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CLASSIFICATION OF THE LOCAL SHADOWS OF MOVING SURFACES

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ABSTRACT. We classify the bifurcation of generic local pictures of shadows for one-parameter families of surfaces in the Euclidean 3-space.

§1. INTRODUCTION

In this paper we consider the problem: how dose the bifurcation of shadows for moving surface look like ?

One of the motivations for the study of the shadows of surfaces is given in Vision Theory([4],[9]). In [9], Lions et. al. studied the so-called Shape-from-Shading problem. This problem corresponds, roughly speaking, to the reconstruction of a shape (a surface) from the brightness of the two-dimensional image.

Firstly they considered that shape of the surface is related to the image brightness by the Horn image irradiance equation (see Horn [5], chap. 10) which relates the brightness of the image $I(y_1, y_2)$ to the reflectance

$$R(n) = I(y_1, y_2) \tag{0.1}$$

where R is the reflectance map which specifies the reflectance of a surface as a function of its orientation (or unit normal) n. The reflectance depends in general on the reflectance properties of the surface and on the distribution of light sources.

If the surface is given locally by $x = u(y_1, y_2)$, the equation (0.1) is written explicitly in terms of the unknown function u (see Lions. et. al [9]). Here, we only describe a simple example of this general class of equations. In the case of single vertical light source, the equation (0.1) becomes

$$(1 + |\nabla u|^2)^{-\frac{1}{2}} = I(y_1, y_2) \tag{0.2}$$

where $\nabla u = (\frac{\partial u}{\partial y_1}, \frac{\partial u}{\partial y_2})$ and $y = (y_1, y_2)$, $|\nabla u|$ denotes the Euclidian norm of ∇u. The equation (0.2) is a Hamilton-Jacobi equation. They studied the equation (0.2) as an application of the theory of viscosity solutions for various kinds of boundary value problems. The boundary in these problems was considered as the edge of the shadows of a surface.

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However, they only considered this problem for the simple boundaries. For the detailed study, we need to classify the local shape of shadows of surfaces. A classification of the shadows of generic submanifolds with codimension 1 in $\mathbb{R}^{n+1}$ was given by O. A. Platonova [12]. The result is generalized to classifications of the shadows of generic submanifolds in $\mathbb{R}^{n+1}$ with an arbitrary codimension by K. Watanabe [14].

In this paper we shall study the normal forms of shadows of one parameter families of surfaces and illustrate how shadows of surfaces change when surfaces move along one parameter in $\mathbb{R}^3$.

Let $\mathbb{R}^3$ be the Euclidian space with coordinate (x, y_1, y_2). The subset G in $\mathbb{R}^2$ is called the shadow of a surface H in $\mathbb{R}^3$, if G is the image of projection π along a certain direction (for example, x-axis), where

$$\pi : \mathbb{R}^3 \to \mathbb{R}^2$$

is given by

$$\pi(x, y_1, y_2) = (y_1, y_2).$$

Let H be a closed surface in $\mathbb{R}^3$. We shall denote the set of embeddings from H to $\mathbb{R}^3$ by

$$Emb(H, \mathbb{R}^3) = \{i : H \hookrightarrow \mathbb{R}^3 |\ i \text{ is an embedding}\}$$

which is a Borel-space if we adopt the Whitney topology. We consider the following set

$$\mathcal{P} = \{e : H \times I \hookrightarrow \mathbb{R}^3 \times \mathbb{R} | e(p,t) = (i_t(p), t),\ i_t \in Emb(H, \mathbb{R}^3)\},$$

where I is an open interval in $\mathbb{R}$ which contains the origin. For any $e \in \mathcal{P}$, e is regarded as a family of elements of $Emb(H, \mathbb{R}^3)$ with a parameter t, and the image $e(H \times I)$ is a 3-dimensional submanifold in $\mathbb{R}^3 \times \mathbb{R}$.

We suppose that the moving surfaces have the shadow in $\mathbb{R}^2 \times \mathbb{R}$. For any $e \in \mathcal{P}$, the image of $\Pi \circ e$ is called *a shadow* of e, where $\Pi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^2 \times \mathbb{R}$ is the canonical projection defined by

$$\Pi(x, y_1, y_2, t) = (y_1, y_2, t).$$

Our purpose in this paper is local classification of the bifurcation of the image of $\Pi \circ e$ along the parameter t under the parameterized diffeomorphisms. The precise definition is given as follows.

Definition 1.1. Let D and D' be set germs in $(\mathbb{R}^2 \times \mathbb{R}, 0)$. We say that D and D' are *t-diffeomorphic* if there exist diffeomorphism germs $\hat{\Phi} : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ and $\hat{\phi} : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $\hat{\Phi}(D) = D'$ and $\pi_t \circ \hat{\Phi} = \hat{\phi} \circ \pi_t$, where $\pi_t : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is the projection to the second components.

Under the above notation, we define $D_t = D \cap (\mathbb{R}^2 \times \{t\})$ and $D'_t = D' \cap (\mathbb{R}^2 \times \{t\})$. If D and D' are $t-$diffeomorphic, then $\hat{\Phi}(D_t) = {D'}_{\hat{\phi}(t)}$, that is the bifurcations of $\{D_t\}_{t \in (\mathbb{R},0)}$ and $\{D'_t\}_{t \in (\mathbb{R},0)}$ along the parameter t are diffeomorphic. Our main result in this paper is the following theorem.

Theorem A. *There exists a residual subset $\mathcal{Q} \subset \mathcal{P}$ with the following property: For any $e \in \mathcal{Q}$ and for any point Y_0 of the shadow $\Pi \circ e(H \times I)$, the set germ of the shadow at Y_0 is $t-$diffeomorphic to one of the set germ in the following list:*

$r = 1$

${}^pG_k$	normal forms of set germs of the shadows
${}^0G_0$ ${}^0G_2$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \| y_i \in \mathbb{R}\}$
${}^0G_1$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \| y_1 \leqslant 0\}$
${}^1G_2^+$ ${}^1G_2^-$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \| y_i \in \mathbb{R}\}$
${}^1G_3$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \| 27y_2^4 - 256y_1^3 - 144y_1y_2^2t$ $+4y_2^2t^3 - 16y_1t^4 + 128y_1^2t^2 \leqslant 0\}$

The above classification of shadows is obtaines via a classification of defining functions of embedded surfaces $e(H \times I)$. (See Theorem 2.3. See also Proposition 2.2). The notation ${}^pG_k^{(\pm)}$ for the normal forms of shadows is named after the notation ${}^pA_k^{(\pm)}$ for the normal forms of the defining functions. Therefore Theorem A gives informations about not only the shadows but also the locations of the embedded surfaces $e(H \times I)$ from which the shadows come. The idea of the proof of Theorem A is summarized as follows: Since the image of e is a hypersurface in $\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$, it may be locally considered as a zero point set of a submersion $F : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$. We apply Zakalyukin's classifications([15]) among such function germs up to a certain equivalence relation, which preserves the bifurcation of shadows. We can translate such a classification into the classification of $\Pi_F : (F^{-1}(0), 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ which corresponds to the local classification of $\Pi \circ e$ around a point. After that we apply the Thom's transversality theorem to detect the generic condition on e.

In §2, we study the local properties of submanifold $e(H \times I)$ around a single point. In §3, we give a proof of generic property of Theorem A.

All map germs considered here are differentiable of class C^∞, unless stated otherwise.

2. CLASSIFICATION OF THE LOCAL SHADOWS

In this section we prepare some local theory for the study of shadows.

Let $e \in \mathcal{P}$. For any $(p_0, t_0) \in H \times I$, since $e(H \times I)$ is a 3-dimensional submanifold in $\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$, it follows from the implicit function theorem that there exists a small neighborhood U of $e(p_0, t_0)$ in $\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$ and a function $F : U \to \mathbb{R}$ such that $F|_{U \cap \mathbb{R} \times \mathbb{R}^2 \times \{t_0\}}$ is a submersion and

$$F^{-1}(0) = U \cap \ e(H \times I).$$

We call F *a local equation* of e at $e(p_0, t_0)$.

Since we consider the local theory, It suffices to study submersion $F : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ at the origin.

Definition 2.1. Let $F, F' : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be function germs. We say that F and F' are $t-(P-\mathcal{K})-$*equivalent* if there exists a diffeomorphism germ

$$\Phi : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0)$$

of the form

$$\Phi(x, y_1, y_2, t) = (\phi_1(x, y_1, y_2, t), \phi_2(y_1, y_2, t), \phi_3(t))$$

such that

$$\Phi^* < F >_{\mathcal{E}_{(x,y_1,y_2,t)}} = < F' >_{\mathcal{E}_{(x,y_1,y_2,t)}},$$

where $\mathcal{E}_{(x,y_1,y_2,t)}$ denotes the ring consisting of function germs $(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$.

We remark that the following diagram commutes:

$$\begin{array}{ccc}
(\mathbb{R}, 0) & & (\mathbb{R}, 0) \\
{\scriptstyle F}\uparrow & & \uparrow{\scriptstyle F'} \\
(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) & \xrightarrow{\Phi} & (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \\
{\scriptstyle \Pi}\downarrow & & \downarrow{\scriptstyle \Pi} \\
(\mathbb{R}^2 \times \mathbb{R}, 0) & \xrightarrow{(\phi_2,\phi_3)} & (\mathbb{R}^2 \times \mathbb{R}, 0) \\
{\scriptstyle \pi_t}\downarrow & & \downarrow{\scriptstyle \pi_t} \\
(\mathbb{R}, t_0) & \xrightarrow{\phi_3} & (\mathbb{R}, t_0')
\end{array}$$

It is clear that $(\phi_2, \phi_3) : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ and $\phi_3 : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ are the diffeomorphisms.

Similarly we may define the $t-(P-\mathcal{K})-$ *equivalence* for function germs at arbitrary base points. We have the following proposition.

Proposition 2.2. *Let $F, F' : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be function germs. If F, F' are $t-(P-\mathcal{K})-$equivalent then $\Pi(F^{-1}(0))$ and $\Pi({F'}^{-1}(0))$ are $t-$diffeomorphic.*

Proof. By definition, there exists a diffeomorphism germ $\Phi = (\phi_1, \phi_2, \phi_3)$, such that

$$< F' \circ \Phi >_{\mathcal{E}_{(x,y_1,y_2,t)}} = < F >_{\mathcal{E}_{(x,y_1,y_2,t)}},$$

so that $F^{-1}(0) = \Phi^{-1}({F'}^{-1}(0))$. By the commutative diagram, we obtain

$$(\phi_2, \phi_3)(\Pi(F^{-1}(0))) = \Pi({F'}^{-1}(0)).$$

Set $\hat{\Phi} = (\phi_2, \phi_3)$ and $\hat{\phi} = \phi_3$, then we have $\hat{\Phi}(\Pi(F^{-1}(0))) = \Pi({F'}^{-1}(0))$ and $\pi_t \circ \hat{\Phi} = \hat{\phi} \circ \pi_t$, where $\pi_t : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is the projection to the second component. □

For the local case, by Proposition 2.2, it is sufficient to consider the *local shadows* of local equations F, that is, the image of $\Pi_F = \Pi|_{F^{-1}(0)} : (F^{-1}(0), 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$. For $f = F|_{\mathbb{R} \times \mathbb{R}^2 \times \{0\}}$, we consider the subspaces of $\mathcal{E}_{(x,y_1,y_2)}$ given by

$$T_e(P-\mathcal{K})(f) = \langle \frac{\partial f}{\partial x}, f \rangle_{\mathcal{E}_{(x,y_1,y_2)}} + \langle \frac{\partial f}{\partial y_1}, \frac{\partial f}{\partial y_2} \rangle_{\mathcal{E}_{(y_1,y_2)}}.$$

We also consider its codimensions

$$(P-\mathcal{K})_e - cod(f) = dim_{\mathbb{R}}\mathcal{E}_{(x,y_1,y_2)} \,/\, T_e(P-\mathcal{K})(f).$$

Let $F : (\mathbb{R}\times\mathbb{R}^2\times\mathbb{R},0) \to (\mathbb{R},0)$ be a function germ, we say that F is a $(P-\mathcal{K})-versal\ \ deformation$ of $f = F|_{\mathbb{R}\times\mathbb{R}^2\times\{0\}} : (\mathbb{R}\times\mathbb{R}^2\times\{0\},0) \to (\mathbb{R},0)$ if

$$\langle \frac{\partial F}{\partial t}\,|_{t=0}\rangle_{\mathbb{R}} + T(P-\mathcal{K})_e(f) = \mathcal{E}_{(x,y_1,y_2)}.$$

In [8], Zakalyukin's classification theorem is developed to the following theorem which is useful for classification of local equations.

Theorem 2.3. *Let $F : (\mathbb{R}\times\mathbb{R}^n\times\mathbb{R},0) \to (\mathbb{R},0)$ be a function germ with $(P-\mathcal{K})_e - cod(f) \leqslant 1$, where $f = F|_{\mathbb{R}\times\mathbb{R}^n\times\{0\}}$. If F is $(P-\mathcal{K})-versal$ deformation of f, then F is $t-(P-\mathcal{K})-$equivalent to one of the germs in the following list:*

${}^0A_k$:	$x^{k+1} + \sum_{i=1}^{k} y_i x^{i-1}$	$(0 \leqslant k \leqslant n)$
${}^1A_k$:	$x^{k+1} + x^{k-1}(t \pm y_k^2 \pm\cdots \pm y_n^2) + \sum_{i=1}^{k-1} y_i x^{i-1}$	$(2 \leqslant k \leqslant n+1)$

In the case $n = 2$, by Theorem 2.3, we have the following corollary.

Corollary 2.4. *Let $F : (\mathbb{R}\times\mathbb{R}^2\times\mathbb{R},0) \to (\mathbb{R},0)$ be a function germ with $(P-\mathcal{K})_e - cod(f) \leqslant 1$, where $f = F|_{\mathbb{R}\times\mathbb{R}^2\times\{0\}}$. If F is a $(P-\mathcal{K})-versal$ deformation of f, then F is $t-(P-\mathcal{K})-$equivalent to one of the following function germs:*

${}^0A_0 : x$
${}^0A_1 : x^2 + y_1$
${}^0A_2 : x^3 + xy_2 + y_1$
${}^1A_2^+ : x^3 + xy_2^2 + tx + y_1$
${}^1A_2^- : x^3 - xy_2^2 + tx + y_1$
${}^1A_3 : x^4 + xy_2 + tx^2 + y_1.$

We denote the shadow of ${}^pA_k^{(\pm)}$ by ${}^pG_k^{(\pm)}$. Then by Theorem 2.3 we also have the following corollary.

Corollary 2.5. *Let $F : (\mathbb{R}\times\mathbb{R}^2\times\mathbb{R},0) \to (\mathbb{R},0)$ be a function germ with $(P-\mathcal{K})_e - cod(f) \leqslant 1$, where $f = F|_{\mathbb{R}\times\mathbb{R}^2\times\{0\}}$. If F is a $(P-\mathcal{K})-versal$ deformation of f, then $\Pi(F^{-1}(0))$ is $t-$diffeomorphism to one of the set germs in the above list ${}^pG_k^{(\pm)}$ (See the following table).*

${}^pG_k$	normal forms of set germs of the shadows
${}^0G_0$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid y_i \in \mathbb{R}\}$
${}^0G_1$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid y_1 \leqslant 0\}$
${}^0G_2$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid y_i \in \mathbb{R}\}$
${}^1G_2^+$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid y_i \in \mathbb{R}\}$
${}^1G_2^-$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid y_i \in \mathbb{R}\}$
${}^1G_3$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid 27y_2^4 - 256y_1^3 - 144y_1y_2^2t$ $+4y_2^2t^3 - 16y_1t^4 + 128y_1^2t^2 \leqslant 0\}$

Remark. When $p = 1$ and $k = 3$, we observe that $x^4 + tx^2 + xy_2 + y_1$ is $t - (\mathcal{P} - \mathcal{K}) - equivalent$ to $x^4 - tx^2 + xy_2 + y_1$.

In order to study the generic properties of $e \in \mathcal{P}$ which respect to the local equation F at $e(p_0, t_0)$, we need some preparations.

Let $g : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a C^∞ germ. In [2], two types of codimensions of g are defined as follows:

$$(\mathcal{A}) - cod(g) = dim_{\mathbb{R}} \mathfrak{M}_2 \times \mathfrak{M}_2 \,/\, T(\mathcal{A})(g)$$

and

$$(\mathcal{A})_e - cod(g) = dim_{\mathbb{R}} \mathcal{E}_2 \times \mathcal{E}_2 \,/\, T_e(\mathcal{A})(g),$$

where

$$T(\mathcal{A})(g) = \mathfrak{M}_2 \langle \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2} \rangle_{\mathcal{E}_2} + g^*\mathfrak{M}_2 \times g^*\mathfrak{M}_2$$

and

$$T_e(\mathcal{A})(g) = \langle \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2} \rangle_{\mathcal{E}_2} + g^*\mathcal{E}_2 \times g^*\mathcal{E}_2.$$

Remark. $T(\mathcal{A})(g)$ and $T(\mathcal{A})_e(g)$ do not depend on the choice of the local coordinates on the source and the target.

In ([1],[8]), versality of deformations is defined as follows.

Let $G : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ be a $C^\infty-$ map germ and $g = G|_{\mathbb{R}^2 \times \{0\}} : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$. We say that G is an $\mathcal{A} - versal\ \ deformation$ of g if

$$\langle \frac{\partial G}{\partial t} \,|_{t=0} \rangle_{\mathbb{R}} + T(\mathcal{A})_e(g) = \mathcal{E}_2 \times \mathcal{E}_2.$$

We now consider a map germ

$$j_1^\ell G : (\mathbb{R}^2 \times \mathbb{R}, 0) \to J^\ell(\mathbb{R}^2, \mathbb{R}^2) \cong \mathbb{R}^2 \times \mathbb{R}^2 \times J^\ell(2, 2)$$

given by

$$j_1^\ell G(x, t) = j^\ell G_t(x).$$

Let $z = j^\ell g(0)$ and $L^\ell(2) \times L^\ell(2)(z)$ be the $\mathcal{A}-$ orbit through z in $J^\ell(2, 2)$ (See [2],[6],[7]). We can prove the following lemma.

Lemma 2.6. *Suppose that $g = G|_{t=0}$ is $\mathcal{A}-$finitely determined (i.e. $(\mathcal{A})_e - cod(g) < +\infty$). Under the above notations, for sufficiently large ℓ, the following conditions are equivalent.*
(i) $\tilde{\pi} \circ j_1^\ell G \,\overline{\pitchfork}\, (L^\ell(2) \times L^\ell(2))(z)$.
(ii) *G is an $\mathcal{A}-versal\ deformation$ of g,*
where, $\tilde{\pi} : \mathbb{R}^2 \times \mathbb{R}^2 \times J^\ell(2,2) \to J^\ell(2,2)$ is the canonical projection.

Let $F : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a function germ such that $f = F|_{\mathbb{R}\times\mathbb{R}^2\times\{0\}} : (\mathbb{R} \times \mathbb{R}^2 \times \{0\}, 0) \to (\mathbb{R}, 0)$ is a submersion germ. We consider the local projection $\Pi_F = \Pi|_{F^{-1}(0)} : (F^{-1}(0), 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$. and $\pi_f = \pi|_{f^{-1}(0)\times\{0\}} : (f^{-1}(0), 0) \to (\mathbb{R}^2 \times \{0\}, 0)$.
By the above remark, $T(\mathcal{A})(\pi_f)$ and $T(\mathcal{A})_e(\pi_f)$ are well-defined. Therefore $\mathcal{A}-$versality of deformation Π_F of π_f is also well-defined.
Under the above notations, we have the following proposition.

Proposition 2.7. *The following conditions are equivalent.*
(i) *F is a $(P - \mathcal{K}) - versal\ deformation$ of f.*
(ii) *$\pi_2 \circ \Pi_F$ is an $\mathcal{A} - versal\ deformation$ of π_f.*
Here $\pi_2 : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ is the canonical projection.

Proof. Since f is a submersion, we may suppose that $\frac{\partial F}{\partial y_1} \neq 0$ (for the case $\frac{\partial F}{\partial x} \neq 0$ or $\frac{\partial F}{\partial y_2} \neq 0$ are similar), then we may suppose that F has the form $F(x, y_1, y_2, t) = y_1 - h(x, y_2, t)$, for some function $h : (\mathbb{R} \times \mathbb{R} \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ and $f(x, y_1, y_2) = F(x, y_1, y_2, 0) = y_1 - h_0(x, y_2)$, where $h_0(x, y_2) = h(x, y_2, 0)$. Define $G_F : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ by $G_F(x, y_2, t) = (h(x, y_2, t), y_2)$ and $g_f(x, y_2) = (h_0(x, y_2), y_2)$. Then $G_F = \pi_2 \circ \Pi_F$ and $g_f = \pi_f$. We consider the map germ $I_{h_0} : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ defined by

$$I_{h_0}(x, y_2) = (x, h_0(x, y_2), y_2)$$

and we also consider the *pull-back* homomorphism

$$I_{h_0}^* : \mathcal{E}_{(x,y_1,y_2)} \to \mathcal{E}_{(x,y_2)}.$$

Then $ker I_{h_0}^* = \langle y_1 - h_0(x, y_2)\rangle_{\mathcal{E}_{(x,y_1,y_2)}}$ and

$$I_{h_0}^*(T(P - \mathcal{K})_e(f)) = \langle\frac{\partial h_0}{\partial x}\rangle_{\mathcal{E}_{(x,y_2)}} + \langle 1, \frac{\partial h_0}{\partial y_2}\rangle_{I_{h_0}^*\mathcal{E}_{(y_1,y_2)}} \tag{4}$$

We now verify the following equality

$$\mathcal{E}_{(x,y_2)} \times \{0\} \cap T(\mathcal{A})_e(g_f) = \langle(\frac{\partial h_0}{\partial x}, 0)\rangle_{\mathcal{E}_{(x,y_2)}} + \langle(1, 0), (\frac{\partial h_0}{\partial y_2}, 0)\rangle_{I_{h_0}^*\mathcal{E}_{(y_1,y_2)}} \tag{5}$$

By the definition of $T(\mathcal{A})_e(g_f)$ and the equality (4), we may assume that any $(\zeta, 0) \in \mathcal{E}_{(x,y_2)} \times \{0\} \cap T(\mathcal{A})_e(g_f)$ has the form

$$(\zeta, 0) = (\xi\frac{\partial h_0}{\partial x}, 0) + (\lambda\frac{\partial h_0}{\partial y_2}, \lambda) + (I_{h_0}^*\eta_1, I_{h_0}^*\eta_2)$$

for some $\eta_1, \eta_2 \in \mathcal{E}_{(y_1,y_2)}$ and $\xi, \lambda \in \mathcal{E}_{(x,y_2)}$. Hence $(\zeta, 0) = (\xi\frac{\partial h_0}{\partial x} - (I^*_{h_0}\eta_2)\frac{\partial h_0}{\partial y_2} + (I^*_{h_0}\eta_1)\cdot 1, 0) \in \langle(\frac{\partial h_0}{\partial x}, 0)\rangle_{\mathcal{E}_{(x,y_2)}} + \langle(\frac{\partial h_0}{\partial y_2}, 0), (1,0)\rangle_{I^*_{h_0}\mathcal{E}_{(y_1,y_2)}}$, that is $(\zeta, 0) \in$ the right hand side of (5). The converse can be verified similarly, so we omit its proof.

By (4) and (5), we have

$$\mathcal{E}_{(x,y_2)} \times \{0\} \cap T(\mathcal{A})_e(g_f) = \langle(\frac{\partial h_0}{\partial x}, 0)\rangle_{\mathcal{E}_{(x,y_2)}} + \langle(1,0), (\frac{\partial h_0}{\partial y_2}, 0)\rangle_{I^*_{h_0}\mathcal{E}_{(y_1,y_2)}}$$
$$= I^*_{h_0}(T(P-\mathcal{K})_e(f) \times \{0\}$$

Then

$$I^*_{h_0}T(P-\mathcal{K})_e(f) \cong I^*_{h_0}T(P-\mathcal{K})_e(f) \times \{0\} = \mathcal{E}_{(x,y_2)} \times \{0\} \cap T(\mathcal{A})_e(g_f),$$

and $I^*_{h_0}$ induces an $\mathbb{R}$-isomorphism:

$$\mathcal{E}_{(x,y_1,y_2)} \,/\, T(P-\mathcal{K})_e(f) \cong \mathcal{E}_{(x,y_2)} \times \{0\} \,/\, \mathcal{E}_{(x,y_2)} \times \{0\} \cap T(\mathcal{A})_e(g_f).$$

On the other hand, since $g_f(x, y_2) = (h_0(x, y_2), y_2)$, it is clear that

$$\mathcal{E}_{(x,y_2)} \times \mathcal{E}_{(x,y_2)} = \mathcal{E}_{(x,y_2)} \times \{0\} + T(\mathcal{A})_e(g_f).$$

Then

$$\begin{aligned}
&\mathcal{E}_{(x,y_1,y_2)} \,/\, T(P-\mathcal{K})_e(f) \\
&\cong \mathcal{E}_{(x,y_2)} \times \{0\} \,/\, T(\mathcal{A})_e(g_f) \cap \mathcal{E}_{(x,y_2)} \times \{0\} \\
&\cong \mathcal{E}_{(x,y_2)} \times \{0\} + T(\mathcal{A})_e(g_f) \,/\, T(\mathcal{A})_e(g_f) \\
&= \mathcal{E}_{(x,y_2)} \times \mathcal{E}_{(x,y_2)} \,/\, T(\mathcal{A})_e(g_f).
\end{aligned}$$

If $\mathcal{E}_{(x,y_1,y_2)} = T(P-\mathcal{K})_e(f)$, by the above equality we have $\mathcal{E}_{(x,y_2)} \times \mathcal{E}_{(x,y_2)} = T(\mathcal{A})_e(g_f)$. Hence (i) holds if and only if (ii) holds. On the other hand, since

$$\frac{\partial G_F}{\partial t}\,|_{t=0} = (\frac{\partial h}{\partial t}\,|_{t=0}, 0) \in \mathcal{E}_{(x,y_2)} \times \mathcal{E}_{(x,y_2)}$$

and

$$\frac{\partial F}{\partial t}\,|_{t=0} = -\frac{\partial h}{\partial t}\,|_{t=0} \in \mathcal{E}_{(x,y_2)},$$

the condition

$$dim_{\mathbb{R}}\mathcal{E}_{(x,y_1,y_2)} \,/\, T(P-\mathcal{K})_e(f) = 1$$

is equivalent to

$$dim_{\mathbb{R}}\mathcal{E}_{(x,y_2)} \times \mathcal{E}_{(x,y_2)} \,/\, T(\mathcal{A})_e(g_f) = 1.$$

In this case, F is a $(P-\mathcal{K})-$versal deformation of f if and only if $\frac{\partial F}{\partial t}\,|_{t=0} \notin T(P-\mathcal{K})_e(f)$. Moreover

$$I^*_{h_0}(\frac{\partial F}{\partial t}\,|_{t=0}) = I^*_{h_0}(\frac{-\partial h}{\partial t}\,|_{t=0}) = (\frac{-\partial h}{\partial t}\,|_{t=0}, 0) = \frac{-\partial G_F}{\partial t}\,|_{t=0}$$

so that

$$\frac{\partial F}{\partial t}\,|_{t=0} \notin T(P-\mathcal{K})_e(f) \text{ if and only if } \frac{-\partial G_F}{\partial t}\,|_{t=0} \notin T(\mathcal{A})_e(g_f).$$

The last condition is equivalent to G_F is an $\mathcal{A}-$versal deformation of g_f.

For the other case($\frac{\partial F}{\partial y_2} \neq 0$ or $\frac{\partial F}{\partial x} \neq 0$), the proof is similar. □

3. Generic property of shadows of the moving surface

In this section we use Thom's k-transversal theorem to show generic property of shadows of the moving surface, that is, we shall prove the following Theorem.

Theorem 3.1. *There exists a dense subset $\mathcal{Q} \subset \mathcal{P}$ such that for any $e \in \mathcal{Q}$ and $(p_0, t_0) \in H \times I$, the set germ of the shadow of $e(H \times I)$ at $\Pi \circ e(p_0, t_0)$ is t-diffeomorphic to one of the following normal forms ${}^pG_k^{(\pm)}$:*

$$^pG_k^{(\pm)} = \{(y_1, y_2, t) \in (\mathbb{R}^2 \times \mathbb{R}, 0) | x^{k+1} + \sum_{i=1}^{k-1} y_i x^{i-1} + p x^{k-1}(t \pm y_k^2)$$

$$+(1-p) y_k x^{k-1} = 0, \textit{for some } x \in (\mathbb{R}, 0)\}$$

where $p = 0, 1,$ and $2p \leqslant k \leqslant p+2$.

Proof. Take ℓ to be sufficiently large. Let $\hat{S}_j, j = 0, 1, 2$ or 3, be the set of jets $z = J^1(h)(0,0)$ of $J^\ell(2,2)$ with $(\mathcal{A}) - cod(h) = j$. Let Σ be the compliment of $\cup_{j=0}^3 \hat{S}_j$ in $J^\ell(2,2)$(That is, Σ is the *union* of jets $j^\ell(h)$ with $(\mathcal{A}) - cod(h) \geqslant 4$). Then we have

$$J^\ell(2,2) = \hat{S}_0 \cup \hat{S}_1 \cup \hat{S}_2 \cup \hat{S}_3 \cup \Sigma.$$

Now we consider the subsets $S_j = H^2 \times \mathbb{R}^2 \times \hat{S}_j$ in $J^\ell(H, \mathbb{R}^2)$. For any $e \in \mathcal{P}$, we define the $\ell - jet - extension$ map $j_1^\ell e : H \times I \to J^\ell(H, \mathbb{R}^3)$ given by

$$j_1^\ell e(p,t) := j^\ell(i_t(p)),$$

where $i_t = e|_{H \times \{t\}}$.

We also consider the projection ${}^\ell\pi : J^\ell(H, \mathbb{R}^3) \to J^\ell(H, \mathbb{R}^2)$ defined by

$${}^\ell\pi(j^\ell h(x)) = j^\ell(\Pi \circ h(x))$$

for $h : (H, p_0) \to (\mathbb{R}^3, h(p_0))$ and $\Pi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$.

Since ${}^\ell\pi$ is a submersion and $S_j (j = 0,1,2,3)$ are submanifolds of $J^\ell(H, \mathbb{R}^2)$, ${}^\ell\pi^{-1}(S_j)$ are submanifolds in $J^\ell(H, \mathbb{R}^3)$ and

$$\text{codim of } S_j = \text{codim of } {}^\ell\pi^{-1}(S_j) \qquad (j = 0,1,2,3).$$

Moreover, we can show that

$$j_1^\ell(e) \pitchfork {}^\ell\pi^{-1}(S_j) \text{ if and only if } j_1^\ell(\Pi \circ e) \pitchfork S_j.$$

Set

$$\hat{\mathcal{Q}}_j := \{e \in \mathcal{P} |\ j_1^\ell(e) \pitchfork {}^\ell\pi^{-1}(S_j)\}, \quad (j = 0,1,2,3).$$

and

$$\mathcal{Q}_\Sigma := \{e \in \mathcal{P} | j_1^\ell(e) \cap {}^\ell\pi^{-1}(\Sigma) = \phi\},$$

By [13], $\mathcal{Q}_\Sigma$ is an algebraic subset of $J^\ell(2,2)$ of codimension $\geqslant 4$.

It follows from Thom's $k-$ transversal Theorem(See [3],[11]) that $\hat{\mathcal{Q}}_j$ are residual subsets of $\mathcal{P}$.

Finally we set

$$\mathcal{Q} = (\cap_{j=0}^{3} \hat{\mathcal{Q}}_j) \cap \mathcal{Q}_\Sigma \subset \mathcal{P},$$

then $\mathcal{Q}$ is a residual subset in $\mathcal{P}$.

For any $e \in \mathcal{Q}$ and $(p_0, t_0) \in H \times I$, there exists a neighbourhood U of $e(p_0, t_0)$ and a local equation $F : (U, e(p_0, t_0)) \to (\mathbb{R}, 0)$ of e at $e(p_0, t_0)$, so that $F^{-1}(0) = U \cap e(H \times I)$. Without the loss of generality, $e(p_0, t_0)$ is assumed to be the origin, so that we consider a submersion germ $F : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$. Under the above notation, we may have the following identification:

$$j_1^\ell \Pi \circ e = j_1^\ell \pi_2 \circ \Pi_F.$$

Since $e \in \mathcal{Q}$, $j_1^\ell \pi_2 \circ \Pi_F$ is transversal to S_j. It follows from lemma 2.6, that $\pi_2 \circ \Pi_F$ is an $\mathcal{A}-$ *versal deformation* of f. Moreover, by the Proposition 2.7 F is $P - \mathcal{K}-$versal deformation of $f = F|_{\mathbb{R} \times \mathbb{R}^2 \times \{t_0\}}$. Hence we may apply Corollary 2.5 to get the result. □

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