^2 > 0$, which lies between two distinct stable ranges. Some numerical solutions of equation (5) show "separation" into these two stable states, while the solution of equation (20) for the same initial condition explodes within finite time due to self-focusing. This is the reason why the MKdV-term should be included. We should note that Komatsu and Sasa have obtained the MKdV equation as the lowest order model for traffic flows [6].

5.2 Steady Traveling Solutions

In many nonlinear systems steady traveling solutions play an important role. The triumph of the soliton is too famous to mention here. Pulse dynamics achieved remarkable success in several non-conservative, non-integrable systems, described by the Kuramoto-Sivashinsky equation, the Benney equation etc. [15, 16]

We can obtain a steady traveling pulse solution to eq. (5). By assuming that $\Psi = \Psi(Z)$ with $Z = X - cT$, we obtain an ordinary differential equation

$$(1-c)\partial_Z \Psi + c\partial_Z^3 \Psi + \Psi \partial_Z \Psi - \mu \Psi^2 \partial_Z \Psi - (1-c)\gamma \partial_Z^2 \Psi = 0, \tag{55}$$

which poses a nonlinear eigenvalue problem under the boundary condition $\Psi(z_{\min}) = \Psi(z_{\max}) = \Psi_b$. The "eigenvalue" c is easily determined as follows. Let us multiply (55) by Ψ and integrate with respect to Z. This leads to

$$(1-c)\gamma \int dZ \, (\partial_Z \Psi)^2 = 0 \tag{56}$$

by partial integration. Obviously $c = 1$ if Ψ is not trivial. Then the terms with γ completely cancels out each other, so that we obtain a family of exact solutions which travel in constant shape with $c = 1$. Especially, when $\mu = 0$, a family of cnoidal wave solutions (22) is obtained. Note that l can take any positive value if Ψ_b is given in accord.

5.3 Comparison with Other Models of Non-Conservative Waves

According to the linearized expression (54), the sign of Ψ_b determines the sign of $\Re\sigma$ for long waves. This is called "baseline effect". When $\Psi_b < 0$, the dynamics is similar to that of the KdV-Burgers equation. On the other hand, the dynamics for $\Psi_b > 0$ resembles that of the Benney equation in the existence of positive growth in long-wave region. Weakly nonlinear analysis of waves in two-phase fluids has been yielded either the KdV-Burgers equation or the Benney equation, depending on the setting. We may say that our equation unifies these two cases.

The Benney equation and the Kuramoto-Sivashinsky equation involve an intrinsic length, determined by the coefficients of $\partial_X^2 \Psi$ and $\partial_X^4 \Psi$. This length, defining the width of the steady pulse solution, seems to be influential to the time evolution, though it is a little modified due to nonlinearity. On the contrary eq. (5) does not exhibit such a finite intrinsic length, as is clear from (22) or (54). The presence of intrinsic length, independent of Ψ, is thought to be an artifact as far as waves in two-phase fluids or traffic flows are concerned, because under the set of eqs. (1) and (25) wave length seems to have no limit.

Due to the lack of intrinsic wave length at criticality, we cannot apply the (time-dependent) Ginzburg-Landau equation, except when a finite wave length is supplied through the initial condition. Such a case is numerically tested. Something like a finite-amplitude analogue of the modulational instability is observed, which accords with the presence of an inflection point in $\Im\sigma$ in (54).

5.4 Implicit Inclusion of Higher-Order Terms by the New Expansion

By introducing $\hat{L}$ we included infinite number of linear and nonlinear terms. The inclusion of linear terms is understood as a straightforward extension of the Padé approximation. The inclusion of nonlinearity must be checked by expanding up to such a high order that overlooked nonlinear terms, if any, can be gleaned. We can either begin the expansion of ϕ by the first order of ϵ and calculate up to ϵ^5, or begin ϕ by the second order and calculate up to ϵ^8. In this paper we adopted the first strategy.

Formally we can operate an inverse of $\hat{L}' = 1 - \gamma\partial_X + \partial_X^2$ upon eq. (5), to rewrite it as

$$\partial_T \Psi + \left[1 + \gamma\partial_X + (\gamma^2 - 1)\partial_X^2 + \cdots\right] \left[\partial_X \Psi + \Psi\partial_X\Psi - \mu\Psi^2\partial_X\Psi - \gamma\partial_X^2\Psi\right] = 0. \tag{57}$$

Thus we return to the Benney equation with many higher-order terms. However, the expansion of $\hat{L}'^{-1}$ is not guaranteed to converge. It may be conjectured that our method realizes a kind of non-convergent summation, as a generalization of the Padé approximation.

6 Conclusion

We have derived a new weakly-nonlinear model equation (5) which describes generic behaviour of density waves subject to the continuity equation (1) and the velocity-density conjuncting equation (3). At first we found eq. (5) by extending Whitham's idea of "wave hierarchies" to include the dispersion and the nonlinearity. The nonlinearity was incorporated by means of the frozen coefficient method, whose validity should be due to the slow variation of the variables. This idea could be formulated by multiple-scale expansion, but in order to include sufficient degree of nonlinearity, it was necessary to improve the expansion method in a way analogous to the Padé approximation. Numerical simulation of initial value problems, both for the fully-nonlinear set of equation and for the weakly-nonlinear model equation, revealed that the model equation is capable of describing behaviours such as pulse formation, baseline effect, growth saturation and even something like the modulational instability.

The baseline effect is the outstanding feature of our model equation. It is, roughly speaking, a triangle interaction of $(0, k, k)$ (not necessarily conservative). It should be noted that whether there are growing modes or not in eq.

(5) depends on the initial condition, and is not determined solely by the equation itself. In this sense our model equation unifies the KdV-Burgers equation and the Benney equation, as a model describing the same system with different initial conditions.

Some part of the algebraic calculations were performed with REDUCE at the Kyoto University Data Processing Center. The authors would like to thank Prof. Sekimito at Yukawa Institute and Mr. Ichiki at Tohoku University for some fruitful discussions and a wonderful demonstration of granular pipe flow. It was during discussion with the Nagoya group [17] on traffic flows when the idea of the MKdV term came to one of the authors. We thank also Prof. Toh, Ms. Suzuki, Mr. Iima and Mr. Goto for valuable discussions and suggestions.

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