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Generic knots in tight contact 3-manifolds

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0 Introduction

There are two well known notions of knots in contact 3-manifolds. A notion of Legendrian knots, which are tangent to the contact structure, is one of them. That of transversal knots, which are transversal to the contact structure, is the other. In this note, a notion of generic knots is defined. Generic knots are defined to be knots which are simple tangent to the contact structure at finite points. For each type of knots, there is the classification problem up to isotopy preserving each structure. They are more complicated than the topological knot theory. Only trivial Legendrian and transversal knots in tight contact manifolds are classified by Ya. Eliashberg ([E2]). In this note, we classify trivial generic knots in tight contact 3-manifolds.

A *contact structure* on a 3-manifold M is a completely non-integrable tangent plane field ξ. In other words, contact structure ξ is defined as a kernel of a 1-form α on M which satisfies $\alpha \wedge d\alpha \neq 0$ everywhere. This 1-form is called a *contact form.*

For an embedded surface F in contact manifold (M, ξ), ξ traces a singular foliation on F. That is called a *characteristic foliation* on F with respect to ξ, and we note it F_ξ. At singular points of F_ξ, ξ is tangent to F. When ξ and F are oriented, a singular point is called *positive* or *negative* depending on whether the orientation of them coincide at the point or not. Generically, singular points of F_ξ are isolated, finite, and the indices of them are ± 1. A singular point $p \in F$ is called *elliptic* if its index is $+1$, and *hyperbolic* if it is -1.

A contact structure ξ is called *tight* , if for any embedded disc D in (M, ξ) D_ξ never have limit cycle.

Knots mean embeddings of S^1 into M ; $f : S^1 \longrightarrow (M, \xi)$. A knot $f : S^1 \longrightarrow (M, \xi)$ is *Legendrian*, if the pull-back of a contact form α by f, we note $f^*\alpha$, vanishes for all points of S^1. A knot f is *transversal*, if $f^*\alpha$ never vanishes on S^1. We define that a knot f is *generic*, if $f^*\alpha$ vanishes on finite points of S^1 and they are simple zero.

For transversal knots a transversal isotopy invariant is defined (see [B], [E1], [E2]). Let Γ be a transversal knot in a contact manifold (M, ξ) which is homologue to zero. Fix a relative homology class $\beta \in H_2(M, \Gamma)$. Let F be a surface bounded by Γ which represents β ; $[F] = \beta \in H_2(M, \Gamma)$. Let ν be a vector field tangent to $\xi|_F$. Then ν is transversal to Γ and we can perturb Γ slightly along ν to a curve Γ'. We define $l(\Gamma|\beta)$ be the intersection number of Γ' and β. It is well defined and we call it the *self-linking number* of Γ with respect to β.

A knot is called topologically *trivial* , if there exists an embedded disc D in M whose boundary is the knot. In this note, we fix a relative homology class represented by this disc D ; $[D] \in H_2(M, \partial D)$, and self-linking numbers are considered with respect to $[D]$.

The main results of this note are the following.

Theorem A *Any generic trivial knot in tight contact 3-manifold is represented as a result of alternating connected summation of positive and negative transversal trivial knots.*

We suppose that the generic knot Γ has $2k$ non-transversal points. We may assume that there are $(k+1)/2$ positive and negative transversal knots if k is odd, there are $(k+2)/2$ positive and $k/2$ negative transversal knots if k is even. Let $l_g(\Gamma)$ be the summation of self-linking numbers of positive transversal knots minus the summation of that of negative ones. We call it the *essential self-linking number* of the generic knot Γ.(see Definition 3.2.)

Theorem B *Generic knots are classified by their numbers of non-transversal points and essential self-linking numbers.*

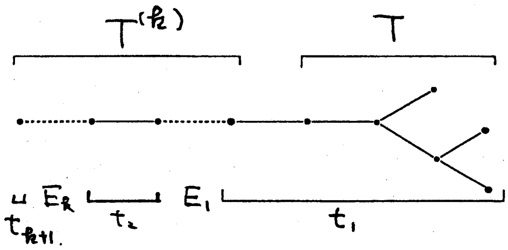

Figure 1:

Moreover, we obtain complete list of generic trivial knots in tight contact 3-manifolds. Self-linking numbers for transversal knots can take only negative odd integers. So, essential self-linking numbers for generic knots can take even integers if k is odd, and odd integers if k is even. Examples of generic knots which have these numbers of non-transversal points and essential self-linking numbers are constructed in the following.

1 Generic surroundings

Let $T^{(k)}$ be a linear tree (i.e., without branches) which has $2k$ vertices and alternate indicated edges $E_1, E_2, \ldots, E_k$. For an abstract tree T we define ${}^kT$ be the result of connection of $T^{(k)}$ and T with an edge. Subtrees between E_{i-1} and E_i are named t_i. (see Figure 1.)

Let (M,ξ) be a contact 3-manifold. An embedding $\alpha : {}^kT \to (M,\xi)$ is called *Legendrian* if the restriction of α to each edge of ${}^kT$ is Legendrian and the edges of the embedded tree $\widehat{{}^kT} := \alpha({}^kT)$ are not tangential at the embedded vertices. The embedded tree $\widehat{{}^kT}$ is also called Legendrian.

For any Legendrian tree $\widehat{{}^kT}$ in (M,ξ) there exists an embedded oriented surface $F \subset M$ which contains $\widehat{{}^kT}$ and satisfies the following conditions.

(a) Vertices of ${}^kT$ corresponds to elliptic points of F_ξ. Vertices of t_i is positive (resp., negative) if i is odd (resp., even).

(b) Each edge between two positive (resp., negative) elliptic points has exactly one negative (resp., positive) hyperbolic point of F_ξ.

(c) The edges $E_1, E_2, \ldots, E_k$ has no critical point F_ξ.

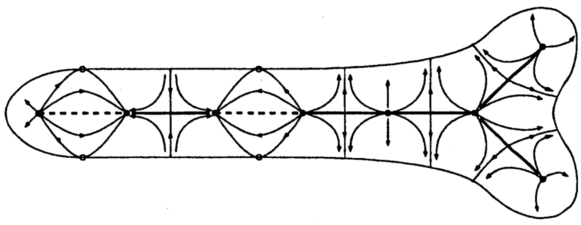

Figure 2:

A germ of the above surface F along $\widehat{^kT}$ is called the *thickening* of $\widehat{^kT}$. (see Figure 2.)

First we note the following simple facts.

Lemma 1.1 **([E2])** *Let T be a given abstract tree. The space of Legendrian embeddings of T is connected.*

Lemma 1.2 **([E2])** *Let $\widehat{^kT}$ be a given Legendrian tree. The space of thickenings of $\widehat{^kT}$ is connected.*

For a Legendrian tree $\widehat{^kT}$, there exists an arbitrary small neighborhood $U \subset F$ of $\widehat{^kT}$ whose boundary $\partial U =: \Gamma$ is transversal to F_ξ except $2k$-points. We call this generic trivial knot Γ the *generic surrounding* of the Legendrian tree $\widehat{^kT}$.

On account of Lemma 1.2, the generic isotopy class of Γ, which is the generic surrounding of $\widehat{^kT}$, is independent of the choice of a thickening, and therefore is an invariant of the Legendrian tree $\widehat{^kT}$. Moreover, according to Lemma 1.1, this class depends only on the tree ^kT and will be denoted by $\Gamma_{^kT}$.

Similarly, there exists a thickening of an abstract tree T which satisfies the above conditions (a), (b). So, we obtain a transversal knot Γ_T which depends only on T. The following is known for this transversal knot.

Lemma 1.3 **([E2])**

$$l(\Gamma) = 1 - 2|T| \; .$$

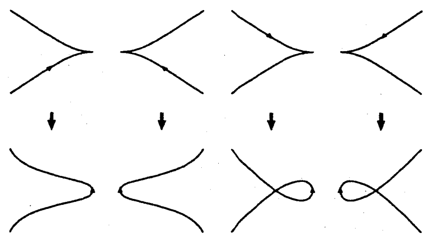

Figure 3:

2 Generic knot associated with a Legendrian tree Γ_{kT}

2.1 Front projections of transversals.

Let ξ_0 be the standard contact structure on $\mathbb{R}^3$: $\{dz - y \cdot dx = 0\}$. For a Legendrian curve L in $(\mathbb{R}^3, \xi_0)$, we call the image of the projection to the coordinate plane $\Pi := \{y = 0\}$ its *front* . Let us first observe the following fact. Let L be a Legendrian knot, and v be a vector field along L which direct normal to L in ξ_0.

Proposition 2.1 ([B], [E1]) *Perturbing L slightly to the direction of v or the opposite direction, we can make it positive or negative transversal to ξ_0.*

These transversal knots are independent, up to transversal isotopy, of the construction. They are denoted by $T_{\pm}(L)$.

We can chose a transversal knot $\tilde{\Gamma}$, up to transversal isotopy, in such a way that its projection Φ_+ onto the plane $\Pi := \{y = 0\}$ is different from the front of L (denote Φ) only near cusp points of Φ. Each cusp point in Φ is replaced in Φ_+ with immersed smooth curve as is shown in Figure 3.

2.2 Legendrian knot associated with a tree.

Let T be an abstract tree. Let $\alpha : T \to \Pi = \mathbb{R}^2$ be an embedding whose composition with the first projection increases monotonously on edges of T, and has exactly one minimum. With the embedded graph $\hat{T} := \alpha(T)$ we associate the front Φ_T of a Legendrian curve L_T as Figure 4. (i.e.,

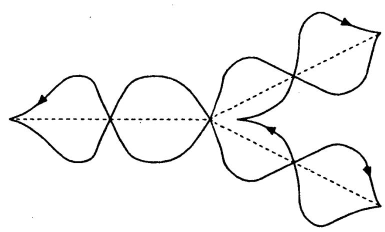

Figure 4:

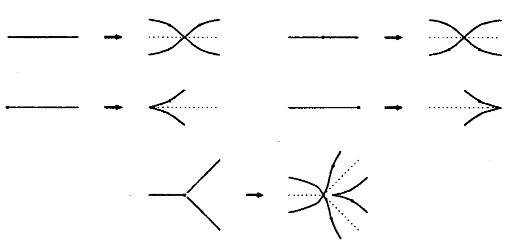

Figure 5:

Each edge of $\hat{T}$ is replaced with a pair of intersecting branches of Φ_T, and each vertex of T corresponds in Φ_T to the following 3 types of cusps and intersecting branches of Φ_T depending on there types as vertices. (see Figure 5.))

The following Lemma means that Legendrian knots L_T depend only on the number $|T|$ of vertices of T up to a Legendrian isotopy. Therefore, transversal knots $\tilde{\Gamma}_T := T_+(L_T)$ also depend only on $|T|$ up to a transversal isotopy.

Lemma 2.2 (Eliashberg-Fuchs [E2]) *If* $|T| = |T'|$ *then* L_T *is Legendrian isotopic to* $L_{T'}$.

2.3 Generic knot associated with a tree of type ${}^kT$.

For a generic type tree ${}^kT = T^{(k)} \# T = t_1 \cup E_1 \cup \cdots \cup E_k \cup t_{k+1}$, a generic knot with $2k$ non-transversal points is constructed in the following way. For each subtree t_i we give transversal knot $\tilde{\Gamma}_{t_i}$ constructed as the above section. Give each $\tilde{\Gamma}_{t_i}$, whose number i is even, the reversed orientation. Then they

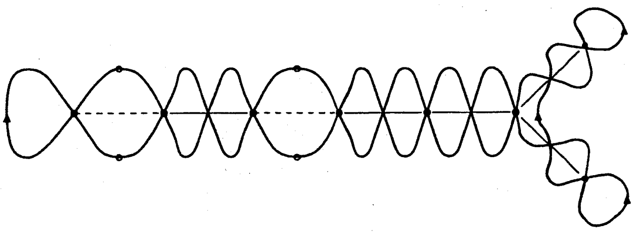

Figure 6:

are negative transversal. Next, taking proper connected summation of $\tilde{\Gamma}_{t_i}$ and $\tilde{\Gamma}_{t_{i+1}}$ near the space corresponding to E_i, we obtain a generic trivial knot with $2k$ non-transversal points. We denote it $\tilde{\Gamma}_{{}^kT}$. (see Figure 6.)

Taking a Seifert surface of $\tilde{\Gamma}_{{}^kT}$ properly, the characteristic foliation on it becomes as Figure 2. Then, the following proposition holds.

Proposition 2.3 *$\tilde{\Gamma}_{{}^kT}$ is isotopic to $\Gamma_{{}^kT}$ as generic knots.*

The main result of this section is the following.

Proposition 2.4 *Two generic knots of this type : $\Gamma_{{}^kT}$, $\Gamma_{{}^{k'}T'}$ are isotopic as generic knots if and only if $k = k'$ and $|T| = |T'|$.*

Proof. If $k = k'$, then $T^{(k)} = T^{(k')}$. By applying Lemma 2.2 to the parts of knots corresponding to T and T', $\Gamma_{{}^kT=T^{(k)}\#T}$ is isotopic to $\Gamma_{{}^{k'}T'=T^{(k')}\#T'}$ as generic knots, if $|T| = |T'|$.

Let us show the sufficient condition. Suppose that $\Gamma_{{}^kT}$ is isotopic to $\Gamma_{{}^{k'}T'}$ as generic knots. Their numbers of non-transversal points are $2k$ and $2k'$. On account of the definition, the number of non-transversal points is invariant under isotopy as generic knots. So, $k = k'$. According to Lemma 1.1 and Lemma 1.2, there is an isotopy from $\Gamma_{T^{(k)}}$ to $\Gamma_{T^{(k')}}$ preserving the characteristic foliation on the Seifert surface. New isotopy from $\Gamma_{{}^kT}$ to $\Gamma_{{}^{k'}T'}$ is given by exchanging the part of the given isotopy corresponding to $T^{(k)} = T^{(k')}$ with the above isotopy. This new isotopy induces a transversal isotopy from Γ_{t_1} to $\Gamma_{t'_1}$. (see Figure 7.)

As the self-linking number is a transversal isotopy invariant, $|T| = |t_1| - 1 = |t_2| - 1 = |T'|$, on account of Lemma 1.3. $\square$

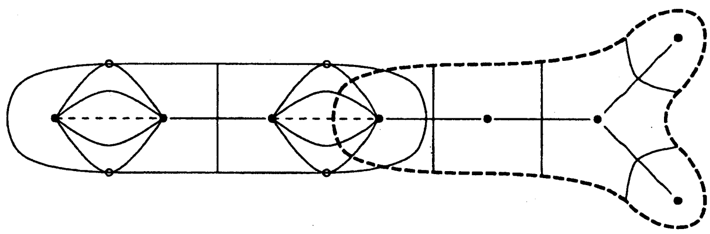

Figure 7:

3 Proof of Theorems

In this section, we show that for any generic knot Γ in tight contact 3-manifold (M, ξ) there exists a tree of type ${}^kT$ and Γ is isotopic to $\Gamma_{{}^kT}$ as generic knots. Last of all, we obtain the complete list of generic trivial knots in tight contact 3-manifolds.

3.1 Reduction to the $\Gamma_{{}^kT}$ type.

Let Γ be a generic trivial knot in a tight contact 3-manifold (M, ξ). Topologically trivial knot has a embedded disc $D \subset M$ which is bounded by Γ. We suppose that D is embedded generically (i.e., D_ξ has no separatrices connections). In this section we treat generic knots up to orientation.

The aim of this section is to observe the following proposition.

Proposition 3.1 *There exists an embedded disc $D \subset (M, \xi)$ bounded by Γ, which satisfy the following conditions.*

1. *There develop an tree of type ${}^kT$ whose edges are leaf of D_ξ.*
2. *The characteristic foliation D_ξ means that Γ belongs to the class $\Gamma_{{}^kT}$ the generic surrounding of $\widehat{{}^kT}$.*

(see Figure 8.)

This Proposition is proved in the following 5 steps. We will perturb D and observe that there develop the tree of type ${}^kT$ in the characteristic foliation. We use the technique which Eliashberg used in the case of transversal knots.

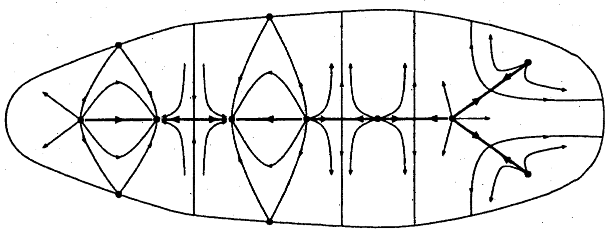

Figure 8:

• **Step 1** *The embedded disc D can be perturbed so that the characteristic foliation D_ξ on D may become as Figure 9(a). Where the figure $\oplus, \ominus$ means a simply connected domain whose boundary is transversal to the characteristic foliation. The sign corresponds to the orientation of the characteristic foliation at the boundary. (looking outward or inward.)*

Remark 1 *On account of [E2], a tree corresponds to each $\oplus$, $\ominus$-domain. This tree has positive (resp., negative) elliptic points of D_ξ as vertices and stable (resp., unstable) separatrices of negative (resp., positive) hyperbolic points as edges.*

• **Step 2** *The embedded disc D can be perturbed so that the characteristic foliation D_ξ may become as Figure 9(b).*

On D of Step 2 we can take $k+1$ transversal trivial knots $\Gamma_1, \ldots, \Gamma_{k+1}$ as the broken line in Figure 9(b). They are positive or negative alternately. This completes the proof of Theorem A.

• **Step 3** *The embedded disc D can be perturbed so that the characteristic foliation D_ξ may become as Figure 9(c).*

We may suppose that the number of vertices of a tree corresponding to the $\oplus$-domain of D_ξ is greater than or equal to that of $\ominus$-domain, by changing orientation if necessary.

• **Step 4** *The embedded disc D can be perturbed so that the characteristic foliation D_ξ may become as Figure 9(d).*

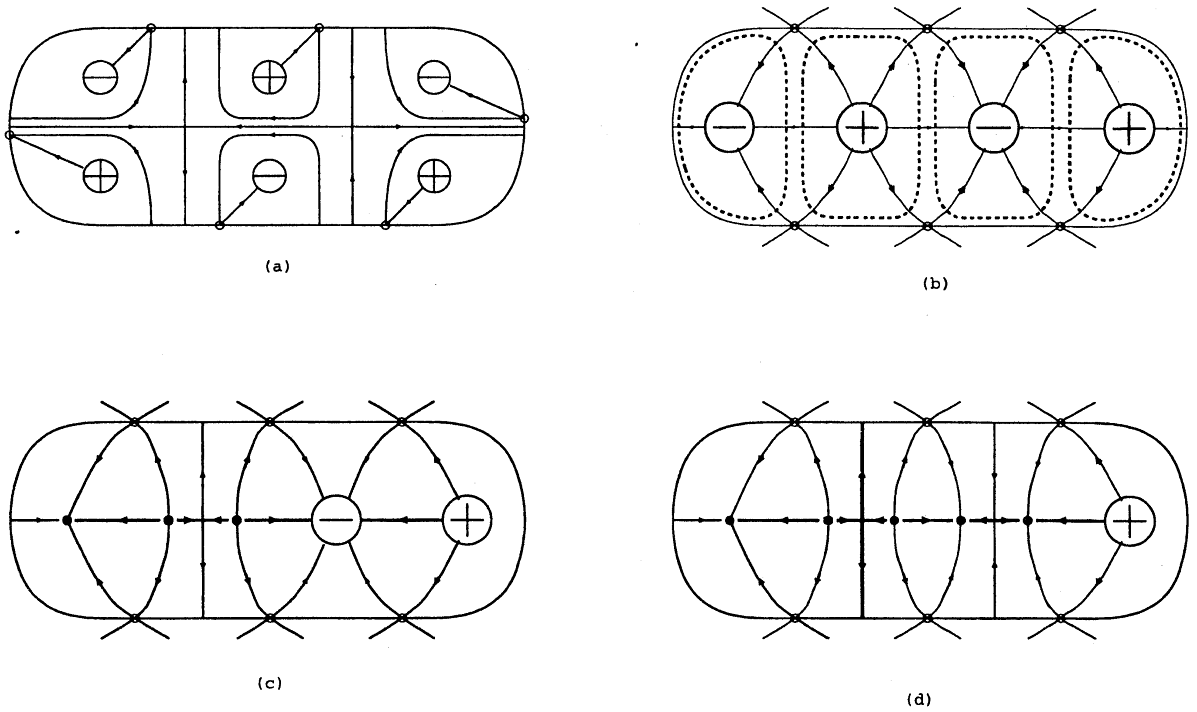

Figure 9:

• **Step 5** *The embedded disc D can be perturbed so that the characteristic foliation D_ξ may become as Figure 8 of Proposition 3.1.*

This completes the proof of Proposition 3.1. □

3.2 The essential self-linking number.

Finally, we observe the essential self-linking number of a generic trivial knot Γ with $2k$ non-transversal points in a tight contact 3-manifold. First, we define the essential self-linking number precisely. Let γ be a generic trivial knot having $2l$ non-transversal points in a tight contact 3-manifold. On account of Theorem A γ is represented as a connected sum of $l+1$ transversal trivial knots $\gamma_1, \gamma_2, \ldots, \gamma_{l+1}$; $\gamma = \gamma_1 \# \gamma_2 \# \cdots \# \gamma_{l+1}$. We may suppose that γ_i is a positive transversal knot if i is odd and, negative one if i is even.

Definition 3.2 *We call the following integer the* essential self-linking number *of γ.*

$$l_g(\gamma) := \sum_{i:odd} l(\gamma_i) - \sum_{j:even} l(\gamma_j)$$

Where l is the self-linking number of transversal knots.

According to Proposition 3.1, there exists a tree T for which Γ is isotopic to $\Gamma_{^kT}$ as a transversal knots. By Lemma refs-l number we obtain

$$(*) \qquad l_g(\Gamma) = \frac{1+(-1)^k}{2} - 2|T| \,.$$

Consequently, the pair $(k, |T|)$ corresponds to $(k, l_g(\Gamma))$ one to one. Therefore, to complete the proof of Theorem B we can apply Proposition 2.4. □

3.3 Complete list of generic knots.

First of all, we note that, for a transversal knot, the self-linking number can take only negative odd integer (see [E1], [E2], [E3]). So, by the definition, the essential self-linking number can take only odd integers if k is even, and even integers if k is odd.

Let $T_{(n)}$ be a tree having n vertices without branches. We write $\Gamma_n^k := \Gamma_{^kT_{(n)}}$ for convenience. $\Gamma_{^kT_{(n)}}$ is a generic trivial knot constructed from $T_{(n)}$ as in Section 2. According to the above equation $(*)$,

$$l_g(\Gamma_n^k) = \frac{1+(-1)^k}{2} - 2n \,.$$

Therefore, $l_g(\Gamma_n^k)$ for $k = 1, 2, 3, \ldots, n = 0, 1, 2, \ldots$ takes all possible values which are allowed for trivial generic knots, by taking reversed orientation if necessary. Note that

$$\Gamma_0^k = -\Gamma_0^k \qquad \text{if } k \text{ is odd,}$$
$$\Gamma_1^k = -\Gamma_0^k, \quad \Gamma_0^k = -\Gamma_1^k \qquad \text{if } k \text{ is even}$$

Where $-\Gamma$ means Γ with the reversed orientation. Therefore, according to Theorem B, Γ_n^k for $n = 0, 1, 2, \ldots$ if k is odd, and for $n = 1, 2, 3, \ldots$ if k

is even, form a complete list of generic isotopy class of topologically trivial generic knots in tight contact 3-manifolds, up to orientation.

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