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Shape Optimization Problem on the Lateral Boundary for Thermodynamical Phase Separation

角谷 敦
Atsushi KADOYA (Hiroshima Shudo University)

1. Formulation of an optimization problem

This paper is concerned with an optimization problem on the lateral boundary $\partial\Omega$ for a thermodynamical phase separation model formulated in a domain Ω.

Ω is a bounded domain in $\mathbb{R}^N$ ($N=2$ or 3) with smooth boundary $\partial\Omega$ and T is a fixed positive number. Our state problem $SP(\Gamma)$ is of the form

$$\begin{cases} \rho(u)_t+\lambda(w)_t-\Delta u=f \quad \text{in } Q:=(0,T)\times\Omega, \\ w_t-\Delta\{-\mu\Delta w_t-\kappa\Delta w+\xi+g(w)-\lambda'(w)u\}=0 \quad \text{in } Q, \\ \xi\in\beta(w) \quad \text{in } Q, \\ u=h_D \quad \text{on } \Sigma_D:=(0,T)\times\Gamma, \\ \dfrac{\partial u}{\partial n}+n_0u=h_N \quad \text{on } \Sigma_N:=(0.T)\times\Gamma',\ \Gamma':=\partial\Omega\setminus\Gamma, \\ \dfrac{\partial w}{\partial n}=0,\ \dfrac{\partial}{\partial n}\{-\mu\Delta w_t-\kappa\Delta w+\xi+g(w)-\lambda'(w)u\}=0 \quad \text{on } \Sigma:=(0,T)\times\partial\Omega, \\ u(0,\cdot)=u_0, w(0,\cdot)=w_0 \quad \text{in } \Omega. \end{cases}$$

Throughout this paper, we use the following notation.

For a general (real) Banach space Y, we denote by $|\cdot|_Y$ the norm in Y and by Y^* the dual of Y. Also, for a positive finite number T, we denote by $C_w([0,T];Y)$ the space of all weakly continuous functions $u:[0,T]\to Y$, and by definition "$u_n\to u$ in $C_w([0,T];Y)$ as $n\to+\infty$" means that for each $z^*\in Y^*$, $\langle z^*,u_n(t)\rangle_{Y^*,Y}$ converges to $\langle z^*,u(t)\rangle_{Y^*,Y}$ uniformly in $t\in[0,T]$ as $n\to+\infty$, where $\langle\cdot,\cdot\rangle_{Y^*,Y}$ is the duality pairing between Y^* and Y.

For simplicity we put

$$H:=L^2(\Omega),\ V:=H^1(\Omega),\ H_0:=\{v\in H;\int_\Omega zdx=0\}, V_0:=V\cap H_0,$$

and

$$\Pi:=\{\Gamma\subset\partial\Omega;\ \Gamma \text{ is compact in } \partial\Omega,\ \sigma(\Gamma)>0\}.$$

For each $\Gamma\in\Pi$, we put

$$V(\Gamma):=\{z\in V; z=0 \text{ a.e. on } \Gamma\}$$

which is a closed subspace of V, and

$$(v,w) := \int_\Omega vw dx \qquad \text{for } v,w \in H,$$
$$(v,w)_{\partial\Omega} := \int_{\partial\Omega} vw d\sigma \qquad \text{for } v,w \in L^2(\partial\Omega),$$
$$a(v,w) := \int_\Omega \nabla v \cdot \nabla w dx \quad \text{for } v,w \in V.$$

In general, given a subset E of $\overline{\Omega}$, χ_E denotes the characteristic function of E defined on $\overline{\Omega}$.

We now introduce a notion of convergence in Π. By definition, a sequence $\{\Gamma_n\} \subset \Pi$ converges to $\Gamma \in \Pi$, denoted by $\Gamma_n \to \Gamma$ in Π as $n \to +\infty$, if the following conditions (C1) – (C3) are satisfied:

(C1) If $\{n_k\}$ is a subsequence of $\{n\}$, $z_k \in V(\Gamma_{n_k})$ and $z_k \to z$ weakly in V as $k \to +\infty$, then $z \in V(\Gamma)$.

(C2) For any $z \in V(\Gamma)$, there is a sequence $\{z_n\} \subset V$ such that $z_n \in V(\Gamma_n)$, $n = 1, 2, \cdots$, and $z_n \to z$ in V as $n \to +\infty$.

(C3) $\chi_{\Gamma_n} \to \chi_\Gamma$ in $L^1(\partial\Omega)$ as $n \to +\infty$.

Also, a subset Π' of Π is said to have property (C), if Π' is compact in the sense of (C1) – (C3), namely, any sequence $\{\Gamma_n\}$ of Π' contains a subsequence convergent to a certain $\Gamma \in \Pi'$.

We suppose precise assumptions on the data as follows.

(H1) ρ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ whose domain $D(\rho)$ and range $R(\rho)$ are open in $\mathbb{R}$, and it is locally bi-Lipschitz continuous as a function from $D(\rho)$ onto $R(\rho)$, and there are constants $A_0 > 0$ and α with $1 \le \alpha < 2$ such that

$$|\rho(r_1) - \rho(r_2)| \ge \frac{A_0|r_1 - r_2|}{|r_1 r_2|^\alpha + 1} \quad \text{for all } r_1, r_2 \in D(\rho).$$

(H2) β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ such that $\overline{D(\beta)} = [\sigma_*, \sigma^*]$ for constants σ_*, σ^* with $-\infty < \sigma_* < \sigma^* < +\infty$.

(H3) λ is a C^2-function from $\mathbb{R}$ into itself and g is a C^1-function from $\mathbb{R}$ into itself; λ' is the derivative of λ.

(H4) (i) $f \in W^{1,2}(0,T;H)$;

(ii) $h_D \in W^{1,2}(0,T;H^{1/2}(\partial\Omega))$ such that there is a function $\tilde{h}_D \in W^{1,2}(0,T;V)$ with $\rho(\tilde{h}_D) \in W^{1,2}(0,T;V)$;

(iii) $h_N \in W^{1,2}(0,T;L^2(\partial\Omega)) \cap L^\infty(\Sigma)$ such that

$$n_0 \inf D(\rho) \le h_N(t,x) \le n_0 \sup D(\rho) \quad \text{for a.e. } (t,x) \in \Sigma$$

and there are positive constants A_1 and A_1' such that

$$\rho(r)(n_0 r - h_N(t,x)) \ge -A_1|r| - A_1' \quad \text{for all } r \in D(\rho) \text{ and a.e. } (t,x) \in \Sigma.$$

(H5) (i) $u_0 \in V$ such that $\rho(u_0) \in H$ and $u_0 = h_D(0,\cdot)$ a.e. on $\partial\Omega$;

(ii) $w_0 \in H^2(\Omega)$ such that

$$\sigma_* < \frac{1}{|\Omega|}\int_\Omega w_0 dx =: m < \sigma^*$$

and $\dfrac{\partial w_0}{\partial n} = 0$ a.e. on $\partial\Omega$ and there is $\xi_0 \in H$ satisfying

$$\xi_0 \in \beta(w_0) \quad \text{a.e. in } \Omega, \ -\kappa\Delta w_0 + \xi_0 \in V.$$

Corresponding to functions h_D, h_N and $\Gamma \in \Pi$, we consider the function $h_\Gamma : [0,T] \to V$ given by

$$\begin{cases} h_\Gamma(t) = h_D(t) \quad \text{a.e. on } \Gamma, \\ a(h_\Gamma(t), z) + (n_0 h_\Gamma(t) - h_N(t), z)_{\partial\Omega} = 0 \ \text{ for all } z \in V(\Gamma); \end{cases}$$

note under condition (H4) and $\sigma(\Gamma) \ge \sigma_0$ for a positive constant σ_0 that such a function h_Γ exists in $W^{1,2}(0,T;V)$ and $|h_\Gamma|_{W^{1,2}(0,T;V)} \le K$ for a certain constant K depending only on quantities in (H4) and σ_0. Moreover, if $\Gamma_n \to \Gamma$ in Π as $n \to +\infty$, then $h_{\Gamma_n} \to h_\Gamma$ in $C([0,T];V)$ as $n \to +\infty$ (cf. [6]).

We now give the weak formulation for state problem $SP(\Gamma)$ for each $\Gamma \in \Pi$.

Definition 1.1. A couple $\{u, w\}$ of functions $u : [0,T] \to V$ and $w : [0,T] \to H^2(\Omega)$ is called a (weak) solution of $SP(\Gamma)$, if the following properties (w1) – (w4) are fulfilled:

(w1) $u - h_\Gamma \in C_w([0,T];V(\Gamma))$, $\rho(u) \in C_w([0,T];H)$, $\rho(u)' \in L^2(0,T;V(\Gamma)^*)$, $w \in C_w([0,T];H^2(\Omega))$ with $\dfrac{\partial w(t)}{\partial n} = 0$ a.e. on $\partial\Omega$ for all $t \in [0,T]$, and $w' \in L^2(0,T;H)$.

(w2) $u(0) = u_0$ and $w(0) = w_0$.

(w3) For all $z \in V(\Gamma)$ and a.e. $t \in [0,T]$,

$$\frac{d}{dt}(\rho(u)(t) + \lambda(w)(t), z) + a(u(t), z) + n_0(u(t) - h_\Gamma(t), z)_{\partial\Omega} = (f(t), z).$$

(w4) There exists a function $\xi \in L^2(0,T;H)$ such that $\xi \in \beta(w)$ a.e. in Q and

$$\frac{d}{dt}(w(t), \eta - \mu\Delta\eta) + \kappa(\Delta w(t), \Delta\eta) - (g(w(t)) + \xi(t) - \lambda'(w(t))u(t), \Delta\eta) = 0$$

$$\text{for all } \eta \in H^2(\Omega) \text{ with } \frac{\partial\eta}{\partial n} = 0 \text{ a.e. on } \partial\Omega \text{ and a.e. } t \in [0,T].$$

According to a result [5, Theorem 2.2], problem $SP(\Gamma)$ has an unique solution $\{u, w\}$ for each $\Gamma \in \Pi$. Based on the solvability of $SP(\Gamma)$, we now propose an optimization problem.

For a given non-empty subset Π_c of Π having property (C), our optimization problem, denoted by $P(\Pi_c)$, is to find a set $\Gamma_* \in \Pi_c$ such that

$$J(\Gamma_*) = \inf_{\Gamma \in \Pi_c} J(\Gamma),$$

where

$$J(\Gamma) := A\int_Q |u_\Gamma - u_d|^2 dxdt + B|w_\Gamma - w_d|^2_{C(\overline{Q})} + C\int_{\Sigma(\Gamma')} |h_d|^2 d\sigma dt \quad \Gamma \in \Pi_c,$$

A, B, C are positive constants, u_d, w_d, h_d are given in $L^2(Q)$, $C(\overline{Q})$, $L^2(\Sigma)$, respectively, and $\{u_\Gamma, w_\Gamma\}$ is the solution of state problem $SP(\Gamma)$; $d\sigma$ stands for the surface element on $\partial\Omega$.

Our main results are stated as follows.

Theorem 1.1. *Let Π_c be a non-empty subset of Π having property(C). Then, optimization problem $P(\Pi_c)$ has at least one solution $\Gamma_* \in \Pi_c$.*

The above existence result is obtained from the following theorem on the continuous dependence of the solution $\{u_\Gamma, w_\Gamma\}$ of $SP(\Gamma)$ upon $\Gamma \in \Pi$.

Theorem 1.2. *Let $\{\Gamma_n\}$ be a sequence in Π such that $\Gamma_n \to \Gamma$ in Π as $n \to +\infty$, and $\{u_n, w_n\}$ and $\{u, w\}$ be the solutions of $SP(\Gamma_n)$ and $SP(\Gamma)$, respectively. Then*

$$u_n \to u \text{ in } C_w([0,T]; V), \quad w_n \to w \text{ in } C_w([0,T]; H^2(\Omega))$$

as $n \to +\infty$.

For a detailed proofs, see a forthcoming paper [3].

It is easily seen from Theorem 1.2 that any minimizing sequence $\{\Gamma_n\} \subset \Pi_c$ of the cost functional $J(\cdot)$ on Π_c contains a subsequence convergent to a solution of $P(\Pi_c)$.

2. Regular approximation for $P(\Pi_c)$

In this section, from the numerical point of view we discuss regular approximation of $SP(\Gamma)$ and $P(\Pi_c)$.

At first, we introduce the approximation ρ^ν, β^ε and χ^τ_Γ for ρ, β and χ_Γ, respectively, which are defined below.

(a) Let $D(\rho) := (r_*, r^*)$ for $-\infty \le r_* < r^* \le +\infty$, and choose two families $\{a_\nu; 0 < \nu \le 1\}$ and $\{b_\nu; 0 < \nu \le 1\}$ in $D(\rho)$ such that

$$r_* < a_\nu < a_{\nu'} < a_1 < b_1 < b_{\nu'} < b_\nu < r^* \text{ if } 0 < \nu < \nu' < 1$$

and

$$a_\nu \downarrow r_*, \ b_\nu \uparrow r^* \text{ as } \nu \downarrow 0.$$

Then, $\rho^\nu : \mathbb{R} \to \mathbb{R}$ is defined for each $\nu \in (0, 1]$ by

$$\rho^\nu(r) := \begin{cases} \rho(b_\nu) + r - b_\nu & \text{for } r > b_\nu, \\ \rho(r) & \text{for } a_\nu \le r \le b_\nu, \\ \rho(a_\nu) + r - a_\nu & \text{for } r < a_\nu. \end{cases}$$

(b) For each $0 < \varepsilon \le 1$, β^ε is the Yosida-approximation of β, namely,

$$\beta^\varepsilon(r) := \frac{r - (I + \varepsilon\beta)^{-1} r}{\varepsilon}, \quad r \in \mathbb{R}.$$

(c) Let $\{\chi_\Gamma^\tau\} := \{\chi_\Gamma^\tau; 0 < \tau \le 1, \Gamma \in \Pi_c\}$ be a family of smooth functions on $\partial\Omega$ and suppose that it satisfies the following properties $(\chi 1)$ – $(\chi 3)$:

$(\chi 1)$ $0 \le \chi_\Gamma \le \chi_\Gamma^\tau \le 1$; $\mathrm{supp}(\chi_\Gamma^\tau) \subset \{x \in \partial\Omega; dist(x, \Gamma) \le \tau\}$ for all $\tau \in (0, 1]$ and $\Gamma \in \Pi_c$.

$(\chi 2)$ For each $\tau \in (0, 1]$, $\{\chi_\Gamma^\tau; \Gamma \in \Pi_c\}$ is compact in $L^1(\partial\Omega)$.

$(\chi 3)$ Let $V(\tau, \Gamma) := \{z \in V; \chi_\Gamma^\tau z = 0 \text{ a.e. on } \Gamma\}$ for each $\tau \in (0, 1]$ and $\Gamma \in \Pi_c$. If $\tau_n \downarrow 0$ and $\Gamma_n \in \Pi_c$, then there are a subsequence $\{n_k\}$ of $\{n\}$ and $\Gamma \in \Pi_c$ such that $\chi_{\Gamma_{n_k}}^{\tau_{n_k}} \to \chi_\Gamma$ in $L^1(\partial\Omega)$ as $k \to \infty$, and $V(\tau_{n_k}, \Gamma_{n_k}) \to V(\Gamma)$ in V as $k \to \infty$ in the sense of Mosco [6].

Now we propose a regular approximation for $SP(\Gamma)$, referred as $SP(\Gamma)^{\nu\varepsilon\tau\delta}$, $\nu, \varepsilon, \tau, \delta \in (0, 1]$, by the penalty method:

$$\begin{cases} \rho^\nu(u)_t + \lambda(w)_t - \Delta u = f \quad \text{in } Q, \\ w_t - \Delta(-\mu\Delta w_t - \kappa\Delta w + \beta^\varepsilon(w) + g(w) - \lambda'(w)u) = 0 \quad \text{in } Q, \\ \dfrac{\partial u}{\partial n} = -\dfrac{\chi_\Gamma^\tau}{\delta}(u - h_D) + (1 - \chi_\Gamma^\tau)(h_N - n_0 u) \quad \text{on } \Sigma, \\ \dfrac{\partial w}{\partial n} = 0, \ \dfrac{\partial}{\partial n}(-\mu\Delta w_t - \kappa\Delta w + \beta^\varepsilon(w) + g(w) - \lambda'(w)u) = 0 \quad \text{on } \Sigma, \\ u(0) = u_{0\nu} := \min\{\max\{u_0, a_\nu\}, b_\nu\}, \ w(0) = w_0 \quad \text{in } \Omega. \end{cases}$$

The notion of a weak solution of $SP(\Gamma)^{\nu\varepsilon\tau\delta}$ is given below.

Definition 2.1. A couple $\{u, w\}$ of functions $u : [0, T] \to V$ and $w : [0, T] \to H^2(\Omega)$ is called a solution of $SP(\Gamma)^{\nu\varepsilon\tau\delta}$, if the following conditions (w1)′ – (w4)′ are satisfied:

(w1)′ $u \in W^{1,2}(0, T; H) \cap C([0, T]; V)$,

$w \in W^{1,2}(0, T; H) \cap C_w([0, T]; H^2(\Omega))$ with $\dfrac{\partial w(t)}{\partial n} = 0$ a.e. on $\partial\Omega$ for all $t \in [0, T]$.

(w2)′ $u(0) = u_{0\nu}$, $w(0) = w_0$.

(w3)′ For all $z \in V$ and a.e. $t \in [0, T]$,

$$(\rho^\nu(u)'(t) + \lambda(w)'(t), z) + a(u(t), z) + \left(\frac{\chi_\Gamma^\tau}{\delta}(u(t) - h_D(t)) - (1 - \chi_\Gamma^\tau)(h_N(t) - n_0 u(t)), z\right)_{\partial\Omega} = (f(t), z).$$

(w4)′ For all $\eta \in H^2(\Omega)$ with $\dfrac{\partial \eta}{\partial n} = 0$ a.e. on $\partial\Omega$ and a.e. $t \in [0, T]$,

$$(w'(t), \eta - \mu\Delta\eta) + \kappa(\Delta w(t), \Delta\eta) - (g(w(t)) + \beta^\varepsilon(w(t)) - \lambda'(w(t))u(t), \Delta\eta) = 0.$$

According to a result in [4], $SP(\Gamma)^{\nu\varepsilon\tau\delta}$ has a unique solution $\{u, w\}$. Our regular approximate optimization problem $P(\Pi_c)^{\nu\varepsilon\tau\delta}$ is to find $\Gamma_*^{\nu\varepsilon\tau\delta} \in \Pi_c$ such that

$$J^{\nu\varepsilon\tau\delta}(\Gamma_*^{\nu\varepsilon\tau\delta}) = \inf_{\Gamma\in\Pi_c} J^{\nu\varepsilon\tau\delta}(\Gamma),$$

where

$$J^{\nu\varepsilon\tau\delta}(\Gamma) := A\int_Q |u-u_d|^2 dxdt + B|w-w_d|^2_{C(\overline{Q})} + C\int_\Sigma (1-\chi_\Gamma^\tau)|h_d|^2 d\sigma dt,$$

$\{u, w\}$ is the solution of $SP(\Gamma)^{\nu\varepsilon\tau\delta}$.

Finally, we show a convergence result.

Theorem 2.1. *Let Π_c, $\{\rho^\nu\}$, $\{\beta^\varepsilon\}$, $\{\chi_\Gamma^\tau\}$ be as above. Then:*

(1) For $\nu, \varepsilon, \tau, \delta \in (0, 1]$, $P(\Pi_c)^{\nu\varepsilon\tau\delta}$ has at least one solution $\Gamma_*^{\nu\varepsilon\tau\delta} \in \Pi_c$.

(2) Let $\{\nu_n\}$, $\{\varepsilon_n\}$, $\{\tau_n\}$ and $\{\delta_n\}$ be any null sequences and let $\{\Gamma_n := \Gamma_*^{\nu_n\varepsilon_n\tau_n\delta_n}\}$ be a sequence of solutions of $P(\Pi_c)^{\nu_n\varepsilon_n\tau_n\delta_n}$. Then, $\{\Gamma_n\}$ contains a subsequence convergent in Π and any limit Γ_* is a solution of $P(\Pi_c)$.

For a detailed proof, see a forthcoming paper [3].

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