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Pohozaev-type inequalities for weak solutions of elliptic equations

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1 Introduction

In this note, we are concerned with the following quasilinear elliptic equations

$$
\text{(E)} \begin{cases} -\Delta_p u = |u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(x) \to 0 & (\text{as } |x| \to \infty), \end{cases}
$$

where $\Delta_p u(x) = \operatorname{div}(|\nabla u(x)|^{p-2}\nabla u(x))$ and Ω is a domain in $\mathbb{R}^N$ such that $\Omega := \Omega_d \times \mathbb{R}^{N-d}$, $\Omega_d \subset \mathbb{R}^d$ and has a smooth boundary $\partial\Omega$.

When Ω is a bounded domain, this equation arises from the minimizing problem for Rayleigh quotient $R(v) := \|\nabla v\|_{L^p}/\|v\|_{L^q}$. That is to say, assume that u minimizes $R(\cdot)$, i.e.,

$$
R(u) = \min\{R(v); v \in W_0^{1,p}(\Omega) \setminus \{0\}\}.
$$

This is equivalent to the fact that u attains the best possible constant for the well-known Sobolev-Poincaré-type inequality:

$$
\text{(SP)} \qquad \|v\|_{L^q} \le C\|\nabla v\|_{L^p} \quad \forall v \in W_0^{1,p}(\Omega).
$$

Then u, normalized in a proper way, gives a nontrivial solution for (E).

When Ω is a general unbounded domain, the significance of our equation from this point of view might fade away, since the Sobolev-Poincaré-type inequality (SP) above does not hold any more in general.

However, from the view-point of nonlinear P.D.E., the existence of nontrivial solutions for our type of equation has been studied vigorously by many peoples in unbounded domains. For example we here quote the work by

- Jianfu & Xiping [4, 5] $\Omega = \mathbb{R}^N$ $(d = 0)$, $-\Delta_p u = f(u)$, where $f(u)$ admits much more complicated nonlinearity than ours.
- Schindler [12] $\Omega = \Omega_d \times \mathbb{R}^{N-d}$ $(0 < d < N)$, $\Omega_d \subset \mathbb{R}^d$: bounded.

Therefore, from this point of view, it would be meaningful to investigate the nonexistence of nontrivial solution in unbounded domains. In fact, some attempt in this direction are already done by several peoples, say by [1, Esteban & Lions], [8, Ni & Serrin], [6, Kawano, Ni & Yotsutani] in the class of classical solution or radially symmetric solutions. However it should be noted that the degeneracy of p-Laplacian causes the lack of regularity of solutions of our equation. More precisely, we have the following proposition.

Proposition 1.1 *Let $p > 2$. then any nontrivial solution u of (E) does not belong to* $C^2(\Omega) \cap C(\overline{\Omega})$.

Proof. Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$, $u \not\equiv 0$ be a nontrivial solution of (E). Then without loss of generality, we may assume there exists a point $x_0 \in \Omega$ s.t., $u(x_0) = \max_{x\in\bar{\Omega}} u(x) > 0$ and $\nabla u(x_0) = 0$.

On the other hand, we can write down the equation (E) in the following form

$$-\Delta_p u(x) = -\sum_{i,j=1}^{N} |\nabla u(x)|^{p-2}(\delta_{ij} + (p-2)\frac{u_{x_i}u_{x_j}}{|\nabla u|^2})u_{x_i x_j}(x) = |u|^{q-2}u(x). \quad (1)$$

If $\exists\delta > 0$ s.t. $|\nabla u(x)| = 0$ $\forall x \in B(x_0;\delta)$, then $u(x_0) = 0$. This is a contradiction.

So there exists a sequence $\{x_n\} \subset \Omega$ s.t. $x_n \to x_0$ as $n \to \infty$ and $|\nabla u(x_n)| \neq 0$. Since $\left|\frac{u_{x_i}(x_n)u_{x_j}(x_n)}{|\nabla u(x_n)|^2}\right| \leq 1$, $|u_{x_i x_j}(x_n)| \leq Const.$ and $|\nabla u(x_n)| \to |\nabla u(x_0)| = 0$ as $n \to \infty$, putting $x = x_n$ in (1) and letting $n \to \infty$, we derive $|u|^{q-2}u(x_0) = 0$. This is a contradiction. □

The main purpose of this note is to discuss the nonexistence of nontrivial solutions for (E) when Ω is an exterior domain or a cylindrical domain in a class of weak solutions which is analogous to that introduced in [9]. Most of proofs for nonexistence results for nonlinear elliptic equations such as (E) obtained so far rely essentially on "Pohozaev-type *identity*". As for our case, we introduce a "Pohozaev-type *inequality*" for weak solutions, which is effective enough for discussing the nonexistence in a class of weak solutions.

2 Main results

To formulate our results, we need the notion of starshapedness of the domain Ω. We say that Ω is starshaped if $(x \cdot \vec{n}(x)) \geq 0$ holds for all $x \in \partial\Omega$ with a suitable choice of the origin, where $\vec{n}(x) = (n_1(x), \ldots, n_N(x))$ denotes the outward normal unit vector at $x \in \partial\Omega$.

We also need the following two exponents p' (hölder conjugate) and p^* (Sobolev conjugate):

$$p' := \frac{p}{p-1}, \quad p^* := \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N, \\ \infty & \text{if } p \geq N. \end{cases}$$

Then our main results read as follows.

Theorem 2.1 *Let $\Omega = \mathbb{R}^N \setminus \overline{\Omega_0}$, and let $\Omega_0 := \Omega_d \times \mathbb{R}^{N-d}$, and Ω_d be bounded and starshaped. Put*

$$\mathcal{Q}^e = \{u \in L^q(\Omega); \nabla u \in (L^p(\Omega))^N, u|_{\partial\Omega} = 0\}.$$

Then the following hold.

1. *Let $q < p^*$, then* (E) *has no nontrivial weak solution belonging to Q^e.*

2. *Let $q = p^*$ with $p < N$, then* (E) *has no nontrivial weak solution of definite sign belonging to Q^e.*

Theorem 2.2 *Let $\Omega = \Omega_d \times \mathbb{R}^{N-d}$ and put*

$$\mathcal{Q}^i = \{u \in L^q(\Omega); \nabla u \in (L^p(\Omega))^N, u|_{\partial\Omega} = 0, x_i|u|^{q-1} \in L^{p'}_{loc}(\Omega), i = 1, 2, \ldots, N\}.$$

Then the following hold.

1. *Let $q > p^*$, then* (E) *has no nontrivial weak solution belonging to Q^i.*

2. *Let $q = p^*$ with $p < N$, then* (E) *has no nontrivial weak solution of definite sign belonging to Q^i.*

Remark Above results together with our previous results in [9], [12] and [3] suggest the following duality between the interior problems and the exterior problems for star-shaped (cylindrical) domains. Although the existence of nontrivial positive solutions for the exterior problems with $q < p^*$ is not yet proved, it is likely that it should hold true.

duality between interior and exterior problems			
domain	$q < p^*$	$q = p^*$	$q > p^*$
interior	∃ positive solution	no positive solution	no nontrivial solution
exterior	no nontrivial solution	no positive solution	∃ positive solution ?

3 Pohozaev-type inequality

In this section, we introduce a "*Pohozaev-type inequality*" valid for weak solutions u belonging to a certain class of weak solutions $\mathcal{P}$. To do this, we need some approximation procedures. First of all, we prepare a sequence of bounded domains Ω_n, such that

$$\Omega_n \underset{\neq}{\supset} \Omega_{n-1}, \ \bigcup_{n=1}^{\infty} \Omega_n = \Omega, \ \partial\Omega_n\text{: smooth}$$

(e.g., $\Omega_n = \Omega \cap B_{R_n}$, $B_{R_n} = \{x \in \mathbb{R}^N; |x| < R_n\}$, $R_n \to \infty$ as $n \to \infty$), and the cut-off functions $g_n(\cdot) \in C^1(\mathbb{R})$ such that

$$0 \le g_n'(s) \le 1 \quad s \in \mathbb{R}, \quad g_n(s) = \begin{cases} s & |s| \le n, \\ (n+1)\text{sign } s & |s| \ge n+1. \end{cases}$$

Let u be a weak solution of (E) belonging to $\mathcal{Q} = \mathcal{Q}^e$ or $\mathcal{Q}^i$, and $u_n = g_n(u)$ and $\underline{u_n} := u_n|_{\Omega_n}$.

We first add the term $|u|^{q-2}u$ to both sides of (E) to get the equation (E)$'$: $|u|^{q-2}u - \Delta_p u = 2|u|^{q-2}u$, equivalent to (E). Then we consider the following approximate equation $(\mathrm{E})_n$ for (E)$'$:

$$(\mathrm{E})_n \quad \begin{cases} |w_n|^{q-2}w_n - \Delta_p w_n = 2|\underline{u_n}|^{q-2}\underline{u_n} & \text{in } \Omega_n, \\ w_n = 0 & \text{on } \partial\Omega_n. \end{cases} \tag{2}$$

Since $\underline{u_n} \in L^\infty(\Omega_n)$, we can take a sequence v_n^ε in $C_0^\infty(\Omega_n)$ satisfying

$$\|v_n^\varepsilon\|_{L^\infty} \le C_0 \qquad \text{for all } \varepsilon \in (0,1), \tag{3}$$

$$v_n^\varepsilon \to 2|\underline{u_n}|^{q-2}\underline{u_n} \quad \text{strongly in } L^r(\Omega_n) \text{ as } \varepsilon \to 0 \text{ for all } r \in [1,\infty). \tag{4}$$

We further need another approximate equation $(\mathrm{E})_n^\varepsilon$ for $(\mathrm{E})_n$ of the form:

$$(\mathrm{E})_n^\varepsilon \quad \begin{cases} |w_n^\varepsilon|^{q-2}w_n^\varepsilon + A_\varepsilon w_n^\varepsilon = v_n^\varepsilon & \text{in } \Omega_n, \\ w_n^\varepsilon = 0 & \text{on } \partial\Omega_n, \end{cases} \tag{5}$$

where $A_\varepsilon u(x) = -\mathrm{div}\{(|\nabla u(x)|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u(x)\}$ and $\varepsilon > 0$.

We can show that $(\mathrm{E})_n$ and $(\mathrm{E})_n^\varepsilon$ have unique solutions and that $(\mathrm{E})_n^\varepsilon$ and $(\mathrm{E})_n$ give good approximations for $(\mathrm{E})_n$ and (E) respectively in the following sense.

Lemma 3.1 *The following hold true.*

(1) *For each $\varepsilon \in (0,1)$ and $n \in \mathbb{N}$, there exists a unique solution $w_n^\varepsilon \in C^2(\overline{\Omega_n})$ of* $(\mathrm{E})_n^\varepsilon$.

(2) *For each* $n \in \mathbb{N}$, *there exists a unique solution* $w_n \in C^{1,\alpha}(\overline{\Omega_n}) \cap W_0^{1,p}(\Omega_n)$ *of* $(E)_n$.

(3) w_n^ε *converges to* w_n *as* $\varepsilon \to 0$ *in the following sense.*

$$\begin{aligned} \nabla w_n^\varepsilon \to \nabla w_n \quad & \text{strongly in } (L^p(\Omega_n))^N \quad \text{as } \varepsilon \to 0, & (6) \\ w_n^\varepsilon \to w_n \quad & \text{strongly in } L^r(\Omega_n) \quad \text{for all } r \in [1, \infty) \text{ as } \varepsilon \to 0. & (7) \end{aligned}$$

(4) w_n *converges to* u *as* $n \to \infty$ *in the following sense.*

$$\begin{aligned} \nabla \tilde{w}_n \to \nabla u \quad & \text{strongly in } (L^p(\Omega))^N \quad \text{as } n \to \infty, & (8) \\ \tilde{w}_n \to u \quad & \text{strongly in } L^q(\Omega) \quad \text{as } n \to \infty, & (9) \end{aligned}$$

where $\tilde{w}_n$ *is the zero extension of* w_n *to* Ω.

The proof of this Lemma is shown in [3].

For the integrability of u, we assume only $u \in L^q(\Omega)$, $\left(x_i|u|^{q-1} \in L^{p'}(\Omega_R)\right)$ and $\nabla u \in (L^p(\Omega))^N$, in consequence we encounter serious difficulties concerning the integrability of various integrands in the procedure of deriving the Pohozaev-type inequality. To cope with this difficulty, we introduce the cut-off function $\Psi_R(r) \in C_0^\infty(\mathbb{R})$ satisfying

$$\Psi_R(r) = \begin{cases} 1 & r \le R, \\ 0 & r \ge 2R, \end{cases} \quad 0 \le \Psi_R(r) \le 1, -\frac{C}{R} \le \Psi_R'(r) \le 0.$$

Modifying Pohozaev's idea [11, Pohozaev], we calculate $\displaystyle\lim_{\varepsilon \to 0} \sum_{i=1}^N \int_{\Omega_n} x_i \frac{\partial w_n^\varepsilon}{\partial x_i} \Psi_R(r) (\mathrm{E})_n^\varepsilon dx$. Then for the case of Theorem 2.1, we have:

Lemma 3.2 *Let* Ω *be same as in Theorem 2.1. For any* $R > 0$ *with* $B_R \cap \Omega \neq \emptyset$ *and* $B_{2R} \cap \Gamma_n (\Gamma_n := \partial\Omega_n \setminus \partial\Omega)$, *there exists a number* n_0 *such that the solution* w_n *of* $(\mathrm{E})_n$ *satisfies the following inequality for all* $n \ge n_0$.

$$\begin{aligned} & -\frac{N}{q} \int_{\Omega_n} |w_n|^q \Psi_R(r) dx - \frac{1}{q} \int_{\Omega_n} |w_n|^q r \Psi_R'(r) dx + \frac{p-N}{p} \int_{\Omega_n} |\nabla w_n|^p \Psi_R(r) dx \\ & \quad - \frac{1}{p} \int_{\Omega_n} |\nabla w_n|^p r \Psi_R'(r) dx + \int_{\Omega_n} |\nabla w_n|^{p-2} (x \cdot \nabla w_n)^2 \frac{\Psi_R'(r)}{r} dx \\ & \quad - 2 \int_{\Omega_n} |\underline{u_n}|^{q-2} \underline{u_n} \Psi_R(r) x \cdot \nabla w_n dx + \mathcal{R}_n \le 0, \end{aligned} \tag{10}$$

where $\mathcal{R}_n = \overline{\lim_{\varepsilon \to 0}} \dfrac{\bar{p}-1}{p} \displaystyle\int_{\partial\Omega \cap B_{2R}} (|\nabla w_n^\varepsilon|^2 + \varepsilon)^{\frac{\bar{p}}{2}} (-x \cdot \vec{n}) \Psi_R(r) dS$, $r = |x|$ *and* $\bar{p} = \min(p, 2)$.

Proof Take n_0 so that $\partial B_{2R} \cap \Gamma_n = \emptyset$, then $\Psi_R(r) = 0$ on Γ_n for all $n \geq n_0$. We are going to calculate $\sum_{i=1}^{N} \int_{\Omega_n} x_i \frac{\partial w_n^\varepsilon}{\partial x_i} \Psi_R(r)(\mathrm{E})_n^\varepsilon dx$. By the integration by parts, we get

$$
\begin{aligned}
&-\frac{N}{q}\int_{\Omega_n} |w_n^\varepsilon|^q \Psi_R(r)dx - \frac{1}{q}\int_{\Omega_n} |w_n^\varepsilon|^q r\Psi_R'(r)dx \\
&+\int_{\Omega_n} (|\nabla w_n^\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}} |\nabla w_n^\varepsilon|^2 \Psi_R(r)dx - \frac{N}{p}\int_{\Omega_n} (|\nabla w_n^\varepsilon|^2+\varepsilon)^{\frac{p}{2}} \Psi_R(r)dx \\
&-\frac{1}{p}\int_{\Omega_n} (|\nabla w_n^\varepsilon|^2+\varepsilon)^{\frac{p}{2}} r\Psi_R'(r)dx + \int_{\Omega_n} (|\nabla w_n^\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}} (x\cdot\nabla w_n^\varepsilon)^2 \frac{\Psi_R'(r)}{r}dx \\
&-\int_{\Omega_n} v_n^\varepsilon x\cdot\nabla w_n^\varepsilon \Psi_R(r)dx + \frac{p-1}{p}\int_{\partial\Omega\cap B_{2R}} (|\nabla w_n^\varepsilon|^2+\varepsilon)^{\frac{p}{2}} (-x\cdot\vec{n})\Psi_R(r)dS. \\
&\qquad = \varepsilon\int_{\partial\Omega\cap B_{2R}} (|\nabla w_n^\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}} (-x\cdot\vec{n})\Psi_R(r)dS. \qquad (11)
\end{aligned}
$$

Since w_n^ε converges to w_n strongly in $L^q(\Omega_n)$ and $W_0^{1,p}(\Omega_n)$ by (3) of Lemma 3.1, we can easily show that

$$
\begin{aligned}
|w_n^\varepsilon|(\Psi_R(r))^{\frac{1}{q}} &\to |w_n|(\Psi_R(r))^{\frac{1}{q}} \quad \text{strongly in } L^q(\Omega_n), &(12)\\
|w_n^\varepsilon|(-r\Psi_R'(r))^{\frac{1}{q}} &\to |w_n|(-r\Psi_R'(r))^{\frac{1}{q}} \quad \text{strongly in } L^q(\Omega_n), &(13)\\
(|\nabla w_n^\varepsilon|^2+\varepsilon)^{\frac{1}{2}}(\Psi_R(r))^{\frac{1}{p}} &\to |\nabla w_n|(\Psi_R(r))^{\frac{1}{p}} \quad \text{strongly in } L^p(\Omega_n), &(14)\\
(|\nabla w_n^\varepsilon|^2+\varepsilon)^{\frac{1}{2}}(-r\Psi_R'(r))^{\frac{1}{p}} &\to |\nabla w_n|(-r\Psi_R'(r))^{\frac{1}{p}} \quad \text{strongly in } L^p(\Omega_n), &(15)
\end{aligned}
$$

where we used the fact that $|\Psi_R(r)| \leq 1$ and $|r\Psi_R'(r)| \leq C$.

On the other hand, noting that $(-x\cdot\vec{n}) \geq 0$ on $\partial\Omega$, we get

$$
\begin{aligned}
&\varepsilon\int_{\partial\Omega\cap B_{2R}} (|\nabla w_n^\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}(-x\cdot\vec{n})dS \\
&\leq \begin{cases} \displaystyle\int_{\partial\Omega\cap B_{2R}} \varepsilon^{\frac{p}{2}}(-x\cdot\vec{n}(x))dS & \text{if } 1 < p \leq 2, \\ \displaystyle\frac{p-2}{p}\int_{\partial\Omega\cap B_{2R}} (|\nabla w_n^\varepsilon|^2+\varepsilon)^{\frac{p}{2}}(-x\cdot\vec{n}(x))dS & \\ \displaystyle+\frac{2}{p}\int_{\partial\Omega\cap B_{2R}} \varepsilon^{\frac{p}{2}}(-x\cdot\vec{n}(x))dS & \text{if } 2 < p. \end{cases} \qquad (16)
\end{aligned}
$$

Now, let $\varepsilon \to 0$ in (11), then (10) is derived from (6), (12)–(15), and (16). □

Now we are ready to introduce our "Pohozaev-type inequality", which is formulated in terms of solutions w_n^ε of approximate equations $(\mathrm{E})_n^\varepsilon$.

Theorem 3.3 *Let Ω be same as in Theorem 2.1 and put*

$$\mathcal{P} = \{u \in L^q(\Omega); \nabla u \in (L^p(\Omega))^N, u|_{\partial\Omega} = 0, |u|^{q-1} \in L^{p'}(\Omega_R), \text{ for all } R > 0\},$$

where $\Omega_R = \Omega \cap B_R$. Then every solution of (E) *belonging to $\mathcal{P}$ satisfies the following Pohozaev-type inequality:*

$$\left(\frac{N}{q} + \frac{p-N}{p}\right) \int_\Omega |u|^q dx + \mathcal{R} \leq 0, \tag{17}$$

where

$$\mathcal{R} = \overline{\lim_{R\to\infty}}\ \overline{\lim_{n\to\infty}}\ \overline{\lim_{\varepsilon\to 0}} \frac{p-1}{p} \int_{\partial\Omega\cap B_{2R}} (|\nabla w_n^\varepsilon|^2 + \varepsilon)^{\frac{p}{2}} (-x \cdot \vec{n}) \Psi_R(r) dS,$$

and w_n^ε is the solution of $(\mathrm{E})_n^\varepsilon$ *uniquely determined by u.*

Proof By virtue of the fact that $u_n(x) \to u(x)$, $|u_n(x)| \leq |u(x)|$ for a.e. $x \in \Omega$, and $x_i|u|^{q-2}u \in L^{p'}_{loc}(\overline{\Omega})$, we note

$$x_i \Psi_R(|x|)|u_n|^{q-2}u_n \to x_i \Psi_R(|x|)|u|^{q-2}u \text{ strongly in } L^{p'}(\Omega) \text{ as } n \to \infty.$$

Hence we find

$$\begin{aligned} -2\int_{\Omega_n} |\underline{u_n}|^{q-2}\underline{u_n}\Psi_R(r)x \cdot \nabla w_n dx &\to -2\int_\Omega |u|^{q-2}u\Psi_R(r)x \cdot \nabla u dx \\ &= \frac{2N}{q}\int_\Omega |u|^q\Psi_R(r)dx + \frac{2}{q}\int_\Omega |u|^q r\Psi_R'(r)dx. \end{aligned}$$

as $n \to \infty$.

Since $\tilde{w}_n$, the zero extension of w_n to Ω, converges strongly to u in $L^q(\Omega)$ and $\nabla\tilde{w}_n$ to ∇u in $(L^p(\Omega))^N$, by (4) of Lemma 3.1, we can repeat the same verifications as for (12)–(15), with w_n^ε, w_n and Ω_n replaced by $\tilde{w}_n$, u and Ω. Consequently, by letting $n \to +\infty$ in (10), we obtain

$$\frac{N}{q}\int_\Omega |u|^q\Psi_R(r)dx + \frac{p-N}{p}\int_\Omega |\nabla u|^p\Psi_R(r)dx + I_R + \overline{\lim_{n\to\infty}}\,\mathcal{R}_n \leq 0, \tag{18}$$

$$I_R = \frac{1}{q}\int_\Omega |u|^q r\Psi_R'(r)dx - \frac{1}{p}\int_\Omega |\nabla u|^p r\Psi_R'(r)dx + \int_\Omega |\nabla u|^{p-2}(x\cdot\nabla u)^2\frac{\Psi_R'(r)}{r}dx.$$

Hence it easily follows from the fact that $u \in L^q(\Omega)$, $|\nabla u| \in L^p(\Omega)$, $|\Psi_R'(r)| \leq \dfrac{C}{R}$ and supp $\Psi_R'(r) \subset \{x \in \Omega; R \leq |x| \leq 2R\}$, that $I_R \to 0$ as $R \to 0$. Then to complete the proof, it suffices to let $R \to \infty$ in (18) and use the relation $\|\nabla u\|_{L^p}^p = \|u\|_{L^q}^q$. □

As for the case where $\Omega = \Omega_d \times \mathbb{R}^{N-d}$ (cylindrical domain), we can repeat the same argument as above and derive the following result.

Theorem 3.4 *Let $\Omega = \Omega_d \times \mathbb{R}^{N-d}$, then every solution of* (E) *belonging to $\mathcal{Q}^i$ satisfies the following Pohozaev-type inequality.*

$$\left(\frac{N-p}{p} - \frac{N}{q}\right) \int_\Omega |u|^q dx + \mathcal{R} \leq 0, \tag{19}$$

where

$$\mathcal{R} = \overline{\lim_{R\to\infty}}\ \overline{\lim_{n\to\infty}}\ \overline{\lim_{\varepsilon\to 0}} \frac{p-1}{p} \int_{\partial\Omega\cap B_{2R}} (|\nabla w_n^\varepsilon|^2 + \varepsilon)^{\frac{p}{2}} (x \cdot \vec{n}) \Psi_R(r) dS.$$

4 Proofs of Main Theorems

In this section, we complete the proofs of Theorems 2.1 and 2.2.

Proof of Theorem 2.1 Regularity results in [3] assures that if $q \leq p^*$, then $\mathcal{Q}^e$ is contained in $\mathcal{P}$. Therefore every solution u in $\mathcal{Q}^e$ enjoys the Pohozaev-type inequality (17). Since $\mathcal{R} \geq 0$, (17) with $q < p^*$ (equivalently $N/q + (p-N)/p > 0$), implies that $\|u\|_{L^q} = 0$. Thus the first assertion is verified.

As for the critical case $q = p^*$, we need further delicate arguments. At first, by virtue of Theorem 1.1 of [7, Ladyzhenskaya & Ural'ceva, p.251] and a comparison theorem (see [10, Ôtani & Teshima, Lemma 3]), we find that $w_n(x) \geq 0$ for a.e. $x \in \Omega_n$, whence follows $w_{n+1}|_{\partial\Omega_n} \geq 0 = w_n|_{\partial\Omega_n}$. Hence again by applying the comparison theorem in Ω_n, we observe that $w_{n+1}(x) \geq w_n(x)$ for a.e. $x \in \Omega_n$ Consequently it follows that $\tilde{w}_{n+1}(x) \geq \tilde{w}_n(x)$ in Ω, and then

$$\tilde{w}_n(x) \uparrow u(x) \quad \text{for a.e. } x \in \Omega. \tag{20}$$

Moreover the Harnack principle (see [13, Trudinger]) assures that

$$u(x) \geq w_n(x) > 0 \quad \text{for a.e. } x \in \Omega_n. \tag{21}$$

On the other hand, (17) with $q = p^*$ implies that $\mathcal{R} = 0$. Then for any $\eta > 0$, there exist R_0, N_0 and $\varepsilon_0 > 0$ such that

$$\int_{\partial\Omega\cap B_R} (|\nabla w_n^\varepsilon|^2 + \varepsilon)^{\frac{p}{2}} (-x \cdot \vec{n}) dS < \eta \quad \text{for all } R \geq R_0\ n \geq N_0 \text{ and } \varepsilon \in (0, \varepsilon_0). \tag{22}$$

Since Ω is an exterior of a cylindrical domain, there exist a positive number ρ and a relatively open subset $\Gamma_0 \subset \partial\Omega$ such that $(-x \cdot \vec{n}) \geq \rho > 0$ on $\overline{\Gamma_0}$.
Therefore it follows from (22) that

$$\int_{\Gamma_0} |\nabla w_n^\varepsilon|^p dS < \frac{\eta}{\rho} \quad \text{for all } n \geq N_0 \text{ and } \varepsilon \in (0, \varepsilon_0). \tag{23}$$

Then, by the same argument based on barrier functions

$$v(x) = \alpha(3\ell - r)^{\delta} - \alpha\ell^{\delta} \ (\alpha, \ell, \delta : \text{positive parameters})$$

as in [3] and have

$$(\alpha\delta\ell^{\delta-1})^p|\Gamma_0| \le \int_{\Gamma_0} \varliminf_{\varepsilon\to 0} |\nabla w_n^{\varepsilon}(x)|^p dS \le \varliminf_{\varepsilon\to 0} \int_{\Gamma_0} |\nabla w_n^{\varepsilon}(x)|^p dS < \frac{\eta}{\rho},$$

which leads to a contradiction. □

Proof of Theorem 2.2 The first assertion is a direct consequence of (19) in Theorem 3.4, since $\mathcal{R} \ge 0$.

For the critical case $q = p^*$, we can repeat the same argument as above. (Since Ω is a cylindrical domain, there exist a positive number ρ and a relatively open subset $\Gamma_0 \subset \partial\Omega$ such that $(x \cdot \vec{n}) \ge \rho > 0$ on $\overline{\Gamma_0}$.) □

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