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A free boundary problem for one dimensional hyperbolic equation

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In this note we treat a free boundary problem for a hyperbolic equation. This problem comes from the variational problem which are based on the physical image of "peeling off". This problem originated from an elliptic free boundary problem which is firstly introduced by Alt-Caffarelli [1]: find a minimizer of

$$I(u) := \int_{\Omega}\left(|\nabla u|^2 + Q^2\chi_{\{u>0\}}\right)d\mathcal{L}^n. \tag{1}$$

Here Q is a given positive constant. It is also an important problem to investigate the behavior of $\partial\{u>0\}$. The minimizer of I satisfies the folowing elliptic free boundary problem.

$$\begin{cases} -\triangle u = 0 & \text{in} \quad \{u>0\} \\ |\nabla_x u|^2 = Q^2 & \text{on} \quad \partial\{u>0\}. \end{cases}$$

This is a stationary problem, but the physical model of peeling off requires us to examine the motion of memblane which has been peeled off. Hence we now try to formulate a hyperbolic problem corresponds to (1). In order to do so we should investigate the stationary point u of the action integral corresponding to I. But this functional is not Fréchet differentiable. Then we start from the following hyperbolic problem, which the stationary point u should satisfy.

$$\begin{cases} u_{tt} - \triangle u = 0 & \text{in} \quad \{u>0\} \\ |\nabla_x u|^2 - u_t^2 = Q^2 & \text{on} \quad \partial\{u>0\}. \end{cases}$$

In this note, as the first step of this problem, we consider the one space dimensional case and make a classical formulation. Let f be a given function with $f(0)=0$ and $f(t)>0$ for $t>0$. First we consider the problem which has no informations on initial conditions for u:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in} & \{u>0\} \\ u(t,0) = f(t) & \text{for all} & t>0 \\ u_x^2 - u_t^2 = Q^2 & \text{on} & \partial\{u>0\}. \end{cases} \tag{2}$$

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Here we introduce the standard transformation of variables:

$$\xi = \frac{1}{2}(t+x) \qquad \text{and} \qquad \eta = \frac{1}{2}(t-x).$$

Then our problem is rewritten as

$$\text{(3)} \qquad \begin{cases} u_{\xi\eta} = 0 & \text{in} \quad \{\xi > \eta\} \cap \{u > 0\} \\ u(\eta,\eta) = f(\eta) & \\ -u_\xi \cdot u_\eta = 4Q^2 & \text{on} \quad \partial\{u > 0\}. \end{cases}$$

We consider a slightly generalized problem: let α be a given real constant, then find u which satisfies

$$\begin{cases} u_{\xi\eta} = 0 & \text{in} \quad \{\xi > \alpha\eta\} \cap \{u > 0\} \\ u(\alpha\eta,\eta) = f(\eta) & \\ -u_\xi \cdot u_\eta = 4Q^2 & \text{on} \quad \partial\{u > 0\}. \end{cases}$$

Problem (3) is just the case that $\alpha = 1$. We assume that f satisfies

$$f \in C^0(-\infty,\infty) \text{ and } f|_{[0,\infty)} \in C^2([0,\infty)) \text{ with } \begin{cases} f'(\eta) > 0 & \text{for} \quad \eta \in [0,\infty) \\ f(\eta) = 0 & \text{for} \quad \eta \in (-\infty, 0]. \end{cases}$$

Now our problem is reduced in the following way.

Problem 1. Find u in $C^0(\{(\xi,\eta);\xi \geq \alpha\eta\})$ and h in $C^0([0,\infty)) \cap C^2(0,\infty)$ which satisfy

(1) $h(0) = 0$

(2) $u \in C^2(\{(\xi,\eta);\alpha\eta < \xi < h(\eta),\ \eta > 0\}) \cap C^1(\{(\xi,\eta);\alpha\eta < \xi \leq h(\eta),\ \eta > 0\})$

(3) $u > 0$ in $\{(\xi,\eta);\alpha\eta \leq \xi < h(\eta),\ \eta > 0\}$

(4) $u(\xi,\eta) = 0$ for $(\xi,\eta) \in \{(\xi,\eta);\xi \geq h(\eta)\} \cup \{(\xi,\eta);\xi \geq \alpha\eta,\ \eta < 0\}$

(5) $u_{\xi\eta} = 0$ in $\{(\xi,\eta);\alpha\eta < \xi < h(\eta)\}$

(6) $u(\alpha\eta,\eta) = f(\eta)$ for $\eta \geq 0$

(7) $-u_\xi \cdot u_\eta = 4Q^2$ on $\{(\xi,\eta);\xi = h(\eta)\}$.

Let u and h be the solution of Problem 1. Remark that $u(\xi,\eta) = \varphi(\xi) + \psi(\eta)$ and without loss of generality we may assume $\varphi(0) = \psi(0) = 0$. Differentiate the both side of the equality $u(h(\eta),\eta) = \varphi(h(\eta)) + \psi(\eta) = 0$ by η, we have

$$\varphi'(h(\eta))h'(\eta) + \psi'(\eta) = 0.$$

The free boundary condition implies

$$-\varphi(h(\eta))\psi(\eta) = 4Q^2.$$

Thus we have

$$h'(\eta) = \frac{1}{4Q^2}\psi(\eta)^2.$$

Then

$$h(\eta) = \frac{1}{4Q^2}\int_0^{\eta} \psi'(\tilde{\eta})^2 d\tilde{\eta} \tag{4}$$

holds for $\eta > 0$. Now we consider the following three cases: Case 1 $\alpha = 0$, Case 2 $\alpha < 0$, Case 3 $\alpha > 0$. For cases 1 and 2 Problem 1 has been perfectly solved, but for case 3, which includes the original case, it is still open.

Case 1. In this case we give a boundary condition on the characteristic line. This means that $\psi(\eta) = f(\eta)$. Hence we obtain h by (4). Since $\psi'(\eta) = f'(\eta) > 0$, we have $h'(\eta) > 0$. Thus h is monotonously increasing and h^{-1} exists. Since $u(h(\eta),\eta) = \varphi(h(\eta)) + \psi(\eta) = 0$, we have $\varphi(\xi) = -\psi(h^{-1}(\xi))$. Now we obtain a solution of Problem 1 uniquely.

Case 2. In this case the boundary condition is

$$u(\alpha\eta,\eta) \equiv \varphi(\alpha\eta) + \psi(\eta) = f(\eta). \tag{5}$$

Let $g(\eta)$ be a C^1 class function on $[0,\infty)$. Now we solve $u_{\xi\eta} = 0$ with the initial conditions (5) and

$$g(\eta) = -\frac{1}{\alpha}\varphi'(\alpha\eta) + \psi'(\eta)$$

(the normal derivative to the line $\{\xi = \alpha\eta\}$). Then we have $\varphi(\xi)$ for $\xi < 0$ and $\psi(\eta)$ for $\eta > 0$ as

$$\psi(\eta) = \frac{1}{1+\alpha^2}\left\{f(\eta) + \alpha^2\int_0^{\eta} g(\tilde{\eta})d\tilde{\eta}\right\}$$

and

$$\varphi(\xi) = \frac{1}{1+\frac{1}{\alpha^2}}\left\{f(\frac{\xi}{\alpha}) - \int_0^{\frac{\xi}{\alpha}} g(\tilde{\eta})d\tilde{\eta}\right\}.$$

When

$$f'(\eta) + \alpha^2 g(\eta) > 0, \tag{6}$$

we have $\psi'(\eta) > 0$. Thus in the same way as in that of Case 1 we obtain $\varphi(\xi)$ for $\xi > 0$ and the free boundary $\xi = h(\eta)$ which satisfies Problem 1 for for $\xi > 0$. Moreover when f and g satisfy

$$\alpha^2 g(0)^2 + (1-\alpha^2)f'(0)g(0) - f'(0)^2 + 4Q^2\frac{(1+\alpha^2)^2}{|\alpha|} = 0 \tag{7}$$

and

$$g'(0) = \frac{\{f'(0)+\alpha^2 g(0)\}^4 - 16Q^4(1+\alpha^2)^4}{\{f'(0)+\alpha^2 g(0)\}^4 + 16Q^4\alpha^2(1+\alpha^2)^4} f''(0), \tag{8}$$

we obtain $\varphi \in C^2$ by a straightforward calculus. It follows from (7) that, if $f'(0) \geq 4Q\sqrt{|\alpha|}$, there exists a function g which satisfies (7) and $f'(0) + \alpha^2 g(0) > 0$. Thus we obtain a function g which satisfies (6), (7), and (8), and hence a solution of Problem 1. But for such cases there are infinitly many such g, so the uniqueness does not hold.

Case 3. The solution $u(\xi, \eta) = \varphi(\xi) + \psi(\eta)$ and the free boundary $\xi = h(\eta)$ should satisfy

$$\varphi(\alpha\eta) + \psi(\eta) = f(\eta)$$

and

$$\varphi(\xi) = -\psi(h^{-1}(\xi)).$$

Thus in order to solve Problem 1 for this case we should find functions ψ and h which satisfy

$$-\psi(h^{-1}(\alpha\eta)) + \psi(\eta) = f(\eta)$$

and

$$h(\eta) = \frac{1}{4Q^2}\int_0^\eta \psi'(\tilde{\eta})^2 d\tilde{\eta}.$$

But now we do not have any informations about the existence or uniqueness of such functions. This problem is open.

The original problem (2) is included in Case 3, but it has not been solved.

We next consider the problem which has some informations on initial conditions for u:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in} \quad \{u > 0\} \cap \Omega \times (0, \infty) \\ u(0,t) = \tilde{f}(t) & \text{for all} \quad t \in (0,\infty) \\ u(x,0) = \tilde{e}(x) & \text{for} \quad -l \leq x \leq 0 \\ u_t(x,0) = \tilde{g}(x) & \text{for} \quad -l \leq x \leq 0 \\ u_x^2 - u_t^2 = Q^2 & \text{on} \quad \partial\{u > 0\} \cap \Omega \times (0,\infty). \end{cases}$$

where $\tilde{e}(x)$, $\tilde{g}(x)$, $\tilde{f}(t)$ are ginven functions satisfying $\tilde{f}(0) = \tilde{e}(-l) > \tilde{e}(0) = 0$ and l and Q are ginve positive constants. By the use of same transformation of variables $(t, x) \mapsto (\xi, \eta)$ we rewrite above problem as

$$\begin{cases} u_{\xi\eta} = 0 & \text{in} \quad \{u > 0\} \\ u(\xi, \xi + 2l) = f(\xi) & \text{for} \quad -l \le \xi < \infty \\ u(\xi, -\xi) = e(\xi) & \text{for} \quad -l \le \xi \le 0 \\ u_\eta(\xi, -\xi) + u_\xi(\xi, -\xi) = g(\xi) & \text{for} \quad -l \le \xi \le 0 \\ -u_\xi \cdot u_\eta = 4Q^2 & \text{on} \quad \partial\{u > 0\}. \end{cases}$$

In the same way as in Problem 1 we reduce above problem as follows:

Problem 2. Find u in $C^0(\{(\xi, \eta); \xi \ge \eta - 2l,\ \xi \ge -\eta\})$ and h in $C^0([0, \infty)) \cap C^2(0, \infty)$ which satisfy

(1) $h(0) = 0$

(2) $u \in C^2(\{(\xi, \eta); \eta - 2l < \xi < h(\eta),\ \xi > -\eta\}) \cap C^1(\{(\xi, \eta); \eta - 2l < \xi \le h(\eta),\ \xi \ge -\eta\})$

(3) $u > 0$ in $\{(\xi, \eta); \eta - 2l \le \xi < h(\eta),\ \xi \ge -\eta\}$

(4) $u(\xi, \eta) = 0$ for $(\xi, \eta) \in \{(\xi, \eta); \xi \ge h(\eta)\} \cup \{(\xi, \eta); \xi \ge -\eta,\ \eta < 0\}$

(5) $u_{\xi\eta} = 0$ in $\{(\xi, \eta); \eta - 2l < \xi < h(\eta),\ \xi > -\eta\}$

(6) $u(\xi, \xi + 2l) = f(\xi)$ for $-l \le \xi < \infty$

(7) $u(\xi, -\xi) = e(\xi)$ for $-l \le \xi \le 0$

(8) $u_\eta(\xi, -\xi) + u_\xi(\xi, -\xi) = g(\xi)$ for $-l \le \xi \le 0$

(9) $-u_\xi \cdot u_\eta = 4Q^2$ on $\{(\xi, \eta); \xi = h(\eta)\}$.

Now we require the following conditions on f, g, and e:

Assumption.

(1) $g(\xi) > e'(\xi)$ for $-l < \xi < 0$

(2) $f'(\xi) - \frac{1}{2}(e'(\xi) + g(\xi)) > 0$ for $-l < \xi < 0$

(3) $e'(0)^2 - g(0)^2 = 16Q^2$

(4) $(e''(0) + g'(0))(e'(0) - g(0))^4 = 256Q^4(e''(0) - g'(0))$

(5) $f(-l) = e(-l)$

(6) $f'(-l) = g(-l)$

(7) $f''(-l) = e''(-l)$.

Under these assumptions we have the following theorem by the use of result of Problem 1.

Theorem 1 *There exists a C^2 solution $u(\xi, \eta)$ of Problem 2 for $(\xi, \eta) \in (-l, h(h(l) + 2l)) \times (0, h(l) + 2l)$.*

Here we put

$$\begin{cases} h_0(l) = -l \\ h_n(l) = h(h_{n-1}(l) + 2l) \quad (n \geq 1). \end{cases}$$

Note that $h_1(l) = h(l)$. Inductively we obtain a C^2 solution of Problem 2 for $(\xi, \eta) \in (-l, h_n(l)) \times (0, h_{n-1}(l) + 2l)$ $(n = 1, 2, \cdots)$. Moreover we have

Proposition 2 *$h_n(l) > h_{n-1}(l)$ for $n \geq 1$.*

Proof. For $n = 1$, it holds that $h_1(l) = h(l) > 0 > -l$. If we assume $h_{n-1}(l) > h_{n-2}(l)$ holds, then $h_n(l) = h(h_{n-1}(l) + 2l) > h(h_{n-2}(l) + 2l) = h_{n-1}(l)$ holds. Thus we have the proposition. Q.E.D.

Proposition 2 implies that $h^* := \lim_{n \to \infty} h_n(l) \leq \infty$ exists. Finally we obtain

Theorem 3 *There exists a C^2 solution $u(\xi, \eta)$ of Problem 2 for $(\xi, \eta) \in (-l, h^*)) \times (0, h^* + 2l)$.*

References

[1] H. W. Alt - L. A. Caffarelli, *Existence and regularity for a minimum problem with free boundary*, J. Reine Angew. Math., **325** (1981), 105-144.

[2] K. Kikuchi - S. Omata, *A free boundary problem for one dimensional hyperbolic equation*, preprint.